COLLOQUIUM MATHEMATICUM

VOL. 127 2012 NO. 1

DIRECT SUMS OF SEMI-PROJECTIVE MODULES

BY

DERYA KESKİN TÜTÜNCÜ (Ankara), BERKE KALEBOĞAZ (Ankara)
and PATRICK F. SMITH (Glasgow)

Abstract. We investigate when the direct sum of semi-projective modules is semi-projective. It is proved that if R is a right Ore domain with right quotient division ring $Q \neq R$ and X is a free right R-module then the right R-module $Q \oplus X$ is semi-projective if and only if there does not exist an R-epimorphism from X to Q.

1. Introduction. In this work all rings have an identity and all modules are unital right modules. Following [2, 4.20] and [10, p. 260], an R-module M is called *semi-projective* provided for all endomorphisms α and β of M with $\beta(M) \subseteq \alpha(M)$ there exists an endomorphism γ of M such that $\beta = \alpha\gamma$. As Wisbauer [10, p. 260] observes, an R-module M with endomorphism ring $S = \mathrm{End}(M_R)$ is semi-projective if and only if $\alpha S = \mathrm{Hom}_R(M, \alpha(M))$. In [11, Examples 5.6], the semi-projectivity notion has been discussed, as well as a stronger condition called *intrinsically projective*. Examples 5.6 of [11] say in particular that a module M_R with endomorphism ring S is semi-projective if S is a right PP-ring and the kernels of endomorphisms of M are M-generated. In particular, if ${}_SM$ is flat and S is right semi-hereditary, then M is semi-projective (see [11, Examples 5.6]). For an endomorphism α of an R-module M we define $D(\alpha)$ as $\mathrm{Hom}_R(M, \alpha(M))$.

The remainder of our work is organized as follows. In Section 2, we give some basic properties of semi-projective modules, and provide some characterizations. We prove that every nonsingular extending module is semi-projective (Corollary 2.6). Let R be a Dedekind domain and let M be an R-module which is a direct sum of cyclic modules. Then M is quasi-projective iff it is semi-projective iff it is direct projective (Theorem 2.11). It is shown that every direct summand of a semi-projective module inherits this property (Lemma 2.7), while a direct sum of semi-projective modules need not be semi-projective (Corollary 2.10). We show that a module

2010 *Mathematics Subject Classification*: Primary 16D40, 16U10; Secondary 16D50, 13C11, 13F10.

Key words and phrases: semi-projective module, direct sum, right Ore domain, quotient ring.

DOI: 10.4064/cm127-1-5

$M = \bigoplus_{i \in I} M_i$ with $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$ in I is semi-projective iff M_i is semi-projective for all $i \in I$.

The focus in Section 3 is on studying direct sums of semi-projective modules over right Ore domains. We prove that if R is a right Ore domain with right quotient division ring $Q \neq R$ and X is a free right R-module then the right R-module $Q \oplus X$ is semi-projective if and only if there does not exist an R-epimorphism from X to Q (Corollary 3.6).

In Section 4, we observe that if R is a PID with field of fractions Q, and X is a proper submodule of Q such that $R \subseteq X$, then M is finitely generated iff it is projective iff it is semi-projective iff it is direct projective (Theorem 4.3).

2. Semi-projective modules. Let R be a ring. An R-module M is called *direct projective* if for every direct summand K of M every epimorphism from M to K splits (see [2, 4.21] or [10, p. 365]). It is pointed out in [2, p. 33] that M is direct projective if every submodule N such that M/N is isomorphic to a direct summand of M is also a direct summand of M. In [7, p. 57], direct projective modules are called *modules which satisfy condition* (D_2). Note the following elementary fact.

LEMMA 2.1. *A module M is direct projective if and only if for all endomorphisms α and β of M with $\beta(M) \subseteq \alpha(M)$ and $\alpha(M)$ a direct summand of M there exists an endomorphism γ of M such that $\beta = \alpha\gamma$.*

Proof. Suppose first that M is direct projective. Let α and β be endomorphisms of M with $\beta(M) \subseteq \alpha(M)$ and $K = \alpha(M)$ a direct summand of M. Because M is direct projective, there exists a homomorphism $\delta : K \to M$ such that $\alpha\delta = 1$. Now $\gamma = \delta\beta$ is an endomorphism of M such that $\beta = \alpha\gamma$.

Conversely, suppose that M has the stated condition. Let L be a direct summand of M and $\varphi : M \to L$ be an epimorphism. There exists a submodule L' of M such that $M = L \oplus L'$. Let $\theta : M \to L$ be the canonical projection. Clearly $\theta(M) = L = \varphi(M)$. By hypothesis, there exists an endomorphism λ of M such that $\theta = \varphi\lambda$. Let $\iota : L \to M$ denote the inclusion mapping. For all $y \in L$, $y = \theta(y) = \varphi\lambda(y) = \varphi\lambda\iota(y)$. It follows that $\varphi(\lambda\iota) = 1$ and hence $\varphi : M \to L$ splits. Thus M is direct projective. ∎

Lemma 2.1 shows that we have the following hierarchy:

projective $\Rightarrow$ quasi-projective $\Rightarrow$ semi-projective $\Rightarrow$ direct projective.

In particular, every semisimple module, being quasi-projective, is semi-projective (see, for example, [1, p. 191, Ex. 17]).

Let $\mathbb{N}$ denote the set of natural numbers $1, 2, \dots$, $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ the rational field. It is clear that, for any prime p in $\mathbb{Z}$, the Prüfer p-group $\mathbb{Z}(p^\infty)$ is not direct projective and hence not semi-projective. In contrast, every nonsingular injective module is semi-projective. First we prove a simple lemma.

Lemma 2.2. *Let ϵ be any idempotent endomorphism of a module M with endomorphism ring S. Then $\epsilon S = D(\epsilon)$.*

Proof. Let $\beta \in D(\epsilon)$. This means that β is an endomorphism of M such that $\beta(M) \subseteq \epsilon(M)$. Then $\epsilon = \epsilon^2$ implies that

$$(1-\epsilon)\beta(M) \subseteq (1-\epsilon)\epsilon(M) = 0.$$

Thus $(1-\epsilon)\beta = 0$ and hence $\beta = \epsilon\beta \in \epsilon S$. ∎

Let α be an endomorphism of a module M with endomorphism ring S such that $\alpha(M)$ is a direct summand of M. Then $\alpha(M) = \epsilon(M)$ for some idempotent endomorphism ϵ of M. If $\beta \in D(\alpha)$ then $\beta(M) \subseteq \alpha(M) = \epsilon(M)$. It follows that $D(\alpha) \subseteq D(\epsilon)$. Now we consider an endomorphism of M whose kernel is a direct summand of M.

Lemma 2.3. *Let α be an endomorphism of a module M with endomorphism ring S such that the kernel of α is a direct summand of M. Then $D(\alpha) = \alpha S$.*

Proof. Let $K = \ker\alpha$. Then there exists a submodule L of M such that $M = K \oplus L$. Note that $\alpha(M) = \alpha(K) + \alpha(L) = \alpha(L)$. Let $\lambda : L \to \alpha(M)$ be the homomorphism defined by $\lambda(x) = \alpha(x)$ for all $x \in L$. Note that λ is an isomorphism. If β is any endomorphism of M such that $\beta(M) \subseteq \alpha(M)$ then $\gamma = \lambda^{-1}\beta$ is an endomorphism of M such that $\beta = \alpha\gamma$. It follows that $D(\alpha) = \alpha S$. ∎

A module M is called *Rickart* if the kernel of any endomorphism of M is a direct summand of M. Thus we have

Corollary 2.4. *Let M be a Rickart module. Then M is semi-projective.*

Note that any Rickart module satisfies the sufficient condition of [11, Examples 5.6] (see [10, 39.10 (1)]).

Corollary 2.5. *Let M be a module with endomorphism ring S such that S is a von Neumann regular ring. Then M is semi-projective.*

Proof. By [8, Theorem 4], M is a Rickart module. Thus M is semi-projective by Corollary 2.4. ∎

A module M is called *extending* provided every submodule is essential in a direct summand of M. For example, semisimple modules are extending, as are uniform modules and injective modules.

Corollary 2.6. *Every nonsingular extending module is semi-projective.*

Proof. Let M be any nonsingular extending module. Let α be any endomorphism of M and let $K = \ker\alpha$. There exists a direct summand L of

M such that K is an essential submodule of L. Now $M/K \cong \alpha(M)$, which is nonsingular. Thus L/K is nonsingular and hence $K = L$. This means that K is a direct summand of M. Therefore M is a Rickart module. By Corollary 2.4, M is semi-projective. ∎

Note that the $\mathbb{Z}$-module $\mathbb{Q}$ is semi-projective (Corollary 2.6) but not quasi-projective (see, for example, [3, Theorem]). It is not difficult to check that every direct summand of a semi-projective (respectively, direct projective) module is semi-projective (respectively, direct projective), as we show next for completeness.

LEMMA 2.7. *Every direct summand of a semi-projective* (*respectively, direct projective*) *module is also semi-projective* (*respectively, direct projective*).

Proof. Let a semi-projective module M be a direct sum of submodules M_1, M_2. Let α and β be endomorphisms of M_1 such that $\beta(M_1) \subseteq \alpha(M_1)$. Now define endomorphisms λ and μ of M as follows: $\lambda(m_1 + m_2) = \alpha(m_1)$ and $\mu(m_1 + m_2) = \beta(m_1)$ for all $m_1 \in M_1$ and $m_2 \in M_2$. Clearly $\mu(M) \subseteq \lambda(M)$. By hypothesis, there exists an endomorphism ν of M such that $\mu = \lambda\nu$. If $\iota : M_1 \to M$ denotes the inclusion mapping and $\pi : M \to M_1$ the canonical projection then let γ denote the endomorphism $\pi\nu\iota$ of M_1. It is easy to check that $\beta = \alpha\gamma$. It follows that M_1 is a semi-projective module. The case of a direct summand of a direct projective module can be proved similarly. ∎

It is stated in [9] that the direct sum of any collection of semi-projective modules is also semi-projective. This is not true in general although it is true sometimes. For example, Haghany and Vedadi [5, p. 490] prove that if R is a commutative domain with field of fractions F then the R-module $R \oplus F$ is semi-projective. We shall show that the direct sum of semi-projective modules need not be semi-projective, nor even direct projective. Then we shall go on to investigate when the direct sum of semi-projective modules is semi-projective.

First we shall show that the direct sum of semi-projective modules need not be direct projective.

LEMMA 2.8. *Let R be a ring and let X and Y be R-modules such that the R-module $X \oplus Y$ is direct projective. Then every epimorphism $\varphi : X \to Y$ splits.*

Proof. Clear by [7, Lemma 4.6(i)]. ∎

COROLLARY 2.9. *Given any semi-projective R-module Y which is not projective, there exists a projective R-module X such that the R-module $X \oplus Y$ is not direct projective* (*and hence not semi-projective*).

Proof. There exists a free R-module X and a nonsplitting epimorphism $\varphi : X \to Y$. By Lemma 2.8, the module $X \oplus Y$ is not direct projective. ∎

COROLLARY 2.10. *The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is semi-projective but the $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})} \oplus \mathbb{Q}$ is not direct projective (and hence not semi-projective).*

Proof. The module $\mathbb{Z}\oplus\mathbb{Q}$ is semi-projective by [5, p. 490]. Because there is an epimorphism from $\mathbb{Z}^{(\mathbb{N})}$ to $\mathbb{Q}$, Lemma 2.8 shows that the $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})} \oplus \mathbb{Q}$ is not direct projective. ∎

Now we show that every finitely generated direct projective $\mathbb{Z}$-module is quasi-projective. In fact, more is true. Let R be a (commutative) Dedekind domain and let M be a nonzero torsion cyclic R-module. It is well known that M is a direct sum of primary cyclic R-modules. Let X be a nonzero primary cyclic R-module. Being cyclic, $X \cong R/A$ for some proper ideal A of R and being primary, $P^n \subseteq A$ for some positive integer n. Now every nonzero ideal of R is invertible and A is a product of maximal ideals. It follows that $A = P^k$ for some positive integer k with $1 \le k \le n$.

THEOREM 2.11. *Let R be any Dedekind domain. Then the following statements are equivalent for an R-module M which is a direct sum of cyclic submodules:*

(i) *M is quasi-projective.*
(ii) *M is semi-projective.*
(iii) *M is direct projective.*

Proof. (i)⇒(ii)⇒(iii). Clear.

(iii)⇒(i). Let M be a direct sum of cyclic submodules M_i ($i \in I$) and suppose that M is direct projective. Suppose that M is not torsion. Then $M_j \cong R$ for some $j \in I$. If M is not free then there exists $k \in I$ such that M_k is torsion cyclic and hence there exists a nonsplitting epimorphism $\varphi : M_j \to M_k$. By Lemma 2.8, $M_j \oplus M_k$ is not direct projective and, by Lemma 2.7, neither is M. Thus M is free.

Now suppose that M is a torsion R-module. Let P be any maximal ideal in R and let N denote the P-primary component of M. Suppose that $N \neq 0$. By the above remarks, $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$ for some index set Λ and nonzero cyclic P-primary submodules N_λ ($\lambda \in \Lambda$). Again by the above remarks, for each $\lambda \in \Lambda$ there exists a positive integer m_λ such that $N_\lambda \cong R/P^{m_\lambda}$. If $m_\mu < m_\nu$ for some $\mu \neq \nu$ then there is a nonsplitting epimorphism $R/P^{m_\nu} \to R/P^{m_\mu}$. By Lemmas 2.7 and 2.8, N is not a direct projective module and hence neither is M. Thus $m_\mu = m_\nu$ for all $\mu \neq \nu$ in Λ. It follows that N is quasi-projective. We have proved that every primary component of M is quasi-projective and hence so also is M. This proves the result. ∎

Corollary 2.12. *Every finitely generated direct projective $\mathbb{Z}$-module is quasi-projective.*

Proof. By Theorem 2.11. ∎

Let R be a ring and let X and M be (right) R-modules. Then we shall say that X is *M-sprojective* provided for every endomorphism α of M and homomorphism $\beta : X \to M$ with $\beta(X) \subseteq \alpha(M)$ there exists a homomorphism $\gamma : X \to M$ such that $\beta = \alpha\gamma$. It is clear that a module M is semi-projective if and only if M is M-sprojective. Note the following elementary fact which should be compared with [1, Proposition 16.7]. We give the proof for completeness.

Proposition 2.13. *Given R-modules X and M, X is M-sprojective if and only if for every submodule L of M such that M/L embeds in M and for every homomorphism $\beta : X \to M/L$ there exists a homomorphism $\gamma : X \to M$ such that $\beta = \pi\gamma$, where $\pi : M \to M/L$ is the canonical projection.*

Proof. The necessity is clear. Conversely, suppose that X and M have the stated condition. Let α be an endomorphism of M and let $\beta : X \to M$ be a homomorphism such that $\beta(X) \subseteq \alpha(M)$. Let $N = \alpha(M)$ and let K denote the kernel of α. Then $N \cong M/K$. For each $x \in N$ there exists $m \in M$ such that $x = \alpha(m)$. Define the isomorphism $\theta : N \to M/K$ by $\theta(x) = m + K$. Note that $\pi = \theta\alpha$. By hypothesis, there exists a homomorphism $\gamma : X \to M$ such that $\pi\gamma = \theta\beta$. This implies that $\beta = \theta^{-1}\pi\gamma = \alpha\gamma$. It follows that X is M-sprojective. ∎

Proposition 2.14. *Given a module M, every direct sum of M-sprojective modules is also M-sprojective.*

Proof. Adapt the proof of [1, Proposition 16.10(1)]. ∎

It is not clear if there are analogues of [1, Proposition 16.12] for M-sprojective modules. By Lemma 2.8 if R is a commutative domain which is not a field and U a simple R-module then the R-module $R \oplus U$ is not semi-projective. Note that $\mathrm{Hom}_R(U, R) = 0$ but $\mathrm{Hom}_R(R, U) \neq 0$. Compare this fact with the following result.

Remark 2.15. *Let a module M be a direct sum of submodules M_i $(i \in I)$ such that* $\mathrm{Hom}_R(M_i, M_j) = 0$ *for all $i \neq j$ in I. Then M is semi-projective if and only if M_i is semi-projective for all $i \in I$.*

Proof. The necessity follows by Lemma 2.7. Conversely, suppose that M_i is semi-projective for all $i \in I$. For each $k \in I$, let $\iota_k : M_k \to M$ denote the inclusion mapping and let $\pi_k : M \to M_k$ denote the canonical projection. Let α be any endomorphism of M. For all $j \neq k$ in I, $\pi_j\alpha\iota_k \in \mathrm{Hom}_R(M_k, M_j) = 0$. Thus $\alpha(M_k) \subseteq M_k$ for all $k \in I$. But this implies that $\alpha(M) = \bigoplus_{i\in I} \alpha(M_i)$. Now let β be an endomorphism of M such that $\beta(M) \subseteq \alpha(M)$. For each $k \in I$, $\beta(M_k) \subseteq \alpha(M_k)$ and hence there exists an endomorphism γ_k of M_k such that $\alpha\iota_k\gamma_k = \beta\iota_k$. Define $\gamma = \sum_{k\in I} \gamma_k\pi_k$, which is an endomorphism of M. It is easy to check that $\beta = \alpha\gamma$. It follows that M is semi-projective. ∎

Recall that an element c of a ring R is called *regular* provided $cr \neq 0$ and $rc \neq 0$ for all $0 \neq r \in R$. Following [4, p. 104] an R-module X is called *divisible* in case $X = Xc$ for every regular element c of R. An R-module Y is called *torsion* if for all $y \in Y$ there exists a regular element c in R such that $yc = 0$. On the other hand, an R-module Z is called *torsion-free* if whenever $z \in Z$ satisfies $zd = 0$ for some regular element d of R then $z = 0$. Note the following corollary of Remark 2.15 which provides many examples of semi-projective modules.

Corollary 2.16. *Let R be a prime right Goldie ring such that R is not right primitive and let a right R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is semi-projective.*

Proof. Let Q denote the classical right quotient ring of R. Then it is well-known that X is isomorphic to a direct sum of isomorphic copies of the R-module Q and that X is nonsingular injective (see, for example, [4, Propositions 6.12 and 6.13]). Let $\varphi \in \mathrm{Hom}_R(Y, X)$ and let $y \in Y$. There exists a regular element $d \in R$ such that $yd = 0$ and hence $\varphi(y)d = \varphi(yd) = 0$. It follows that $\varphi(y) = 0$ for all $y \in Y$ and hence $\varphi = 0$. Thus $\mathrm{Hom}_R(Y, X) = 0$. Now suppose that $\mathrm{Hom}_R(X, Y) \neq 0$. Then $\mathrm{Hom}_R(Q, V) \neq 0$ for some simple R-module V. Let $\alpha : Q \to V$ be a nonzero homomorphism. Because R is not right primitive, V has nonzero annihilator in R and hence $Vc = 0$ for some regular element c of R. Then $\alpha(Q) = \alpha(Qc) = \alpha(Q)c = Vc = 0$, a contradiction. It follows that $\mathrm{Hom}_R(X, Y) = 0$. By Corollary 2.6 and Remark 2.15, M is semi-projective. ∎

In particular, if R is a prime ring and R satisfies a polynomial identity (a *PI ring* for short) then we have the following result.

Corollary 2.17. *Let R be a prime PI ring which is not Artinian and let a right R-module M be a direct sum of a torsion-free divisible submodule X and a torsion semisimple submodule Y. Then M is semi-projective.*

Proof. By [6, Corollary 13.6.6] R is right Goldie and by [6, Theorem 13.3.8] R is not right primitive. Apply Corollary 2.16. ∎

A module M is called *semi-Hopfian* if the kernel of every epimorphism $\varphi : M \to M$ is a direct summand of M. Note the following fact.

Lemma 2.18. *Every direct projective module is semi-Hopfian.*

Proof. This is clear since every epimorphism from M to M splits. ∎

Semi-Hopfian modules are semi-projective in the case of divisible modules over prime PI rings and this may be true more widely.

Proposition 2.19. *Let R be a prime PI ring. Then the following statements are equivalent for a divisible R-module X:*

(i) *X is semi-projective.*
(ii) *X is direct projective.*
(iii) *X is semi-Hopfian.*
(iv) *X is nonsingular.*

Moreover, in this case X is injective.

Proof. (i)⇒(ii). By Lemma 2.1.

(ii)⇒(iii). By Lemma 2.18.

(iii)⇒(iv). Suppose that X is not nonsingular. There exist a nonzero element $x \in X$ and a nonzero central element $c \in R$ such that $xc = 0$ (see, for example, [6, Theorem 13.6.4 and Corollary 13.6.6]). Let $Y = \{u \in X : uc = 0\}$. It is easy to check that Y is a submodule of X. Now $X = Xc$ because c is a regular element of the prime ring R. Define a mapping $\theta : X \to X$ by $\theta(w) = wc$ for all $w \in X$. It is easy to check that θ is an epimorphism with kernel Y. Suppose that Y is a direct summand of X. Then $X = Xc$ implies that $Y = Yc = 0$, a contradiction. Thus Y is not a direct summand of X and hence X is not semi-Hopfian.

(iv)⇒(i). By [4, Proposition 6.12], X is injective. Then X is semi-projective by Corollary 2.6.

The last part follows by [4, Proposition 6.12]. ∎

3. Modules over right Ore domains. Following [6, 3.1.1], a ring Q is called a *quotient ring* if every regular element of Q is a unit. Given a quotient ring Q a subring R of Q is called a *right order* in Q if for each element $q \in Q$ there exist $r \in R$ and a regular element c of R such that $q = rc^{-1}$. Given a submodule X of the right R-module Q we define $\mathcal{O}(X) = \{q \in Q : qX \subseteq X\}$. Note that $\mathcal{O}(X)$ is a subring of Q. Compare the next result with [6, Proposition 3.1.15].

Lemma 3.1. *Let a ring R be a right order in a quotient ring Q and let X be a submodule of the right R-module Q such that X contains a regular element of R. Then α is an endomorphism of the right R-module X if and only if there exists $q \in \mathcal{O}(X)$ such that $\alpha(x) = qx$ for all $x \in X$.*

Proof. Given $q \in \mathcal{O}(X)$ it is clear that the mapping $\alpha : X \to X$ defined by $\alpha(x) = qx$ $(x \in X)$ is an R-homomorphism. On the other hand, let β be an endomorphism of X. Let c be a regular element of R such that $c \in X$. There exists $p \in X$ such that $\beta(c) = p$. Let $x \in X$. Then $x = ab^{-1}$ for some $a \in R$ and regular element $b \in R$. Note that $xb = a \in R$. There exist $a_1 \in R$ and a regular element $c_1 \in R$ such that $ac_1 = ca_1$. Then $xbc_1 = ca_1$ and hence

$$\beta(x)bc_1 = \beta(xbc_1) = \beta(ca_1) = \beta(c)a_1 = pa_1.$$

It follows that $\beta(x) = pa_1c_1^{-1}b^{-1} = pc^{-1}ab^{-1} = (pc^{-1})x$. Thus $\beta(x) = (pc^{-1})x$ for all $x \in X$. Note that $(pc^{-1})X = \beta(X) \subseteq X$ and hence $pc^{-1} \in \mathcal{O}(X)$. ∎

Proposition 3.2. *Let R be a right Ore domain with right quotient division ring Q. Then every submodule of the right R-module Q is semi-projective.*

Proof. Let X be any submodule of Q_R. If $X = 0$ then X is clearly semi-projective. Suppose that $X \neq 0$. Let $S = \mathrm{End}(X_R)$ and let $\alpha, \beta \in S$ with $\beta(X) \subseteq \alpha(X)$. If $\alpha = 0$ then $\beta = 0$ and hence $\beta \in \alpha S$. Suppose that $\alpha \neq 0$. By Lemma 3.1, there exist $p, q \in \mathcal{O}(X)$ with $\alpha(x) = px$ and $\beta(x) = qx$ for all $x \in X$. Clearly $p \neq 0$ and

$$qX = \beta(X) \subseteq \alpha(X) = pX \subseteq Q.$$

Because p is nonzero we have $p^{-1}q \in Q$. Moreover, $p^{-1}q \in \mathcal{O}(X)$. Define a mapping $\gamma : X \to X$ by $\gamma(x) = (p^{-1}q)x$ $(x \in X)$. Then $\gamma \in S$ and $\beta = \alpha\gamma \in \alpha S$. It follows that X is semi-projective. ∎

The next lemma is elementary but is included for completeness.

Lemma 3.3. *Let a module M be the direct sum of a projective submodule X and a submodule Y. Then M is semi-projective if and only if for all endomorphisms α, β of M with $\beta(X) = 0$ and $\beta(Y) \subseteq \alpha(M)$ there exists an endomorphism γ of M such that $\beta = \alpha\gamma$.*

Proof. The necessity is clear. Conversely, suppose that M, X and Y have the stated property. Let φ, θ be endomorphisms of M with $\varphi(M) \subseteq \theta(M)$. Let $\iota : X \to M$ denote the inclusion mapping. Because X is projective, there exists a homomorphism $\lambda : X \to M$ such that $\varphi\iota = \theta\lambda$. Let μ be the endomorphism $\lambda\pi$ of M, where $\pi : M \to X$ is the canonical projection. Then $\nu = \varphi - \theta\mu$ is also an endomorphism of M. It is clear that $\nu(X) = 0$ and $\nu(M) \subseteq \theta(M)$. By hypothesis, there exists an endomorphism γ of M such that $\nu = \theta\gamma$ and hence $\varphi = \theta(\mu + \gamma)$. Thus M is semi-projective. ∎

Before proving the next result we note the following well known fact which we shall prove for completeness.

Lemma 3.4. *Let R be a right Ore domain with right quotient division ring $Q \neq R$. Then* $\mathrm{Hom}_R(Q, R) = 0$.

Proof. Let $\varphi \in \mathrm{Hom}_R(Q, R)$. For each nonzero element c of R, $Q = Qc$ and hence $\varphi(Q) = \varphi(Qc) = \varphi(Q)c \subseteq Rc$. Suppose that $\varphi(Q) \neq 0$. Then R contains a minimal left ideal and hence $R = Q$, a contradiction. Thus $\mathrm{Hom}_R(Q, R) = 0$. ∎

Let R be a ring and M an R-module. We shall denote by $g(M_R)$ the least cardinal κ such that there exists an index set Λ of cardinality κ and elements m_λ ($\lambda \in \Lambda$) with $M = \sum_{\lambda\in\Lambda} m_\lambda R$. We have already noted that the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}^{(\mathbb{N})}$ is not semi-projective. Compare this fact with the following result.

THEOREM 3.5. *Let R be a right Ore domain with right quotient division ring Q and let X be a projective right R-module such that $g(X_R) < g(Q_R)$. Then the right R-module $M = Q \oplus X$ is semi-projective.*

Proof. Note that X is a direct summand of a free R-module Y such that $g(X_R) \leq g(Y_R)$. By Lemma 2.7 we can suppose without loss of generality that X is free. Let e_i ($i \in I$) be a basis of X with $|I| = \kappa$. Note next that if φ is an endomorphism of M then $\pi_Q\varphi\iota$ is an endomorphism of the R-module Q, where $\iota : Q \to Q \oplus X$ is the inclusion mapping and $\pi_Q : Q \oplus X \to Q$ the canonical projection. By Lemma 3.1 there exists $p \in Q$ such that $\pi_Q\varphi\iota(u) = pu$ for all $u \in Q$. Next note that if $\pi_X : Q \oplus X \to X$ is the canonical projection then $\pi_X\varphi\iota : Q \to X$ is an R-homomorphism. Because X is free, Lemma 3.4 gives $\pi_X\varphi\iota = 0$. Thus $\varphi(u, 0) = (pu, 0)$ for all $u \in Q$.

Let α and β be nonzero endomorphisms of M such that $\beta(M) \subseteq \alpha(M)$ and $\beta(X) = 0$. There exist q, q_i ($i \in I$) in Q and a_i ($i \in I$) in R such that $\alpha(u, 0) = (qu, 0)$ ($u \in Q$) and $\alpha(0, e_i) = (q_i, a_i)$ for all $i \in I$. Next there exists $q' \in Q$ such that $\beta(u, 0) = (q'u, 0)$ for all $u \in Q$. Note that $\beta \neq 0$ implies that $q' \neq 0$. For each $u \in Q$, there exist $w \in Q$, a finite nonempty subset F of I and $r_i \in R$ ($i \in F$) such that

$$(q'u, 0) = \beta(u, 0) = \alpha\Big(w, \sum_{i\in F} e_i r_i\Big) = \Big(qw + \sum_{i\in F} q_i r_i, \sum_{i\in F} a_i r_i\Big).$$

It follows that $q'u = qw + \sum_{i\in F} q_i r_i$. Suppose that $q = 0$. Then $q'u = \sum_{i\in F} q_i r_i$. This implies that

$$Q = q'Q \subseteq \sum_{i\in I} q_i R.$$

In this case, $g(Q_R) \leq |I| = \kappa$, a contradiction.

Thus $q \neq 0$. There exist $w' \in Q$, a finite nonempty subset G of I and $s_i \in R$ ($i \in G$) such that

$$q' = qw' + \sum_{i\in G} q_i s_i = q\bar{q},$$

where $\overline{q} = w' + \sum_{i \in G} q^{-1} q_i s_i \in Q$. Define a mapping $\gamma : M \to M$ by $\gamma(u, z) = (\overline{q}u, 0)$ for all $u \in Q$ and $z \in X$. It is clear that γ is an endomorphism of M. Moreover, for all $u \in Q, z \in X$ we have

$$\beta(u, z) = \beta(u, 0) = (q'u, 0) = (q\overline{q}u, 0) = \alpha\gamma(u, z).$$

Thus $\beta = \alpha\gamma$. By Lemma 3.3, the module M is semi-projective. ∎

Theorem 3.5 has a number of immediate corollaries.

Corollary 3.6. *Let R be a right Ore domain with right quotient division ring $Q \neq R$ and let X be a free right R-module. Then the right R-module $M = Q \oplus X$ is semi-projective if and only if there does not exist an epimorphism from X to Q.*

Proof. Suppose first that M is not semi-projective. By Theorem 3.5, $g(Q) \leq g(X)$ and hence there is an epimorphism from X to Q. Conversely, suppose that there is an epimorphism $\varphi : X \to Q$ and M is semi-projective. By Lemma 2.8, φ splits and hence Q_R is projective. It follows that $\mathrm{Hom}_R(Q, R) \neq 0$, contradicting Lemma 3.4. Thus M is not semi-projective. ∎

Corollary 3.7. *Let R be a right Ore domain with right quotient division ring Q. Then the R-module $Q \oplus R$ is semi-projective.*

Proof. Suppose that $g(Q_R) \leq g(R_R)$. Clearly $g(R_R) = 1$ and hence $Q = qR$ for some $q \in Q$. In this case $Q \cong R$ as right R-modules and thus $Q \oplus R$ is a projective, and hence semi-projective, R-module. If $g(R_R) < g(Q_R)$ then $Q \oplus R$ is semi-projective by Theorem 3.5. ∎

Corollary 3.8. *Let R be a right Ore domain with right quotient division ring Q and let X be a finitely generated projective right R-module. Suppose that R is right noetherian or left Ore. Then the R-module $Q \oplus X$ is semi-projective.*

Proof. The result follows by Theorem 3.5 if Q is not a finitely generated right R-module. Suppose that Q_R is finitely generated. If R is right noetherian then Q_R is noetherian. For any nonzero $c \in R$, the ascending chain

$$R \subseteq c^{-1}R \subseteq c^{-2}R \subseteq \cdots$$

must terminate: there exists a positive integer n such that $c^{-n}R = c^{-n-1}R$. This gives $c^{-n-1} = c^{-n}b$ and hence $cb = 1$ for some $b \in R$. It follows that $Q = R$ and hence $Q \oplus X$ is a projective R-module. Now suppose that R is a left Ore domain. In this case there exists a positive integer k such that $Q = (c_1^{-1}r_1)R + \cdots + (c_k^{-1}r_k)R$ for some $r_i \in R$, $0 \neq c_i \in R$ $(1 \leq i \leq k)$. By a standard argument we can suppose without loss of generality that

$c_1 = \cdots = c_k$. Then $Q = c_1Q = r_1R + \cdots + r_kR \subseteq R$. Thus $Q = R$ and again $Q \oplus X$ is a projective R-module. In any case, $Q \oplus X$ is semi-projective. ∎

4. Some examples. We saw in Proposition 3.2 that if R is a right Ore domain with right quotient division ring Q then every R-submodule X of Q is semi-projective. Moreover, Corollary 3.7 shows that if $X = Q$ then the R-module $X \oplus R$ is semi-projective. Of course, if $X = R$ then the R-module $X \oplus R$ is projective and hence semi-projective. We shall show in this section that in case $R = \mathbb{Z}$ these are the only possible choices for a submodule X of $\mathbb{Q}$ so that the R-module $X \oplus R$ is semi-projective.

Let R be any ring and consider an R-module $M = X \oplus R$ where X is an R-module such that $\mathrm{Hom}_R(X, R) = 0$. Let φ be any endomorphism of the R-module M. Let $\iota_X : X \to M$ denote the inclusion mapping and let $\pi_X : M \to X$ and $\pi_R : M \to R$ denote the canonical projections. Note that $\pi_R\varphi\iota_X \in \mathrm{Hom}_R(X, R) = 0$ and $f = \pi_X\varphi\iota_X \in \mathrm{End}(X_R)$. Thus $\varphi(x, 0) = (f(x), 0)$ for all $x \in X$. Next there exist $y \in X$ and $a \in R$ such that $\varphi(0, 1) = (y, a)$. It follows that

$$\varphi(x, r) = (f(x) + yr, ar) \quad (x \in X,\ r \in R).$$

It is now easy to prove the following result.

LEMMA 4.1. *With the above notation, φ is an endomorphism of M if and only if there exists an endomorphism f of X and elements $y \in X$ and $a \in R$ such that $\varphi(x, r) = (f(x) + yr, ar)$ for all $x \in X$ and $r \in R$.*

COROLLARY 4.2. *Let R be a right Ore domain with right quotient division ring Q and let X be a nonzero submodule of the right R-module Q such that* $\mathrm{Hom}_R(X, R) = 0$. *Let $M = X \oplus R$. Then φ is an endomorphism of the R-module M if and only if there exist $q \in \mathcal{O}(X)$, $y \in X$ and $a \in R$ such that $\varphi(x, r) = (qx + yr, ar)$ for all $x \in X$ and $r \in R$.*

Proof. By Lemmas 3.1 and 4.1. ∎

Now we prove a theorem about modules over a commutative principal ideal domain (PID for short) (see also Theorem 2.11).

THEOREM 4.3. *Let R be a PID with field of fractions Q and let X be a proper submodule of Q such that $R \subseteq X$. Then the following statements are equivalent for the R-module $M = X \oplus R$:*

(i) *M is finitely generated.*
(ii) *M is projective.*
(iii) *M is semi-projective.*
(iv) *M is direct projective.*

Proof. (i)⇒(ii)⇒(iii)⇒(iv). Clear by Lemma 2.1.

(iv)⇒(i). Suppose that X is not finitely generated. Then X being uniform implies that X is not projective. Note that $\mathrm{Hom}_R(X, R) = 0$, for if $\varphi : X \to R$ is a nonzero homomorphism then $\varphi(X)$ is a nonzero projective ideal of R. Hence $X \cong \varphi(X)$ because X is uniform, which is a contradiction. Let $T = \mathrm{End}(X_R)$. Suppose that $T = Q$. Then for any $0 \neq a \in R$, $X = aX$. It follows that the R-module X is divisible and hence $X = Q$, a contradiction. Thus $T \neq Q$, so there exists a prime element p of R such that p is not a unit in T.

Now suppose that $X/R = p(X/R)$. Then $X = pX + R$. Let α denote the endomorphism of M defined by $\alpha(x, r) = (px + r, 0)$ for all $x \in X$ and $r \in R$. Clearly $\alpha(M) = X \oplus 0 = \pi(M)$ where $\pi : M \to X \oplus 0$ is the canonical projection. Suppose that $\pi = \alpha\gamma$ for some endomorphism γ of M. By Lemma 4.1, $1 = pq$ for some $q \in T$, a contradiction. Thus in this case M is not direct projective.

Next we suppose that $X/R \neq p(X/R)$. Recall that Q/R is isomorphic to the direct sum of the injective envelopes of the simple modules R/Rq, where Rq is a maximal ideal of R. The module X/R is torsion and hence is a direct sum of its primary components. It follows that if Y is the submodule of X containing R such that Y/R is the p-primary component of X/R then $Y = R(1/p^n)+R = R(1/p^n)$ for some positive integer n. If Y' is the submodule of X containing R such that Y'/R is the sum of the other primary components of X/R then $X = Y + Y'$. Moreover $Y'/R = p(Y'/R)$ so that $Y' = pY' + R$. Let β be the endomorphism of M defined by $\beta(x, r) = (px + r/p^n, 0)$ for all $x \in X$ and $r \in R$. For all $y' \in Y'$ there exist $z \in Y'$ and $b \in R$ such that $y' = pz + b$ and hence $(y', 0) = \beta(z, p^n b)$. Next note that $(1/p^n, 0) = \beta(0, 1)$. It follows that $\beta(M) = X \oplus 0 = \pi(M)$. If $\pi = \beta\delta$ for some endomorphism δ of M then Lemma 4.1 gives $1 = pq'$ for some $q' \in T$, a contradiction. Thus M is not direct projective in this case also. We conclude that M is not direct projective if M, and hence X, is not finitely generated. ■

Corollary 4.4. *Let R be a PID with field of fractions Q and let X be any nonzero submodule of Q. Then the following statements are equivalent for the R-module $M = X \oplus R$:*

(i) *M is semi-projective.*
(ii) *M is direct projective.*
(iii) *$X \cong R$ or $X \cong Q$.*

Proof. (i)⇒(ii). By Lemma 2.1.

(ii)⇒(iii). There exists a nonzero $c \in R$ such that $c \in X$. Clearly

$$M \cong Mc^{-1} = Xc^{-1} \oplus Rc^{-1} \cong Xc^{-1} \oplus R.$$

In addition, $R = (Rc)c^{-1} \subseteq Xc^{-1}$. By Theorem 4.3, $Xc^{-1} \cong R$ or $Xc^{-1} \cong Q$ and it follows that $X \cong R$ or $X \cong Q$.

(iii)⇒(i). By [5, p. 490] or Corollary 3.7. ∎

In this paper we have been concerned with rings R and R-modules M such that $M = X \oplus R$ for some R-module X with $\mathrm{Hom}_R(X, R) = 0$. We have seen that such modules M need not be semi-projective. In other words, if S is the endomorphism ring of the R-module M then in many cases there exists $\alpha \in S$ such that $\alpha S \neq D(\alpha)$.

Now we show that αS is an essential submodule of the right S-module $D(\alpha)$.

THEOREM 4.5. *Let R be a ring, X an R-module, M the R-module $X \oplus R$ and let S be the endomorphism ring of the R-module M. Then αS is an essential submodule of the S-module $D(\alpha)$ for every $0 \neq \alpha \in S$.*

Proof. There exists an epimorphism $\varphi : F = R^{(\Lambda)} \to M$. Let $0 \neq \alpha \in S$ and $0 \neq g \in D(\alpha)$. By the projectivity of F, there exists a homomorphism $h : F \to M$ such that $\alpha h = g\varphi$. Moreover since $g \neq 0$ and φ is surjective, there exists $\lambda \in \Lambda$ such that $g\varphi\epsilon_\lambda \neq 0$, where ϵ_λ is the inclusion map from R to F. Consider the projection map $\pi : M \to R$. Then $\alpha(h\epsilon_\lambda\pi) = g(\varphi\epsilon_\lambda\pi)$ is a nonzero element of $\alpha S \cap gS$, which shows that αS is essential in $D(\alpha)$. ∎

Acknowledgements. This work resulted from the third author's visit to the Department of Mathematics of Hacettepe University in 2010 and he would like to thank the department for its hospitality. All three authors gratefully acknowledge financial support from the Turkish Scientific Research Council (TÜBİTAK) which made the visit possible. The authors would also like to thank the referee for carefully reading this paper and for the numerous valuable comments in the report.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer, New York, 1974.

[2] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules*, Birkhäuser, Basel, 2006.

[3] L. Fuchs and K. M. Rangaswamy, *Quasi-projective abelian groups*, Bull. Soc. Math. France 98 (1970), 5–8.

[4] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Student Texts 16, Cambridge Univ. Press, Cambridge, 1989.

[5] A. Haghany and M. R. Vedadi, *Study of semi-projective retractable modules*, Algebra Colloq. 14 (2007), 489–496.

[6] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester, 1987.

[7] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Ser. 147, Cambridge Univ. Press, Cambridge, 1990.

[8] K. M. Rangaswamy, *Abelian groups with endomorphic images of special types*, J. Algebra 6 (1967), 271–280.

[9] H. Tansee and S. Wongwai, *A note on semi-projective modules*, Kyungpook Math. J. 42 (2002), 369–380.

[10] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.

[11] R. Wisbauer, *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Monogr. Surveys Pure Appl. Math. 81, Longman, 1996.

Derya Keskin Tütüncü, Berke Kaleboğaz
Department of Mathematics
Hacettepe University
06800 Beytepe, Ankara, Turkey
E-mail: keskin@hacettepe.edu.tr
bkuru@hacettepe.edu.tr

Patrick F. Smith
Department of Mathematics
Glasgow University
Glasgow G12 8QW, Scotland
E-mail: pfs@maths.gla.ac.uk

Received 23 November 2011;
revised 17 April 2012 (5580)