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A COMPUTATION OF POSITIVE ONE-PEAK POSETS THAT ARE TITS-SINCERE

BY

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Dedicated to Andrzej Tyc
on the occasion of his seventieth birthday

Abstract. A complete list of positive Tits-sincere one-peak posets is provided by applying combinatorial algorithms and computer calculations using Maple and Python. The problem whether any square integer matrix $A \in \mathbb{M}_n(\mathbb{Z})$ is $\mathbb{Z}$-congruent to its transpose A^{tr} is also discussed. An affirmative answer is given for the incidence matrices C_I and the Tits matrices $\widehat{C}_I$ of positive one-peak posets I.

1. Introduction. In this paper, we continue our study [8] of positive one-peak posets, in close connection with the results of Bondarenko and Stepochkina [3]–[5], and we freely use the terminology and notation introduced in [8], [22], [23], and [30]–[35]. Given $m \geq 1$, we denote by $\mathbb{M}_m(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square m by m integer matrices, and by $E \in \mathbb{M}_m(\mathbb{Z})$ the identity matrix. By a *poset* $J \equiv (J, \preceq)$ we mean a set J endowed with a partial order relation $\preceq$. Obviously, J is uniquely determined by its *incidence matrix* $C_J \in \mathbb{M}_J(\mathbb{Z}) \equiv \mathbb{M}_m(\mathbb{Z})$, where $m = |J|$, that is, the integer square $m \times m$ matrix

$$C_J = [c_{ij}]_{i,j \in J} \quad \text{with} \quad c_{ij} = \begin{cases} 1 & \text{for } i \preceq j, \\ 0 & \text{for } i \not\preceq j. \end{cases} \tag{1.1}$$

Throughout, we make the identifications $\mathbb{M}_J(\mathbb{Z}) \equiv \mathbb{M}_m(\mathbb{Z})$ and $\mathbb{Z}^J \equiv \mathbb{Z}^m$.

Following [22] and [23], a poset I is called a *one-peak poset* if I has a unique maximal element $*$. Throughout we assume that I is a one-peak poset of the form $I = \{1, \ldots, n, * = n+1\}$, with a unique maximal element

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$* = n+1$, and

$$\widehat{C}_I = \left[\begin{array}{c|c} C_T^{\mathrm{tr}} & -u \\ \hline 0 & 1 \end{array}\right] \in \mathbb{M}_{n+1}(\mathbb{Z}), \quad \text{with} \quad u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{1.2}$$

is the *Tits matrix* of I, where $C_T \in \mathbb{M}_T(\mathbb{Z}) = \mathbb{M}_n(\mathbb{Z})$ is the incidence matrix of the poset $T = I \setminus \{*\} = \{1, \dots, n\}$ (see [22]–[30]). The *Tits quadratic form* $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ of I is defined by

(1.3)
$$\widehat{q}_I(x) = x_1^2 + \dots + x_n^2 + x_{n+1}^2 + \sum_{i \prec j \le n} x_i x_j - (x_1 + \dots + x_n)x_{n+1} = x \cdot \widehat{C}_I \cdot x^{\mathrm{tr}}.$$

We denote by $q_I, \overline{q}_I : \mathbb{Z}^I \equiv \mathbb{Z}^{n+1} \to \mathbb{Z}$ the *incidence quadratic form* and the *Euler quadratic form* of I defined by

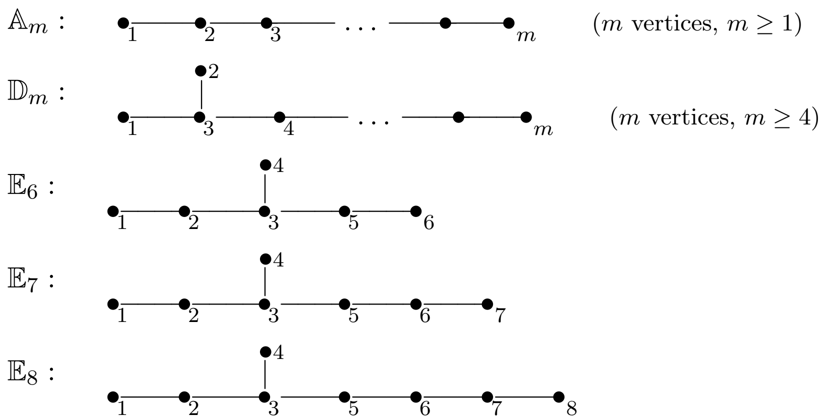
$$\begin{aligned} q_I(x) &= \sum_{j \in I} x_j^2 + \sum_{i \prec j} x_i x_j = x \cdot C_I \cdot x^{\mathrm{tr}}, \\ \overline{q}_I(x) &= x \cdot \overline{C}_I \cdot x^{\mathrm{tr}}, \end{aligned} \tag{1.4}$$

where $\overline{C}_I = C_I^{-1} \in \mathbb{M}_{n+1}(\mathbb{Z})$ is the *Euler matrix* of I (see [22]–[30]). Following [7]–[8] and [17]–[18], we call I *positive* if each (or some) of the forms $\widehat{q}_I$, q_I, and $\overline{q}_I$ is positive, that is, $\widehat{q}_I(v) > 0$ for any $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{Z}^{n+1}$.

The poset I is defined to be *Tits-sincere* if $\widehat{q}_I$ has a *sincere Tits root* $v \in \mathbb{Z}^I$, that is, a vector $v \in \mathbb{Z}^I$ such that $\widehat{q}_I(v) = 1$ and with all coordinates non-zero (see [8], [12], [16], [22]). Given a poset I, we denote by $\mathbf{s}_I$ the number of sincere Tits roots of I, that is, of $\widehat{q}_I$.

In [7]–[8], we have studied the positive one-peak posets I in relation to the simply-laced Dynkin diagrams presented in Table 1.5.

Table 1.5. Simply-laced Dynkin diagrams



It is shown in [8] (see also [14] and [31]–[35]) that, given a positive one-peak poset I, there is a unique simply-laced Dynkin diagram $\Delta_I \in \{\mathbb{A}_{n+1}, \mathbb{D}_{n+1}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (called the *Coxeter–Dynkin type* of I) such that the *Coxeter polynomial*

$$\begin{aligned} \operatorname{cox}_I(t) :=\ & \det(tE + C_I \cdot C_I^{-\mathrm{tr}}) = \det(tE + \widehat{C}_I \cdot \widehat{C}_I^{-\mathrm{tr}}) \\ =\ & \det(tE + C_I^{-1} \cdot C_I^{\mathrm{tr}}) \in \mathbb{Z}[t] \end{aligned}$$

of I (see [30]) coincides with the Coxeter polynomial $F_{\Delta_I}(t)$ of the diagram Δ_I, where

(1.6)

$$F_\Delta(t) := \begin{cases} t^m + t^{m-1} + \cdots + t^2 + t + 1, & \mathbf{c}_\Delta = m+1, \quad \text{for } \Delta = \mathbb{A}_m, \\ t^m + t^{m-1} + t + 1, & \mathbf{c}_\Delta = 2(m-1), \quad \text{for } \Delta = \mathbb{D}_m, \\ t^6 + t^5 - t^3 + t + 1, & \mathbf{c}_\Delta = 12, \quad \text{for } \Delta = \mathbb{E}_6, \\ t^7 + t^6 - t^4 - t^3 + t + 1, & \mathbf{c}_\Delta = 18, \quad \text{for } \Delta = \mathbb{E}_7, \\ t^8 + t^7 - t^5 - t^4 - t^3 + t + 1, & \mathbf{c}_\Delta = 30, \quad \text{for } \Delta = \mathbb{E}_8, \end{cases}$$

$m \geq 1$ for $\Delta = \mathbb{A}_m$ and $m \geq 4$ for $\Delta = \mathbb{D}_m$, and $\mathbf{c}_\Delta$ is the Coxeter number of Δ.

The main result of the paper is the following analogue of Kleiner's theorem [12] (see also [22, Section 10.1], compare with [36]).

THEOREM 1.7. *Let $I \equiv (I, \preceq)$ be a finite positive one-peak poset that is Tits-sincere, and let Δ_I be the Coxeter–Dynkin type of I.*

(a) $|I| \leq 8$.

(b) *If $\Delta_I = \mathbb{A}_{n+1}$ then $|I| = n+1 \leq 3$ and I is one of the posets*

$${}_0\mathbb{A}_0^* : *, \qquad {}_0\mathbb{A}_1^* : \bullet \to *, \qquad {}_0\mathbb{A}_2^* : \begin{tikzcd}[row sep=small, column sep=small] \bullet \arrow[dr] & \\ & * \\ \bullet \arrow[ur] & \end{tikzcd},$$

and, up to multiplication by -1, the vectors ${}_0w_0 = (1)$, ${}_0w_1 = (1,1)$, and ${}_0w_2 = (1,1,1)$ are the only sincere Tits roots of ${}_0\mathbb{A}_0^*$, ${}_0\mathbb{A}_1^*$, and ${}_0\mathbb{A}_2^*$, respectively.

(c) *If $\Delta_I = \mathbb{D}_{n+1}$ then $|I| = n+1 \leq 5$, I is one of the posets listed in* (c1)–(c8) *below, together with their sincere Tits roots, up to multiplication by -1:*

(c1) $\mathbb{D}_3^*$: $\begin{tikzcd}[row sep=small] \overset{2}{\bullet} \arrow[r] & \overset{3}{\bullet} \arrow[r] & \overset{4}{\star} \\ \underset{1}{\bullet} \arrow[ur] & & \end{tikzcd}$ $\qquad w_3 = (1, 1, -1, 1)$,

(c2) $\mathbb{D}_4^*$: $\begin{tikzcd}[row sep=small] \overset{2}{\bullet} \arrow[r] & \overset{3}{\bullet} \arrow[r] & \overset{4}{\bullet} \arrow[r] & \overset{5}{\star} \\ & \underset{1}{\bullet} \arrow[ur] & & \end{tikzcd}$ $\qquad w_4 = (1, 1, 1, -1, 1)$,

(c3) $\widehat{\mathbb{D}}_2^* \diamond \mathbb{A}_1$:
$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] \arrow[dr] & \overset{2}{\bullet} \arrow[r] & \overset{4}{\star} \\ & \underset{3}{\bullet} \arrow[ur] & \end{tikzcd}$$
$w_1^2 = (-1, 1, 1, 1)$,

(c4) $\widehat{\mathbb{D}}_3^* \diamond \mathbb{A}_1$:
$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] \arrow[drrr] & \overset{2}{\bullet} \arrow[r] & \overset{3}{\bullet} \arrow[rr] & & \overset{5}{\star} \\ & & & \underset{4}{\bullet} \arrow[ur] & \end{tikzcd}$$
$w_1^3 = (-1, 1, 1, 1, 1)$,

(c5) ${}_0\mathbb{D}_2^* \diamond \mathbb{A}_1$:
$$\begin{tikzcd} 1\bullet \arrow[dr] & \\ 2\bullet \arrow[r] & \star\, 4 \\ 3\bullet \arrow[ur] & \end{tikzcd}$$
${}_0w_1^2 = (1, 1, 1, 1)$,
${}_0\overline{w}_1^2 = (1, 1, 1, 2)$,

(c6) ${}_0\mathbb{D}_2^* \diamond \mathbb{A}_2$:
$$\begin{tikzcd} & 1\bullet \arrow[dr] & \\ & 2\bullet \arrow[r] & \star\, 5 \\ 3\bullet \arrow[r] & \bullet\, 4 \arrow[ur] & \end{tikzcd}$$
${}_0w_2^2 = (1, 1, 1, 1, 2)$,

(c7) ${}_1\mathbb{D}_3^* \diamond \mathbb{A}_1$:
$$\begin{tikzcd} 1\bullet \arrow[rr] & & \bullet_3 \arrow[rr] & & \star\, 5 \\ & 2\bullet \arrow[ur] & & 4\bullet \arrow[ur] & \end{tikzcd}$$
${}_1w_1^3 = (1, 1, -1, 1, 1)$,

(c8) ${}_0\mathbb{D}_3^* \diamond \mathbb{A}_1$:
$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] \arrow[dr] & \overset{2}{\bullet} \arrow[r] & \overset{5}{\star} \\ & \underset{3}{\bullet} \arrow[ur] & \underset{4}{\bullet} \arrow[u] \end{tikzcd}$$
${}_0w_1^3 = (-1, 1, 1, 1, 1)$.

(d) *If $\Delta_I = \mathbb{E}_6$ then I is one of the posets $\mathbb{P}_1, \ldots, \mathbb{P}_8, \mathbb{P}_{10}, \ldots, \mathbb{P}_{13}$ listed in Table* 1.8 *below. If $\Delta_I \in \{\mathbb{E}_7, \mathbb{E}_8\}$ then I is one of the* 154 *posets $\mathbb{P}_{17}, \ldots, \mathbb{P}_{193}$ listed in Tables* 6.2–6.3 *of* [8], *and distinguished by the symbol* $\mathbf{s}_I$, *with $\mathbf{s}_I \geq 1$, where $\mathbf{s}_I$ is the number of sincere Tits roots of I.*

(e) *The number of positive one-peak posets that are Tits-sincere equals* 177.

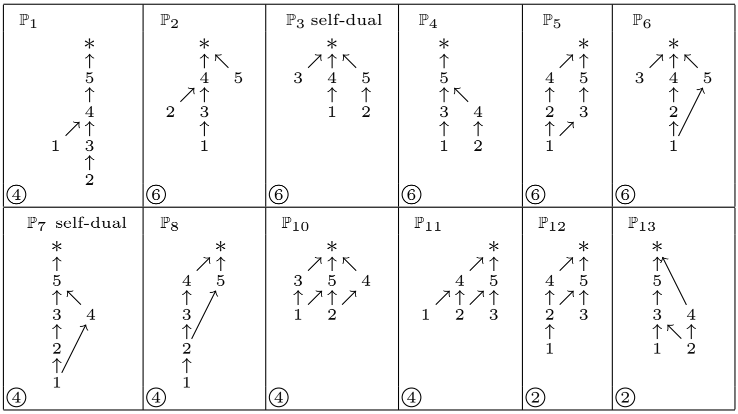
(f) *Assume that $v = (v_1, \ldots, v_n, v_*) \in \mathbb{Z}^{n+1}$ is a Tits root of a positive one-peak poset I, that is, $\widehat{q}_I(v) = 1$ and $v_* \neq 0$. Then*

$$\max\{|v_1|, \ldots, |v_n|, |v_*|\} = \begin{cases} 1 & \textit{if } \Delta_I = \mathbb{A}_{n+1},\ n \geq 0, \\ 2 & \textit{if } \Delta_I = \mathbb{D}_{n+1},\ n \geq 3, \\ 3 & \textit{if } \Delta_I = \mathbb{E}_6, \\ 4 & \textit{if } \Delta_I = \mathbb{E}_7, \\ 6 & \textit{if } \Delta_I = \mathbb{E}_8. \end{cases}$$

The proof of Theorem 1.7 is presented in Section 3. Part (a) is proved by computer calculations using combinatorial algorithms constructing the set of sincere Tits roots, for any positive poset I.

As a byproduct of our Coxeter spectral analysis (see [32]–[35]) of positive posets and their Coxeter–Dynkin types given in [8], we get in Theorem 1.10 a refinement of a result obtained by Horn and Sergeichuk [11] (see also [10],

Table 1.8. Positive Tits-sincere one-peak posets $\mathbb{P}_1, \dots, \mathbb{P}_8$, $\mathbb{P}_{10}, \dots \mathbb{P}_{13}$ of Coxeter–Dynkin type $\mathbb{E}_6$



[33]) asserting that every non-singular matrix $A \in \mathbb{M}_m(\mathbb{R})$ is $\mathbb{R}$-congruent to its transpose $A^{\rm tr}$ and there exists a matrix $B \in \mathrm{Gl}(m, \mathbb{R})$ such that $B^2 = E$ and

$$A^{\rm tr} = B \cdot A \cdot B^{\rm tr}. \tag{1.9}$$

The following theorem shows that if $A = C_I \in \mathbb{M}_{n+1}(\mathbb{Z})$ is the incidence matrix of a positive one-peak poset I then A is $\mathbb{Z}$-congruent to $A^{\rm tr}$. Moreover, by applying numerical and graphical algorithms constructed in [8, Section 7], we are able to construct (by a computer calculation) a family of matrices $B \in \mathbb{M}_{n+1}(\mathbb{Z})$ implementing the $\mathbb{Z}$-congruence of A and $A^{\rm tr}$.

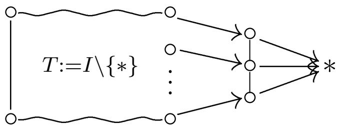
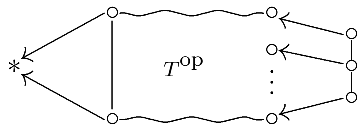
THEOREM 1.10. *Let $I \equiv (I, \preceq)$ be a finite positive one-peak poset with $|I| = n+1$. If $A \in \mathbb{M}_{n+1}(\mathbb{Z})$ is the incidence matrix C_I of I, or its Euler matrix $\overline{C}_I := C_I^{-1}$, or its Tits matrix $\widehat{C}_I$, then there exists $B \in \mathbb{M}_{n+1}(\mathbb{Z})$ such that $B^2 = E$ and* (1.9) *holds.*

The proof is presented in Section 4.

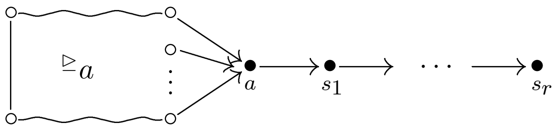
The reader is referred to [1], [2], [6], [15], [19], [20], [22]–[29], [34], and [36] for applications of finite posets in representation theory of algebras and coalgebras.

2. Reflections. In the Coxeter analysis of one-peak posets $I \equiv (I, \preceq)$ we use the following two operations defined in [8] and [21] (see also [24]–[25]).

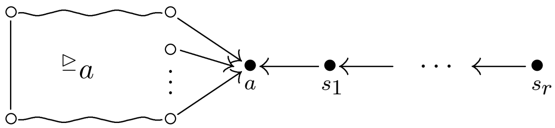
The *reflection-duality* $I \mapsto \widetilde{I} = \widehat{\mathbf{s}}_* I$ associates to any one-peak poset I with peak $* = n+1$ its reflection-dual $\widehat{\mathbf{s}}_* I := (I, \preceq_\bullet)$, where we set $a \preceq_\bullet *$ for $i \in T := I \setminus \{*\}$, and $a \preceq_\bullet b$ if $b \preceq a$ in T. The passage $I \mapsto \widetilde{I} = \widehat{\mathbf{s}}_* I$ can be visualised as follows:

$I:$  $\widetilde{I} = \widehat{\mathbf{s}}_* I:$ 

The *waist reflection* $I \mapsto \delta_a I$ associates to the poset

$I:$ 

with right waist $a \in I$ the one-peak poset

$\delta_a I:$ 

called the waist reflection of I at a, where ${}^{\trianglelefteq}a = \{i;\, i \preceq a\}$, the points $\bullet_{s_1}, \dots, \bullet_{s_r}$ are incomparable with all points $\circ_j$ in ${}^{\trianglelefteq}a \setminus \{a\} \subseteq \delta_a I$, and $\bullet_a$ is the unique maximal element of $\delta_a I$ (see [21]).

We show in the following proposition that the Euler matrix $\overline{C}_I = C_I^{-1}$ of a poset I with waist a, and the Euler matrix $\overline{C}_{\delta_a I}$ of the waist reflection poset $\delta_a I$, have the forms

$$
\begin{aligned}
\overline{C}_I &= \left[\begin{array}{ccc|c|ccccc}
\overline{c}_{1\,1} & \cdots & \overline{c}_{1\,a-1} & \overline{c}_{1\,a} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\overline{c}_{a-1\,1} & \cdots & \overline{c}_{a-1\,a-1} & \overline{c}_{a-1\,a} & 0 & 0 & \cdots & 0 & 0 \\
\hline
\overline{c}_{a\,1} & \cdots & \overline{c}_{a\,a-1} & 1 & -1 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & 0 & 1 & -1 & & & \\
0 & \cdots & 0 & 0 & & 1 & -1 & & \mbox{\Large 0} \\
\vdots & \ddots & \vdots & \vdots & & & \ddots & \ddots & \\
0 & \cdots & 0 & 0 & & & & 1 & -1 \\
0 & \cdots & 0 & 0 & \mbox{\Large 0} & & & & 1
\end{array}\right], \\
(2.1)\qquad \overline{C}_{\delta_a I} &= \left[\begin{array}{ccc|c|ccccc}
\overline{c}_{1\,1} & \cdots & \overline{c}_{1\,a-1} & \overline{c}_{1\,a} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\overline{c}_{a-1\,1} & \cdots & \overline{c}_{a-1\,a-1} & \overline{c}_{a-1\,a} & 0 & 0 & \cdots & 0 & 0 \\
\hline
\overline{c}_{a\,1} & \cdots & \overline{c}_{a\,a-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & -1 & 1 & & & & \mbox{\Large 0} \\
0 & \cdots & 0 & 0 & -1 & 1 & & & \\
\vdots & \ddots & \vdots & \vdots & & \ddots & \ddots & & \\
0 & \cdots & 0 & 0 & & & -1 & 1 & \\
0 & \cdots & 0 & 0 & \mbox{\Large 0} & & & -1 & 1
\end{array}\right].
\end{aligned}
$$

PROPOSITION 2.2. *Let $I \equiv (I, \preceq)$ be a one-peak poset with peak $* = n+1$.*

(a) *The poset I is positive if and only if its reflection-dual $\widetilde{I}$ is positive. In this case the Tits matrices $\widehat{C}_I$ and $\widehat{C}_{\widetilde{I}}$ are $\mathbb{Z}$-congruent and $\mathrm{cox}_{\widetilde{I}}(t) = \mathrm{cox}_I(t)$.*

(b) *Assume that $a \in I$ is a waist and $I'_a := \delta_a I$.*

(b1) *The matrices in* (2.1) *are the Euler matrices of I and $\delta_a I$.*

(b2) *I is positive* (*resp. non-negative, principal*) *if and only if $\delta_a I$ is positive* (*resp. non-negative, principal*).

(b3) *The matrices $\overline{C}_I$ and $\overline{C}_{\delta_a I}$ are $\mathbb{Z}$-congruent,* $\mathrm{cox}_{\delta_a I}(t) = \mathrm{cox}_I(t)$, *and $\overline{C}_{\delta_a I} = B \cdot \overline{C}_I \cdot B^{\mathrm{tr}}$, where*

(2.3)

$$B = \left[\begin{array}{ccc|c|ccc} 1 & & 0 & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \hline 0 & \cdots & 0 & 0 & 0 & & -1 \\ \vdots & \ddots & \vdots & \vdots & & \iddots & \\ 0 & \cdots & 0 & 0 & -1 & & 0 \end{array}\right], \quad B^{\mathrm{tr}} = \left[\begin{array}{ccc|c|ccc} 1 & & 0 & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & & -1 \\ \vdots & \ddots & \vdots & \vdots & & \iddots & \\ 0 & \cdots & 0 & 1 & -1 & & 0 \end{array}\right].$$

Proof. (a) The first statement is a consequence of the equality $\widehat{q}_{\widetilde{I}} = \widehat{q}_I$. Although the remaining ones follow from [8, Proposition 3.5], we give an alternative proof by applying Theorem 1.10. For this purpose we note that $\widehat{C}^{\mathrm{tr}}_{\widetilde{I}} = \widehat{S}^{\mathrm{tr}}_* \cdot \widehat{C}_I \cdot \widehat{S}_*$, where $\widehat{S}_* = \left[\begin{array}{c|c} E & 0 \\ \hline u^{\mathrm{tr}} & -1 \end{array}\right] \in \mathbb{M}_{n+1}(\mathbb{Z})$ and $u^{\mathrm{tr}} = [1, \ldots, 1] \in \mathbb{Z}^n$. Since I is positive if and only if $\widetilde{I}$ is positive, $\widehat{C}_{\widetilde{I}}$ is $\mathbb{Z}$-congruent to $\widehat{C}_I$, by Theorem 1.10.

(b) Let $\overline{C}_I$ and $\overline{C}_{\delta_a I}$ be the matrices in (2.1). Since a direct check yields $C_I \cdot \overline{C}_I = E$ and $C_{\delta_a I} \cdot \overline{C}_{\delta_a I} = E$, statement (b1) follows. Alternatively, by applying [29, Proposition 2.12], one shows that $\overline{C}_I$ is the inverse of C_I, and $\overline{C}_{\delta_a I}$ is the inverse of $C_{\delta_a I}$.

(b2) By (b3), the Euler quadratic forms $\overline{q}_I$ and $\overline{q}_{\delta_a I}$ are $\mathbb{Z}$-equivalent. Hence (b2) follows.

(b3) By a direct calculation we get

$$B \cdot \overline{C}_I = \left[\begin{array}{ccc|c|ccccc} \overline{c}_{1\,1} & \cdots & \overline{c}_{1\,a-1} & \overline{c}_{1\,a} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{c}_{a-1\,1} & \cdots & \overline{c}_{a-1\,a-1} & \overline{c}_{a-1\,a} & 0 & 0 & \cdots & 0 & 0 \\ \hline \overline{c}_{a\,1} & \cdots & \overline{c}_{a\,a-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & & & & -1 \\ 0 & \cdots & 0 & 0 & & & & -1 & 1 \\ \vdots & \ddots & \vdots & \vdots & & & \iddots & \iddots & \\ 0 & \cdots & 0 & 0 & & -1 & 1 & & \\ 0 & \cdots & 0 & 0 & -1 & 1 & & & 0 \end{array}\right], \quad B \cdot \overline{C}_I \cdot B^{\mathrm{tr}} = \overline{C}_{\delta_a I}.$$

To prove $\mathrm{cox}_{\delta_a I}(t) = \mathrm{cox}_I(t)$, we note that, as $\overline{C}_{\delta_a I} = B \cdot \overline{C}_I \cdot B^{\mathrm{tr}}$, the Coxeter matrices of I and $\delta_a I$ are adjoint, and hence their characteristic polynomials $\mathrm{cox}_I(t)$ and $\mathrm{cox}_{\delta_a I}(t)$ coincide (see [30] and [31]). ∎

3. Tits-sincere positive posets. The aim of this section is to prove Theorem 1.7 and present a complete list of all finite positive one-peak posets that are Tits-sincere. We start with an elementary fact on arbitrary sincere positive unit forms $q : \mathbb{Z}^m \to \mathbb{Z}$, with $m \geq 2$. Here we follow Ovsienko [16] and Kosakowska [13] (see also [19]).

PROPOSITION 3.1. *Let $q : \mathbb{Z}^m \to \mathbb{Z}$, $m \geq 2$, be a positive unit form, $b_q : \mathbb{Z}^m \times \mathbb{Z}^m \to \frac{1}{2} \cdot \mathbb{Z}$ the polarisation of q defined by $b_q(x, y) = \frac{1}{2} \cdot [q(x + y) - q(x) - q(y)]$, $v = (v_1, \dots, v_m) \in \mathbb{Z}^m$ a root of q, and $e_1, \dots, e_m \in \mathbb{Z}^m$ the standard basis of the group $\mathbb{Z}^m$.*

(a) *If $j \in \{1, \dots, m\}$ is such that $v \neq e_j$ and $v \neq -e_j$, then*
   (a1) $-1 \leq \frac{\partial q}{\partial x_j}(v) \leq 1$,
   (a2) $q(v - e_j) = 1$ *if and only if* $v_j \neq 0$ *and* $\frac{\partial q}{\partial x_j}(v) = 1$, *and*
   (a3) $q(v + e_j) = 1$ *if and only if* $v_j \neq 0$ *and* $\frac{\partial q}{\partial x_j}(v) = -1$.

(b) *If $v \notin \{e_1, \dots, e_m, -e_1, \dots, -e_m\}$ then there exists $j \in \{1, \dots, m\}$ such that $v_j \neq 0$ and $\frac{\partial q}{\partial x_j}(v)$ is -1 if $v_j < 0$, and 1 if $v_j > 0$.*

(c) *If v is sincere, there exists a sincere root $v' = (v'_1, \dots, v'_m) \in \mathbb{Z}^m$ of q and an index $j \leq m$ such that $v'_j \in \{-1, 1\}$ and $\frac{\partial q}{\partial x_j}(v') = v'_j$.*

Proof. We recall that, given $j \leq m$, we have $2b_q(v, e_j) = \frac{\partial q}{\partial x_j}(v)$.

(a1) Since $v \neq e_j$ and $v \neq -e_j$, we have

$$0 < q(v - e_j) = b_q(v - e_j, v - e_j) = q(v) - 2b_q(v, e_j) + q(e_j)$$
$$= 2 - 2b_q(v, e_j) = 2 - \frac{\partial q}{\partial x_j}(v)$$

and

$$0 < q(v + e_j) = b_q(v + e_j, v - e_j) = 2 + 2b_q(v, e_j) = 2 + \frac{\partial q}{\partial x_j}(v).$$

Hence, we get $-1 \leq \frac{\partial q}{\partial x_j}(v) \leq 1$.

(a2) If $v_j \neq 0$ and $2b_q(v, e_j) = \frac{\partial q}{\partial x_j}(v) = 1$, we get

$$q(v - e_j) = b_q(v - e_j, v - e_j) = q(v) - 2b_q(v, e_j) + q(e_j) = 1 - 1 + 1 = 1.$$

Since the inverse implication follows in a similar way, (a2) is proved.

(a3) If $2b_q(v, e_j) = \frac{\partial q}{\partial x_j}(v) = -1$, then $2b_q(-v, e_j) = -1$ and we get $q(v + e_j) = q(-v - e_j) = 1$, by (a2) applied to the root $-v$.

(b) Assume that $m \geq 2$, $q : \mathbb{Z}^m \to \mathbb{Z}$ is a positive unit form, and $v \notin \{e_1, \dots, e_m, -e_1, \dots, -e_m\}$ is a root of q. We associate to $v = (v_1, \dots, v_m) \in \mathbb{Z}^m$ the vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ with $\varepsilon_j = 1$ if $v_j \geq 0$, and $\varepsilon_j = -1$ if $v_j < 0$. Obviously, the vector

$$w := v^{\varepsilon} = (v_1\varepsilon_1, \dots, v_m\varepsilon_m) \in \mathbb{Z}^m$$

is a positive root of the unit form

$$q^{\varepsilon}(z) := q(z \cdot \varepsilon^{\mathrm{tr}}) = q(z_1\varepsilon_1, \ldots, z_m\varepsilon_m)$$

and $w \notin \{e_1, \ldots, e_m\}$. The equality $2 = 2q^{\varepsilon}(w) = 2b_{q^{\varepsilon}}(w, w)$ yields

$$2 = w_1 \cdot \frac{\partial q^{\varepsilon}}{\partial x_1}(w) + \cdots + w_n \cdot \frac{\partial q^{\varepsilon}}{\partial x_n}(w).$$

Since the root w is positive and $w \notin \{e_1, \ldots, e_m\}$, by (a) applied to q^{ε} and w, we have $-1 \le \frac{\partial q^{\varepsilon}}{\partial x_j}(w) \le 1$ for $j = 1, \ldots, m$. Hence, there exists $j \in \{1, \ldots, m\}$ such that $w_j > 0$ and $\frac{\partial q^{\varepsilon}}{\partial x_j}(w) = 1$. Since $w = v^{\varepsilon}$, we see that $\frac{\partial q}{\partial x_j}(v)$ is -1 if $v_j < 0$, and 1 if $v_j > 0$.

(c) Assume that v is sincere. Then $w := v^{\varepsilon}$ is a sincere positive root of q^{ε}. It follows that $w \notin \{e_1, \ldots, e_m, -e_1, \ldots, -e_m\}$ and, by (b), there exists $j \in \{1, \ldots, m\}$ such that $w_j > 0$ and $\frac{\partial q^{\varepsilon}}{\partial x_j}(w) = 1$. If $w_j = 1$, we are done. If $w_j \ge 2$, the vector $w - e_j < w$ is a sincere positive root of q^{ε}, by (a2).

Continuing this procedure as in [13, Corollary 4.7], we find a sincere positive root $w' = (w'_1, \ldots, w'_m) \in \mathbb{Z}^n$ of q^{ε} and $j \in \{1, \ldots, m\}$ such that $w'_j = 1$ and $\frac{\partial q^{\varepsilon}}{\partial x_j}(w') = w'_j = 1$. Hence it follows easily that the vector

$$v' = w' \cdot \varepsilon^{\mathrm{tr}} = (w'_1\varepsilon_1, \ldots, w'_m\varepsilon_m) \in \mathbb{Z}^m$$

satisfies the required conditions. ∎

COROLLARY 3.2. *If $m \ge 3$ and $q : \mathbb{Z}^m \to \mathbb{Z}$ is a sincere positive unit form, then there exist a sincere root $v = (v_1, \ldots, v_m) \in \mathbb{Z}^n$ of q and $j \in \{1, \ldots, m\}$ such that*

- *$v_j \in \{-1, 1\}$ and $\frac{\partial q}{\partial x_j}(v) = v_j$, and*
- *the vector $v^{(j)} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \in \mathbb{Z}^{m-1}$ is a sincere root of the jth restriction $q^{(j)} : \mathbb{Z}^{m-1} \to \mathbb{Z}$ of q.*

Proof. By Proposition 3.1(c), there exists a sincere root $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m$ of q with $v_j \in \{-1, 1\}$ and $\frac{\partial q}{\partial x_j}(v) = v_j$ for some $j \in \{1, \ldots, m\}$. Then Proposition 3.1(a) yields $q(v - v_j \cdot e_j) = 1$, that is, the non-zero vector

$$v - v_je_j = (v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_m) \in \mathbb{Z}^m$$

is a root of q. It follows that $v^{(j)} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \in \mathbb{Z}^{m-1}$ is sincere and $q^{(j)}(v^{(j)}) = q(v - v_je_j) = 1$. Hence the corollary follows. ∎

In the proof of Theorem 1.7, we essentially use the following result that will also be applied in [18] in an algorithmic procedure constructing all one-peak posets I (and their Coxeter types) such that the Tits form $\widehat{q}_I$ is almost P-critical, that is, $\widehat{q}_I$ is not positive and the restriction $\widehat{q}^{(j)}$ is positive for any $j \in I \setminus \{*\}$. This class of posets is described by Bondarenko and Stepochkina in [4] and [5].

PROPOSITION 3.3. *There is no one-peak poset $I = \{1, \dots, n, * = n+1\}$ such that the following three conditions are satisfied:*

(i) *the Tits quadratic form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ is positive,*
(ii) $|I| \geq 9$, *and*
(iii) *there is an almost sincere Tits root of $\widehat{q}_I$, that is, a vector $u = (u_1, \dots, u_n, u_{n+1}) \in \mathbb{Z}^I$ such that $u_1 \neq 0, \dots, u_n \neq 0$ and $\widehat{q}_I(u) = 1$.*

Proof. We apply induction on $n = |I| - 1 \geq 8$. In case $n = 8$, we show by a computer calculation that there is no one-peak poset I with nine elements and satisfying (i) and (iii).

We do it by applying Algorithms 5.2 and 5.5 below as follows. First, running Algorithm 5.2, we calculate the set **posit**[9] of all one-peak positive posets I with $|I| = 9$. Then we generate the list $\mathbf{roots}_9$ that contains the roots of $\widehat{q}_I(x)$ for every $I \in$ **posit**[9]. The last step is a routine computer check that there is no $u = (u_1, \dots, u_9) \in \mathbf{roots}_9$ with $u_1 \neq 0, \dots, u_8 \neq 0$. The complete computing time we need is about 15 min.

To prove the inductive step, assume that the proposition is proved for one-peak posets I' with $|I'| = n \geq 9$, and assume, to the contrary, that I is a one-peak poset such that $|I| = n + 1 \geq 10$, the Tits quadratic form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ is positive, and there is an almost sincere Tits root $u = (u_1, \dots, u_n, u_{n+1})$ of $\widehat{q}_I$. We consider two cases.

CASE 1. Assume that u is not sincere, that is, $u_{n+1} = 0$. Since $\widehat{q}_I$ is assumed to be positive, the form $\widehat{q}_I^{(*)} = q_T$ is positive and the vector $\check{u} = (u_1, \dots, u_n) \in \mathbb{Z}^T$ is a sincere root of q_T, where $T = I \setminus \{*\}$. By Corollary 3.2, there exists a sincere root $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ of q_T, with $w_j \in \{-1, 1\}$ for some $j \in \{1, \dots, n\}$, and the vector

$$w^{(j)} = (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n) \in \mathbb{Z}^{n-1}$$

is a sincere root of the jth restriction $q_T^{(j)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ of q_T, where $n - 1 \geq 8$. Obviously, the vector

$$\widehat{w}^{(j)} = (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n, 0) \in \mathbb{Z}^n$$

is an almost sincere root of the positive Tits form $\widehat{q}_{I'} = \widehat{q}_I^{(j)}$ of the one-peak subposet $I' := I \setminus \{j\}$, because

$$\widehat{q}_{I'}(\widehat{w}^{(j)}) = \widehat{q}_I^{(j)}(\widehat{w}^{(j)}) = q_T^{(j)}(w^{(j)}) = 1.$$

Since $|I'| = |I| - 1 \geq 9$, we get a contradiction with the induction hypothesis.

CASE 2. Assume that u is sincere, that is, $u_{n+1} \neq 0$. Since $\widehat{q}_I$ is assumed to be positive, there exists a sincere root $w = (w_1, \dots, w_{n+1}) \in \mathbb{Z}^{n+1}$ of $\widehat{q}_I$ with $w_j \in \{-1, 1\}$, for some $j \in \{1, \dots, n+1\}$, and the vector

$$w^{(j)} = (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_{n+1}) \in \mathbb{Z}^n$$

is a sincere root of the jth restriction $\widehat{q}_I^{(j)} : \mathbb{Z}^n \to \mathbb{Z}$, by Corollary 3.2. If $j = n+1$, we are in the situation of Case 1, and we get a contradiction. Assume that $j \le n$. Then the vector

$$\widehat{w}^{(j)} = (w_1, \ldots, w_{j-1}, 0, w_{j+1}, \ldots, w_n, w_{n+1}) \in \mathbb{Z}^n$$

is a sincere root of the positive Tits form $\widehat{q}_{I'} = \widehat{q}_I^{(j)}$ of the one-peak subposet $I' := I \setminus \{j\}$ of I, because $\widehat{q}_{I'}(w^{(j)}) = \widehat{q}_I^{(j)}(w^{(j)}) = 1$. Since $|I'| = |I| - 1 \ge 9$, we again get a contradiction with the induction hypothesis, and the proof is complete. ∎

Proof of Theorem 1.7. Let I be one-peak poset with $n+1 = |I| \ge 1$ such that the Tits form $\widehat{q}_I : \mathbb{Z}^n \to \mathbb{Z}$ is sincere and positive. By [8, Theorem 5.2], there exists a Dynkin diagram $\Delta_I \in \{\mathbb{A}_{n+1}, \mathbb{D}_{n+1}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (uniquely determined by I and called the *Coxeter–Dynkin type* of I) such that the Coxeter polynomial $\mathrm{cox}_I(t) \in \mathbb{Z}[t]$ of I coincides with the Coxeter polynomial $F_{\Delta_I}(t)$ of the diagram Δ_I. Moreover, if $\Delta_I = \mathbb{E}_6$ then I is one of the posets $\mathbb{P}_1, \ldots, \mathbb{P}_8, \mathbb{P}_{10}, \ldots, \mathbb{P}_{13}$ listed in Table 1.8; if $\Delta_I \in \{\mathbb{E}_7, \mathbb{E}_8\}$ then I is one of the posets $\mathbb{P}_{17}, \ldots, \mathbb{P}_{193}$ listed in [8, Tables 6.2–6.3], where each sincere poset is distinguished by the symbol $\mathbf{s}_I$ with $\mathbf{s}_I \ge 1$. Hence (d) follows.

Assume that $\Delta_I \in \{\mathbb{A}_{n+1}, \mathbb{D}_{n+1}\}$. By [8, Theorem 5.2], I is one of the posets in [8, Table 1.6], By a computer calculation we show that, for $n+1 = |I| \le 6$, the only positive Tits-sincere one-peak posets I with $\Delta_I \in \{\mathbb{A}_{n+1}, \mathbb{D}_{n+1}\}$ are those described in (b) and (c). Moreover, we show that there is no positive Tits-sincere one-peak poset I and $|I| = m$ of type $\Delta_I = \mathbb{A}_{n+1}$ with $4 \le n \le 9$ vertices, or of type $\Delta_I = \mathbb{D}_{n+1}$ with $6 \le n \le 9$ vertices.

We do this by applying Algorithms 5.2, 5.5, and 5.6 as follows. First, running Algorithm 5.2, we calculate the sets $\mathbf{posit}[1], \ldots, \mathbf{posit}[9]$ of all one-peak positive posets I with $|I| = 1, \ldots, 9$. Then we generate the list **candidates** of I such that $\Delta_I \in \{\mathbb{A}_n, \mathbb{D}_n\}$ and $I \in \mathbf{posit}[n]$ for $1 \le n \le 9$. Using Algorithm 5.5 we calculate the set of roots of the Tits form $\widehat{q}_I$ for every $I \in$ **candidates**. The final step is a routine computer check that the only Tits-sincere posets $I \in$ **candidates** are those described in (b) and (c). The complete computing time we need is about 16 min.

(a) Apply Proposition 3.3.

(e) Apply statement (c) and [8, Tables 6.1–6.3].

(f) Assume that $v = (v_1, \ldots, v_n, v_*) \in \mathbb{Z}^I \equiv \mathbb{Z}^{n+1}$ is a Tits root of a positive one-peak poset I, that is, $\widehat{q}_I(v) = 1$ and $v_* \ne 0$. Then the support $I_v := \{i \in I; v_i \ne 0\}$ of v contains $*$ and is a one-peak subposet of I. The restriction $\widehat{q}_{I_v} := \widehat{q}|_{I_v} : \mathbb{Z}^{I_v} \to \mathbb{Z}$ is sincere and the sincere vector $\check{v} = (\check{v}_j) \in \mathbb{Z}^{I_v}$, with $\check{v}_j = v_j$ for all $j \in I_v$, is a root of $\widehat{q}_{I_v}$, because

$\widehat{q}_{I_v}(\check{v}) = \widehat{q}(v) = 1$. Since $\widehat{q}_{I_v}$ is positive, the poset I_v is one of the posets

$${}_0\mathbb{A}_0^* : *, \qquad {}_0\mathbb{A}_1^* : \bullet \to *, \qquad {}_0\mathbb{A}_2^* : \begin{smallmatrix} \bullet & & \\ & \searrow & \\ & & * \\ & \nearrow & \\ \bullet & & \end{smallmatrix} ,$$

or (c1)–(c8) listed in (c), or one of the Tits-sincere posets listed in [8, Tables 6.1–6.3]. Since, for each such poset, a complete list of all sincere Tits roots is generated by a computer calculation (using Algorithm 5.5 to calculate the list of all roots and then choosing the sincere ones), statement (f) follows by a case by case inspection of the lists of sincere Tits roots given in [9]. ■

4. A congruence of a matrix with its transpose. We recall from Horn and Sergeichuk [11] (see also [10]) that every non-singular matrix $A \in \mathbb{M}_m(\mathbb{R})$ is $\mathbb{R}$-congruent to its transpose A^{tr} and there exists $B \in \mathrm{Gl}(m,\mathbb{R})$ such that $B^2 = E$ and $A^{\mathrm{tr}} = B \cdot A \cdot B^{\mathrm{tr}}$. Moreover, B can be chosen to be orthogonal.

In the proof of Proposition 2.2 we need a refinement of this theorem which leads to the following question:

PROBLEM 4.1. *For any $A \in \mathrm{Gl}(m,\mathbb{Z})$ find $B \in \mathbb{M}_m(\mathbb{Z})$ such that $B^2 = E$ and $A^{\mathrm{tr}} = B \cdot A \cdot B^{\mathrm{tr}}$.*

Although we are not able to solve this problem for arbitrary $A \in \mathrm{Gl}(n,\mathbb{Z})$, we get in Theorem 1.10 an affirmative solution of 4.1 for a class of matrices connected to one-peak positive posets I, including the incidence matrices $A = C_I$. Moreover, we give an algorithm that constructs a matrix $S \in \mathbb{M}_{n+1}(\mathbb{Z})$ such that $S^2 = E$ and $C_I^{\mathrm{tr}} = S \cdot C_I \cdot S^{\mathrm{tr}}$ for any such poset I with $|I| = n+1$ (see Algorithm 4.5).

Proof of Theorem 1.10. Assume that I is a one-peak poset. It is shown in [30] that the matrices C_I, $\overline{C}_I$, and $\widehat{C}_I$ are $\mathbb{Z}$-congruent to each other. Thus it is sufficient to prove the theorem for the Tits matrix $\widehat{C}_I$ of I, because one easily shows that if $A, A' \in \mathrm{Gl}(n,\mathbb{Z})$ are $\mathbb{Z}$-congruent, then A has the property 4.1 if and only if A' does. Now we split the proof into two steps.

1° First consider the case when the poset is a one-peak Dynkin quiver Q, with $n+1$ vertices. It is easy to see that the transpose $\overline{C}_Q^{\mathrm{tr}}$ of its Euler matrix is the Euler matrix $\overline{C}_{Q^{\mathrm{op}}}$ of the quiver Q^{op} opposite to Q. By applying reflection arguments (as in the proof of [8, Proposition 3.5(b)]) one can find $S \in \mathbb{M}_{n+1}(\mathbb{Z})$ such that $S^2 = E$ and $\overline{C}_Q^{\mathrm{tr}} = \overline{C}_{Q^{\mathrm{op}}} = S \cdot \overline{C}_Q \cdot S^{\mathrm{tr}}$.

Assume that Q is the Dynkin quiver of type $\mathbb{A}_m$ and Q' is the Dynkin quiver of type $\mathbb{D}_m$ obtained from the diagrams $\mathbb{A}_m$ and $\mathbb{D}_m$ of Table 1.5 by replacing any edge ${}_j\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet_{j+1}$ with the arrow ${}_j\bullet\!\!\longrightarrow\!\!\bullet_{j+1}$. Then the matrices $S_m \in \mathbb{M}_m(\mathbb{Z})$ and $S'_m \in \mathbb{M}_m(\mathbb{Z})$ given by

$$(4.2)\qquad S_m = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad S'_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widehat{1} & \widehat{1} & \widehat{1} & \widehat{1} & \widehat{1} & \dots & \widehat{1} & \widehat{1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

with $\widehat{1} = -1$, satisfy the required conditions and $\det S_m = (-1)^{(m(m+3))/2}$, $\det S'_m = (-1)^{(m(m+3)-2)/2}$. Note that the matrices can be read off from the configurations [8, Table 7.3] as in [8, Example 7.6].

Assume next that Q is the Dynkin quiver of type $\mathbb{E}_6$, Q' is the Dynkin quiver of type $\mathbb{E}_7$, and Q'' is the Dynkin quiver of type $\mathbb{E}_8$, obtained from the diagrams $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ of Table 1.5 by replacing any edge ${}_j\bullet\!\!-\!\!-\!\!-\!\!\bullet_{j+1}$ with the arrow ${}_j\bullet\!\!\longrightarrow\!\!\bullet_{j+1}$. Then the matrices $S \in \mathbb{M}_6(\mathbb{Z})$, $S' \in \mathbb{M}_7(\mathbb{Z})$, and $S'' \in \mathbb{M}_8(\mathbb{Z})$ given by

$$(4.3)$$

$$S = \begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ \widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1} \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{bmatrix}, \quad S' = \begin{bmatrix} 0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ \widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1} \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \end{bmatrix}, \quad S'' = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ \widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1}&\widehat{1} \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \end{bmatrix},$$

with $\widehat{1} = -1$, $\det S = -1$, $\det S' = -1$, $\det S'' = 1$, satisfy the required conditions.

2° Finally, consider the case of a general positive one-peak poset. By [8, Theorem 5.2], there exists a Dynkin quiver $Q = Q_I \in \{\mathbb{A}_{n+1}, \mathbb{D}_{n+1}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ such that the Coxeter polynomial $\mathrm{cox}_I(t)$ coincides with the Coxeter polynomial $F_Q(t)$ of Q, the Tits matrix $\widehat{C}_I$ of I is $\mathbb{Z}$-congruent to the Euler matrix $\overline{C}_Q$, and $\widehat{C}_I = B_1 \cdot \overline{C}_Q \cdot B_1^{\mathrm{tr}}$ for some $\mathbb{Z}$-invertible $B_1 \in \mathbb{M}_{n+1}(\mathbb{Z})$. It follows that $\widehat{C}_I^{\mathrm{tr}} = B_1 \cdot \overline{C}_Q^{\mathrm{tr}} \cdot B_1^{\mathrm{tr}}$ and, in view of the equality $\overline{C}_Q^{\mathrm{tr}} = S \cdot \overline{C}_Q \cdot S^{\mathrm{tr}}$ proved in 1°, we get

$$\widehat{C}_I^{\rm tr} = B_1 \cdot S \cdot \overline{C}_Q \cdot S^{\rm tr} \cdot B_1^{\rm tr} = B_1 \cdot S \cdot B_1^{-1} \cdot \widehat{C}_I \cdot B_1^{-\rm tr} \cdot S^{\rm tr} \cdot B_1^{\rm tr} = B \cdot \widehat{C}_I \cdot B^{\rm tr}$$

where $B = B_1 \cdot S \cdot B_1^{-1}$. The equality $S^2 = E$ yields $B^2 = E$ and the proof is complete. ∎

REMARK 4.4. There is an alternative way of constructing the matrix B, presented in [8, Example 7.6] for the Tits-sincere poset $\mathbb{P}_{10}$. The method explained there can be converted to an algorithm as follows.

ALGORITHM 4.5.

Input: The incidence matrix $C_I \in \mathbb{M}_{n+1}(\mathbb{Z})$ of a given positive one-peak poset I.

Output: A matrix $B \in \mathbb{M}_{n+1}(\mathbb{Z})$ such that $C_I^{\rm tr} = B{\cdot}C_I{\cdot}B^{\rm tr}$ and $B^2 = E$.

Step 1. Calculate the incidence quadratic form $q_I\colon \mathbb{Z}^I \equiv \mathbb{Z}^{n+1} \to \mathbb{Z}$, $q_I(x) = x{\cdot}C_I{\cdot}x^{\rm tr}$, the Coxeter transformation $\Phi_I\colon \mathbb{Z}^I \to \mathbb{Z}^I$, $\Phi_I(v) = v{\cdot}{\rm Cox}_I$, where ${\rm Cox}_I = -C_I \cdot (C_I^{-1})^{\rm tr}$, the Coxeter polynomial ${\rm cox}_I(t)$, and determine the Coxeter–Dynkin type Δ_I of I.

Step 2. Compute the finite set $\mathcal{R}_{q_I}$ of roots of q_I using Algorithm 5.5.

Step 3. Split the set $\mathcal{R}_{q_I}$ of roots of $q_I : \mathbb{Z}^I \to \mathbb{Z}$ into Φ_I-orbits.

Step 4. Construct the Φ_I-mesh translation quiver $\Gamma(\mathcal{R}_{q_I}, \Phi_I)$ of roots of q_I by applying the mesh toroidal algorithm described in [31, Proposition 4.5] and [32]–[33] (see also [6]).

Step 5. By using [8, Table 7.3], fix a principal Coxeter Φ_I-orbit configuration $\Gamma_I^{C_I}$ of type Δ_I in the Φ_I-mesh translation quiver $\Gamma(\mathcal{R}_{q_I}, \Phi_I)$. Next, fix a mesh quiver isomorphism $h' : \Gamma_{\Delta_I}^{\check{G}_{\Delta_I}} \to \Gamma_I^{C_I}$ and construct a matrix

$$B' = \begin{bmatrix} h'(e_1) \\ \vdots \\ h'(e_n) \end{bmatrix} \in \mathbb{M}_{n+1}(\mathbb{Z})$$

such that $\check{G}_{\Delta_I} = B' \cdot C_I \cdot B'^{\rm tr}$.

Step 6. Construct $\Phi_{I^{\rm op}} = \Phi_I^{-1}$ and the $\Phi_{I^{\rm op}}$-mesh translation quiver $\Gamma(\mathcal{R}_{q_{I^{\rm op}}}, \Phi_{I^{\rm op}})$ of roots of $q_{I^{\rm op}}\colon \mathbb{Z}^n \to \mathbb{Z}$, $q_{I^{\rm op}}(x) = x \cdot C_I^{\rm tr} \cdot x^{\rm tr}$, by reversing all arrows in the Φ_I-mesh translation quiver $\Gamma(\mathcal{R}_{q_I}, \Phi_I)$.

Step 7. By using [8, Table 7.3], fix a principal Coxeter $\Phi_{I^{\rm op}}$-orbit configuration $\Gamma_{I^{\rm op}}^{C_I^{\rm tr}}$ of type Δ_I in the $\Phi_{I^{\rm op}}$-mesh translation quiver $\Gamma(\mathcal{R}_{q_{I^{\rm op}}}, \Phi_{I^{\rm op}})$. Then fix a mesh quiver isomorphism $h'' : \Gamma_{\Delta_I}^{\check{G}_{\Delta_I}} \to \Gamma_I^{C_I{}^{\rm tr}}$ and construct a matrix

$$B'' = \begin{bmatrix} h''(e_1) \\ \vdots \\ h''(e_n) \end{bmatrix} \in \mathbb{M}_{n+1}(\mathbb{Z})$$

such that $\check{G}_{\Delta_I} = B'' \cdot C_I^{\mathrm{tr}} \cdot B''^{\mathrm{tr}}$.

Step 8. Return $B = B''^{-1} \cdot B'$ and check that $C_I^{\mathrm{tr}} = B \cdot C_I \cdot B^{\mathrm{tr}}$ and $B^2 = E$.

5. Algorithms and computing in Python. In this section we outline the computational algorithms we use in this paper. Their implementations are available from the authors on request.

The first algorithm is a modified version of [8, Algorithm 7.1] and determines all positive one-peak posets that have at most nine elements. Assume that $I = (\{1, \ldots, n, n+1 = *\}, \preceq)$ is a positive one-peak poset with a unique maximal element $*$. Without loss of generality, we can assume that the incidence matrix C_I is upper triangular. Then $1 \in I$ is a minimal element, the subposet $J = I \setminus \{1\}$ of I is positive, and C_I has the triangular form

$$C_I = \left[\begin{array}{c|c} 1 & w \\ \hline 0 & C_J \end{array}\right] \in \mathbb{M}_{n+1}(\mathbb{Z}), \tag{5.1}$$

where $w = [w_2, \ldots, w_n, 1] \in \mathbb{Z}^n$ and $w_2, \ldots, w_n \in \{0, 1\}$. It follows that the description of all one-peak positive posets with $n+1$ elements reduces to the description of posets I with incidence matrix of the shape (5.1), where J is a positive one-peak poset with n elements and upper triangular incidence matrix C_J.

ALGORITHM 5.2.

Input: An integer $n \geq 1$.

Output: Finite sets $\mathbf{posit}[1], \ldots, \mathbf{posit}[n]$ of all one-peak posets such that $I \in \mathbf{posit}[m]$ implies $|I| = m \leq n$ and the quadratic form $q_I : \mathbb{Z}^m \to \mathbb{Z}$, $q_I(x) = x \cdot C_I \cdot x^{\mathrm{tr}}$, is positive.

Step 1. Initialize $\mathbf{posit}[1] = [1] \in \mathbb{M}_1(\mathbb{Z})$.

Step 2. For every m from 2 to n:

Step 2.1. Initialize $\mathbf{candidate}_m$ to be an empty list.

Step 2.2. For every $J \in \mathbf{posit}[m-1]$, generate the list W_J of all vectors $w = [w_2, \ldots, w_{m-1}, 1] \in \{0, 1\}^{m-1}$ such that the matrix

$$C_{J_w} = \left[\begin{array}{c|c} 1 & w \\ \hline 0 & C_J \end{array}\right] = [c_{ij}] \in \mathbb{M}_m(\mathbb{Z})$$

is the incidence matrix of a poset J_w, that is,

$$\text{“}c_{ij} = 1 \text{ and } c_{js} = 1 \text{ implies } c_{is} = 1\text{”}, \quad \text{for } 1 \leq i, j, s \leq n$$

(in other words, $i \preceq j$ and $j \preceq s$ imply $i \preceq s$).

Step 2.3. For every $J \in \mathbf{posit}[m-1]$ and every $w \in W_J$ construct a matrix $C_{J_w} \in \mathbb{M}_m(\mathbb{Z})$ and add the poset J_w to the list $\mathbf{canditate}_m$ if the symmetric matrix $C_{J_w} + C_{J_w}^{\mathrm{tr}}$ satisfies the Sylvester criterion (or equivalently, if the quadratic form $q_{J_w}(x) = x \cdot C_{J_w} \cdot x^{\mathrm{tr}}$ is positive definite).

Step 2.4. Construct the set $\mathbf{posit}[m]$ by selecting the incidence matrices C_I in the list $\mathbf{canditate}_m$ of posets I in such a way that the Hasse quivers $H(I)$ are pairwise non-isomorphic.

Step 3. Return $\mathbf{posit}[1], \ldots, \mathbf{posit}[n]$. ■

REMARKS. (a) Note that Steps 2.3 and 2.4 can be made simultaneously by adding to $\mathbf{posit}[m]$ only those positive posets I that have Hasse quivers not isomorphic to the posets that are already in.

(b) In our Python implementation of the algorithm we use the **igraph** package (http://igraph.sourceforge.net/) to test quiver isomorphism in Step 2.4 and SymPy package (http://sympy.org/) for matrix algorithms.

(c) Although the time complexity of the algorithm is exponential (no polynomial algorithm for testing quiver isomorphism is known), it works pretty fast for small n. The computing time for $n = 6, 7, 8, 9$ was 0.6, 3.6, 16.2 and 65.9 seconds respectively (on a computer with AMD Athlon(tm) II X4 630 processor, using Python 2.7.2, igraph 0.5.4 and SymPy 0.7.1).

Now we recall from [31, Algorithm 4.2] and [32, Algorithm 3.7] how the set $\mathcal{R}_q$ of all roots of a positive definite quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ can be calculated. As a basis of our algorithm we use the following “positive” version of the well-known Lagrange theorem which states that every quadratic form reduces to a canonical form.

THEOREM 5.3 (Lagrange). *For every permutation $\{j_1, \ldots, j_n\}$ of $\{1, \ldots, n\}$, a positive definite quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$, $q(x_1, \ldots, x_n) = q_{11}x_1^2 + \cdots + q_{nn}x_n^2 + \sum_{i<j} q_{ij}x_i x_j$, reduces to the canonical form*

$$q(x_1, \ldots, x_n) = \lambda_1 y_{j_1}^2 + \cdots + \lambda_{n-1} y_{j_{n-1}}^2 + \lambda_n x_{j_n}^2, \tag{5.4}$$

where $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n$ are positive rational numbers and $y_{j_i} = c_{j_i i} x_{j_i} + \cdots + c_{j_i n} x_{j_n}$, with $c_{ij} \in \mathbb{Q}$, such that $\det [c_{j_i i}] \neq 0$.

Proof. Note that each of the coefficients $q_{11}, \ldots, q_{nn}$ is positive, because $q(x)$ is positive definite. We apply induction on $n \geq 1$. For $n = 1$, there is nothing to prove. Assume that $n \geq 2$ and rewrite the form $q(x)$ as

$$q(x_1, \ldots, x_n) = q_{j_1 j_1} y_{j_1}^2 + \check{q}(x_{j_2}, \ldots, x_{j_n}),$$

where

$$y_{j_1} = y_{j_1}(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) := x_{j_1} + \frac{1}{2q_{j_1j_1}} \cdot (q_{j_1j_2}x_{j_2} + \cdots + q_{j_1j_n}x_{j_n}),$$

and

$$\check{q}(x_{j_2}, \ldots, x_{j_n}) = q(x_1, \ldots, x_n) - q_{j_1j_1}y_{j_1}^2$$

depends only on the indeterminates $x_{j_2}, \ldots, x_{j_n}$, and we set $q_{ij} = q_{ji}$ for $j < i$. To finish the proof, it is sufficient to check that the quadratic form $\check{q}(x_{j_2}, \ldots, x_{j_n})$ is positive definite, because then the inductive hypothesis applies to $\check{q}(x_{j_2}, \ldots, x_{j_n})$ and we are done.

To show that $\check{q}(x_{j_2}, \ldots, x_{j_n})$ is positive definite assume, to the contrary, that there exists a non-zero vector $u = (u_{j_2}, \ldots, u_{j_n}) \in \mathbb{R}^{n-1}$ such that $\check{q}(u) \leq 0$. If we set

$$\widehat{u} = (\widehat{u}_{j_1}, u_{j_2}, \ldots, u_{j_n}) \quad \text{with} \quad \widehat{u}_{j_1} := -q_{j_1j_2}u_{j_2} + \cdots + q_{j_1j_n}u_{j_n},$$

then $y_{j_1}(\widehat{u}_{j_1}, u_{j_2}, \ldots, u_{j_n}) = 0$ and we get the contradiction

$$q(\check{u}_1, \ldots, \check{u}_n) = q_{j_1j_1}y_{j_1}(\widehat{u}_{j_1}, u_{j_2}, \ldots, u_{j_n})^2 + 4q_{j_1j_1}^2 \cdot \check{q}(u) = 4q_{j_1j_1}^2 \cdot \check{q}(u) \leq 0.$$

Note that in the final step of induction we get $y_{j_n} = \lambda_n x_{j_n}$ with $\lambda_n > 0$, and consequently

$$q(x_1, \ldots, x_n) = \lambda_1 y_{j_1}^2 + \cdots + \lambda_{n-1} y_{j_{n-1}}^2 + \lambda_n x_{j_n}^2.$$

This finishes the proof. ∎

We can use the canonical form (5.4) to restrict the set $\mathbb{Z}^n$ of all integer vectors to the finite set of vectors that contains all the roots of $q(x)$. We do it as follows.

By Theorem 5.3, given a root $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ of a positive definite form (5.4), the equality $q(v) = 1$ implies $\lambda_n v_{j_n}^2 \leq 1$ and therefore $|v_{j_n}| \leq \sqrt{1/\lambda_n}$. This observation is used in the following restrictively counting algorithm that computes all roots of a given positive definite quadratic form (see also [31, Algorithm 4.2] and [32, Algorithm 3.7]).

Algorithm 5.5.

Input: An integer $n \geq 1$ and a positive definite quadratic form $q(x_1, \ldots, x_n) = q_{11}x_1^2 + \cdots + q_{nn}x_n^2 + \sum_{i<j} q_{ij}x_ix_j$, with $q_{ij} \in \mathbb{Z}$.
Output: The finite set $\mathbf{roots}_n \subset \mathbb{Z}^n$ of all roots of q.

Step 1. Initialize the list **restrictions** of size n, set $q^{[0]}(x) = q(x)$.
Step 2. For every $m \in \{1, \ldots, n\}$ do:
Step 2.1. Fix a permutation $\{j_1, \ldots, j_n\}$ of $\{1, \ldots, n\}$ and interchange j_m and j_n elements.

Step 2.2. For $k = 1, \dots, n-1$ calculate

$$q^{[k]}(x_{j_{k+1}}, \dots, x_{j_n}) = q^{[k-1]}(x_{j_k}, \dots, x_{j_n}) - q^{[k-1]}_{j_k j_k}\left(x_{j_k} + \frac{q^{[k-1]}_{j_k j_{k+1}} x_{j_{k+1}} + \dots + q^{[k-1]}_{j_k j_n} x_{j_n}}{2q^{[k-1]}_{j_k j_k}}\right)^2,$$

setting $q^{[k-1]}_{ij} = q^{[k-1]}_{ji}$ for $j < i$.

Step 2.3. Given $q^{[n-1]}(x_m) = \lambda x_m^2$, where $\lambda = q^{[n-1]}_{mm}$, set **restrictions**$[m] = \lfloor \sqrt{1/\lambda} \rfloor$.

Step 3. Return the list containing every vector $v = (v_1, \dots, v_n) = (v_j) \in \mathbb{Z}^n$ such that $-$**restrictions**$[j] \leq v_j \leq$ **restrictions**$[j]$ and $q(v) = 1$. ∎

To prove Theorem 1.10 for posets I of Coxeter–Dynkin type $\mathbb{E}_6$, we need the following algorithm.

ALGORITHM 5.6.

Input: One-peak poset $I \in \{\mathbb{P}_1, \dots, \mathbb{P}_8, \mathbb{P}_{10}, \dots, \mathbb{P}_{13}\}$ and the non-symmetric Gram matrix $\check{G}_{\mathbb{E}_6} \in \mathbb{M}_6(\mathbb{Z})$ of the Dynkin diagram $\mathbb{E}_6$, with vertices numbered as in Table 1.5.

Output: A $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ such that $\widehat{C}_I^{\mathrm{tr}} = B \cdot \widehat{C}_I \cdot B^{\mathrm{tr}}$, $\det B = -1$ and $B^2 = E$.

Step 1. Apply [8, Algorithm 7.5] to the matrices $\widehat{C}_I$ and $\check{G}_{\mathbb{E}_6}$ in order to obtain $B' \in \mathbb{M}_6(\mathbb{Z})$ such that $\check{G}_{\mathbb{E}_6} = B' \cdot \widehat{C}_I \cdot \widehat{B}'^{\mathrm{tr}}$.

Step 2. Apply [8, Algorithm 7.5] to the matrices $\widehat{C}_I^{\mathrm{tr}}$ and $\check{G}_{\mathbb{E}_6}$ in order to obtain $B'' \in \mathbb{M}_6(\mathbb{Z})$ such that $\check{G}_{\mathbb{E}_6} = B'' \cdot \widehat{C}_I^{\mathrm{tr}} \cdot \widehat{B}''^{\mathrm{tr}}$.

Step 3. Calculate $B = B''^{-1} \cdot B'$ and check that $\det B = -1$. ∎

By applying Algorithm 5.6 to each of the posets $\mathbb{P}_1, \dots, \mathbb{P}_8, \mathbb{P}_{10}, \dots, \mathbb{P}_{13}$ in Table 1.8 we get the following corollary that proves Theorem 1.10 for posets I of Coxeter–Dynkin type $\mathbb{E}_6$ (see also [8, Example 7.6]). The proof for remaining types is analogous; the lists of the corresponding matrices $B = B_j$ can be found in [9].

COROLLARY 5.7. *Assume that* $I = \mathbb{P}_j \in \{\mathbb{P}_1, \dots, \mathbb{P}_8, \mathbb{P}_{10}, \dots, \mathbb{P}_{13}\}$ (*see Table* 1.8). *If* $A = \widehat{C}_I$ *is the Tits matrix of* I *and* $B = B_j \in \mathrm{Gl}(6, \mathbb{Z})$ *is as in Table* 5.8, *then* $\det B = -1$, $B^2 = E$, *and* $A^{\mathrm{tr}} = B \cdot A \cdot B^{\mathrm{tr}}$.

Proof. Apply Algorithm 5.6 and use the idea of the proof of Theorem 1.6. ∎

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Table 5.1. A list of matrices $B = B_j$ such that $A^{\mathrm{tr}} = B \cdot A \cdot B^{\mathrm{tr}}$ with $A = \widehat{C}_{\mathbb{P}_j}$ for the posets $\mathbb{P}_j \in \{\mathbb{P}_1, \ldots, \mathbb{P}_8, \mathbb{P}_{10}, \ldots, \mathbb{P}_{13}\}$ of Table 1.8

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -2 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad B_8 = \begin{bmatrix} 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}.$$

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