ON SELF-INJECTIVE ALGEBRAS OF FINITE REPRESENTATION TYPE

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Abstract. We describe the structure of finite-dimensional self-injective algebras of finite representation type over a field whose stable Auslander–Reiten quiver has a sectional module not lying on a short chain.

Introduction. Throughout the paper, by an algebra we mean a basic indecomposable finite-dimensional associative $K$-algebra with an identity over a (fixed) field $K$. For an algebra $A$, we denote by $\text{mod} A$ the category of finite-dimensional right $A$-modules, and by $D$ the standard duality $\text{Hom}_K(-,K)$ on $\text{mod} A$. We denote by $\Gamma_A$ the Auslander–Reiten quiver of $A$, and by $\tau_A$ and $\tau^{-1}_A$ the Auslander–Reiten translations $D\text{Tr}$ and $\text{Tr}D$, respectively. We will not distinguish between an indecomposable module in $\text{mod} A$ and the vertex of $\Gamma_A$ corresponding to it. An algebra $A$ is called self-injective if $A \cong D(A)$ in $\text{mod} A$, that is, the projective modules in $\text{mod} A$ are injective. In the representation theory of self-injective algebras an important role is played by the self-injective algebras $A$ which admit Galois coverings of the form $\hat{B} \rightarrow \hat{B}/G = A$, where $\hat{B}$ is the repetitive category of an algebra $B$ and $G$ is an admissible group of automorphisms of $\hat{B}$ (see [22], [29]).

We are concerned with the problem of describing the Morita equivalence classes of self-injective algebras of finite representation type, that is, the self-injective algebras $A$ for which $\text{mod} A$ admits only finitely many indecomposable modules up to isomorphism. For $K$ algebraically closed, the problem was solved in the early 1980’s by Riedtmann (see [4], [16], [17], [18]) via the combinatorial classification of the Auslander–Reiten quivers of self-injective algebras of finite representation type over $K$. Equivalently, Riedtmann’s classification can be presented as follows (see [22] Section 3): a non-simple self-injective algebra $A$ over an algebraically closed field $K$ is of finite representation type if and only if $A$ is a socle deformation of an orbit algebra $\hat{B}/G$, where $B$ is a tilted algebra of Dynkin type $A_n$ ($n \geq 1$).

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\(D_n\ (n \geq 4), \ E_6, \ E_7, \ E_8,\) and \(G\) is an admissible infinite cyclic group of automorphisms of \(\hat{B}\). It was conjectured in [29, Problem 2.4] that a non-simple self-injective algebra \(A\) over an arbitrary field \(K\) is of finite representation type if and only if \(A\) is a socle deformation of an orbit algebra \(\hat{B}/G\), where \(B\) is a tilted algebra of Dynkin type \(A_n\ (n \geq 1), \ E_6, \ E_7, \ E_8, \ F_4\) or \(G_2\). This is currently an exciting open problem. An important known result towards solution of this problem is the Riedtmann–Todorov description of the stable Auslander–Reiten quivers of self-injective algebras of finite representation type over an arbitrary field (see [16], [31], [30, Section IV.15]). We also refer to [28] for related results on stable equivalences of self-injective algebras of finite representation type.

The main aim of the paper is to show that a non-simple self-injective algebra \(A\) of a finite representation type whose stable Auslander–Reiten quiver admits a section with good behaviour in the module category \(\text{mod} A\) is isomorphic to an orbit algebra \(\hat{B}/G\), where \(B\) is a tilted algebra of Dynkin type and \(G\) is an infinite cyclic group of automorphisms of \(\hat{B}\).

For basic background on the representation theory applied in this paper we refer to [1] and [30].

1. The main result and related background. Let \(B\) be an algebra and \(1_B = e_1 + \cdots + e_n\) a decomposition of the identity of \(B\) into a sum of pairwise orthogonal primitive idempotents. We associate to \(B\) a self-injective locally bounded \(K\)-category \(\hat{B}\), called the repetitive category of \(B\) (see [11], [20]). The objects of \(\hat{B}\) are \(e_{m,i}, m \in \mathbb{Z}, i \in \{1, \ldots, n\}\), and the morphism spaces are defined as follows:

\[
\hat{B}(e_{m,i}, e_{r,j}) = \begin{cases} 
eq & r = m, \\
D(e_iBe_j), & r = m + 1, \\
0, & \text{otherwise}. \end{cases}
\]

Observe that \(e_jBe_i = \text{Hom}_B(e_iB, e_jB), D(e_iBe_j) = e_jD(B)e_i\) and

\[
\bigoplus_{(r,i) \in \mathbb{Z} \times \{1, \ldots, n\}} \hat{B}(e_{m,i}, e_{r,j}) = e_jB \oplus D(Be_j)
\]

for any \(r \in \mathbb{Z}\) and \(j \in \{1, \ldots, n\}\). We denote by \(\nu_{\hat{B}}\) the Nakayama automorphism of \(\hat{B}\) defined by

\[
\nu_{\hat{B}}(e_{m,i}) = e_{m+1,i} \quad \text{for all } (m, i) \in \mathbb{Z} \times \{1, \ldots, n\}.
\]

An automorphism \(\varphi\) of the \(K\)-category \(\hat{B}\) is said to be:

- **positive** if for each pair \((m, i) \in \mathbb{Z} \times \{1, \ldots, n\}\) we have \(\varphi(e_{m,i}) = e_{p,j}\) for some \(p \geq m\) and some \(j \in \{1, \ldots, n\}\);
• **rigid** if for each pair \((m, i) \in \mathbb{Z} \times \{1, \ldots, n\}\) there exists \(j \in \{1, \ldots, n\}\) such that \(\varphi(e_{m,i}) = e_{m,j}\);
• **strictly positive** if it is positive but not rigid.

Then the automorphisms \(\nu^r_{\hat{B}}, r \geq 1\), are strictly positive automorphisms of \(\hat{B}\).

A group \(G\) of automorphisms of \(\hat{B}\) is said to be **admissible** if \(G\) acts freely on the set of objects of \(\hat{B}\) and has finitely many orbits. Then we may consider the orbit category \(\hat{B}/G\) of \(\hat{B}\) with respect to \(G\) whose objects are the \(G\)-orbits of objects in \(\hat{B}\), and the morphism spaces are given by

\[
(\hat{B}/G)(a, b) = \left\{ f_{y,x} \in \prod_{(x,y) \in a \times b} \hat{B}(x,y) \mid g(f_{y,x}) = f_{gy, gx}, \forall g \in G, (x,y) \in a \times b \right\}
\]

for all objects \(a, b\) of \(\hat{B}/G\). Since \(\hat{B}/G\) has finitely many objects and the morphism spaces in \(\hat{B}/G\) are finite-dimensional, we have the associated finite-dimensional, self-injective \(K\)-algebra \(\bigoplus(\hat{B}/G)\) which is the direct sum of all morphism spaces in \(\hat{B}/G\), called the **orbit algebra** of \(\hat{B}\) with respect to \(G\). We will identify \(\hat{B}/G\) with \(\bigoplus(\hat{B}/G)\). For example, for each positive integer \(r\), the infinite cyclic group \((\nu^r_{\hat{B}})\) generated by the \(r\)th power \(\nu^r_{\hat{B}}\) of \(\nu_{\hat{B}}\) is an admissible group of automorphisms of \(\hat{B}\), and we have the associated self-injective orbit algebra

\[
T(B)^{(r)} = \hat{B}/(\nu^r_{\hat{B}}) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & f_1 & b_1 \\ b_1, \ldots, b_{r-1} \in B, f_1, \ldots, f_{r-1} \in D(B) \end{bmatrix} \right\},
\]

called the **\(r\)-fold trivial extension algebra** of \(B\). In particular, \(T(B)^{(1)} \cong T(B) = B \ltimes D(B)\) is the trivial extension of \(B\) by the injective cogenerator \(D(B)\).

Let \(H\) be a hereditary algebra and \(Q_H\) its valued quiver. Following [3], [9], a module \(T\) in \(\text{mod} \ H\) is called a **tilting module** if \(\text{Ext}^1_H(T, T) = 0\) and \(T\) is a direct sum of \(n\) pairwise non-isomorphic, indecomposable modules, where \(n\) is the rank of the Grothendieck group \(K_0(H)\) of \(H\) (equivalently, the number of vertices of \(Q_H\)). Then the endomorphism algebra \(B = \text{End}_H(T)\) is called a **tilted algebra** of \(H\). Further, the images \(\text{Hom}_H(T, I)\) of indecomposable in-
jective modules \( I \) in \( \text{mod} \, H \) via the functor \( \text{Hom}_H(T, -) : \text{mod} \, H \to \text{mod} \, B \) form a section \( \Delta_T \) of a connected component \( C_T \) of \( \Gamma_B \), called the connecting component of \( \Gamma_B \) determined by \( T \), which connects the torsion-free part \( \mathcal{Y}(T) = \{ Y \in \text{mod} \, B \mid \text{Tor}_1^B(Y, T) = 0 \} \) and the torsion part \( \mathcal{X}(T) = \{ X \in \text{mod} \, B \mid X \otimes_B T = 0 \} \) (see [9]). Moreover, by a criterion of Liu–Skowroński (see [14], [21]), an algebra \( B \) is a tilted algebra of a hereditary algebra \( H \) if and only if the Auslander–Reiten quiver \( \Gamma_B \) of \( B \) admits a connected component \( C \) having a faithful section \( \Delta \) such that \( \text{Hom}_B(U, \tau_B V) = 0 \) for all modules \( U, V \) from \( \Delta \).

Assume now that \( H \) is a hereditary algebra of finite representation type, or equivalently, \( Q_H \) is a Dynkin quiver (see [5], [6], [7]). Then for any tilting module \( T \) in \( \text{mod} \, H \), the associated tilted algebra \( B = \text{End}_H(T) \), called a tilted algebra of Dynkin type, is of finite representation type, and \( \Gamma_B = C_T \).

Further, it follows from [10], [11] that the repetitive category \( \hat{B} \) of a tilted algebra \( B \) of Dynkin type is locally representation-finite in the sense of [8]. In particular, by a theorem of Gabriel [8, Theorem 3.6] the orbit algebra \( A = \hat{B}/G \) of \( \hat{B} \), with respect to an admissible infinite cyclic group \( G \) of automorphisms of \( \hat{B} \), is a self-injective algebra of finite representation type, and the stable Auslander–Reiten quiver \( \Gamma^s_A \) of \( A \) is the orbit quiver \( \mathbb{Z}\Delta/G \), where \( \Delta = Q_H \).

Let \( A \) be a non-simple self-injective algebra of finite representation type. Then by the Riedtmann–Todorov theorem (see [16], [31]) the stable Auslander–Reiten quiver \( \Gamma^s_A \) of \( A \) is isomorphic to the orbit quiver \( \mathbb{Z}\Delta/G \), where \( \Delta \) is a Dynkin quiver and \( G \) is an infinite cyclic group of automorphisms of \( \hat{B} \), is a self-injective algebra of finite representation type, and the stable Auslander–Reiten quiver \( \Gamma^s_A \) of \( A \) is the orbit quiver \( \mathbb{Z}\Delta/G \), where \( \Delta = Q_H \).

Let \( A \) be an algebra. Following [2], [15], a sequence \( N \to M \to \tau_AN \) of non-zero homomorphisms in \( \text{mod} \, A \) with \( N \) indecomposable is called a short chain, and \( M \) is the middle of this chain. We mention that, if \( M \) is a module in \( \text{mod} \, A \) which is not the middle of a short chain, then every indecomposable direct summand \( Z \) of \( M \) is uniquely determined (up to isomorphism) by the simple composition factors (see [15, Corollary 2.2]). It has been recently proved in [12, Theorem] that an algebra \( B \) is a tilted algebra if and only if \( \text{mod} \, B \) contains a sincere module \( M \) which is not the middle of a short chain.
Recall that $M$ is called sincere if every simple module in mod $B$ occurs as a composition factor of $M$. We also refer to [13] for a description of finite-dimensional modules over algebras which are not the middle of a short chain of modules, using injective and tilting modules over hereditary algebras.

The aim of this paper is to prove the following theorem.

**Theorem 1.1.** Let $A$ be a non-simple finite-dimensional basic indecomposable self-injective algebra of finite representation type over a field $K$. The following statements are equivalent:

(i) mod $A$ admits a pure sectional module $M$ which is not the middle of a short chain.

(ii) $A$ is isomorphic to a self-injective orbit algebra $\hat{B}/(\rho \nu^2 \hat{B})$, where $B$ is a tilted algebra of the form $B = \text{End}_H(T)$ with $H$ a hereditary algebra of Dynkin type and $T$ is a tilting module in mod $H$ without indecomposable projective direct summands, and $\rho$ is a positive automorphism of $\hat{B}$.

We note that the module category mod $H$ of a hereditary algebra $H$ of Dynkin type admits a tilting module $T$ without indecomposable projective direct summands if and only if $H$ is not a Nakayama algebra, or equivalently, the quiver $Q_H$ of $H$ is not an equioriented quiver

\[ \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]

of type $\mathbb{A}_n$ ($n \geq 1$).

**2. Self-injective algebras of Dynkin type.** Let $B$ be a triangular algebra (the quiver $Q_B$ has no oriented cycles) and $e_1, \ldots, e_n$ be pairwise orthogonal primitive idempotents of $B$ with $1_B = e_1 + \cdots + e_n$. We identify $B$ with the full subcategory $B_0$ of the repetitive category $\hat{B}$ given by the objects $e_{0,i}$, $1 \leq i \leq n$. For a sink $i$ of $Q_B$, the reflection $S_i^+B$ of $B$ at $i$ is the full subcategory of $\hat{B}$ given by the objects

\[ e_{0,j}, \quad 1 \leq j \leq n, \quad j \neq i, \quad \text{and} \quad e_{1,i} = \nu_{\hat{B}}(e_{0,i}). \]

Then the quiver $Q_{S_i^+B}$ of $S_i^+B$ is the reflection $\sigma_i^+Q_B$ of $Q_B$ at $i$ (see [11]).

Observe that $\hat{B} = S_i^+B$. By a reflection sequence of sinks of $Q_B$ we mean a sequence $i_1, \ldots, i_t$ of vertices of $Q_B$ such that $i_s$ is a sink of $\sigma_{i_s-1}^+ \cdots \sigma_{i_1}^+ Q_B$ for all $s$ in $\{1, \ldots, t\}$. Moreover, for a sink $i$ of $Q_B$, we denote by $T_i^+B$ the full subcategory of $\hat{B}$ given by the objects

\[ e_{0,j}, \quad 1 \leq j \leq n, \quad \text{and} \quad e_{1,i} = \nu_{\hat{B}}(e_{0,i}). \]

Observe that $T_i^+B$ is the one-point extension $B[I_B(i)]$ of $B$ by the indecomposable injective $B$-module $I_B(i)$ at the vertex $i$. By a finite-dimensional
\(\widehat{B}\)-module we mean a contravariant \(K\)-linear functor \(M\) from \(\widehat{B}\) to the category of \(K\)-vector spaces such that \(\sum_{x \in \text{ob} \, \widehat{B}} \dim_K M(x)\) is finite. We denote by \(\text{mod} \, \widehat{B}\) the category of all finite-dimensional \(\widehat{B}\)-modules. Finally, for a module \(M\) in \(\text{mod} \, \widehat{B}\), we denote by \(\text{supp}(M)\) the full subcategory of \(\widehat{B}\) formed by all objects \(x\) with \(M(x) \neq 0\), and call it the support of \(M\).

The following consequence of results proved in \cite{10}, \cite{11} describes the supports of finite-dimensional indecomposable modules over the repetitive categories \(\widehat{B}\) of tilted algebras \(B\) of Dynkin type.

**Theorem 2.1.** Let \(B\) be a tilted algebra of Dynkin type and \(n\) the rank of \(K_0(B)\). Then there exists a reflection sequence \(i_1, \ldots, i_n\) of sinks of \(Q_B\) such that the following statements hold:

(i) \(S_{i_n}^+ \cdots S_{i_1}^+ = \nu_{\widehat{B}}(B)\).

(ii) For every indecomposable non-projective module \(M\) in \(\text{mod} \, \widehat{B}\), \(\text{supp}(M)\) is contained in one of the full subcategories of \(\widehat{B}\) given by

\[\nu^m_{\widehat{B}}(S_{i_r}^+ \cdots S_{i_1}^+ B), \quad r \in \{1, \ldots, n\}, \ m \in \mathbb{Z}.\]

(iii) For every indecomposable projective module \(P\) in \(\text{mod} \, \widehat{B}\), \(\text{supp}(P)\) is contained in one of the full subcategories of \(\widehat{B}\) given by

\[\nu^m_{\widehat{B}}(T_{i_r}^+ S_{i_{r-1}}^+ \cdots S_{i_1}^+ B), \quad r \in \{1, \ldots, n\}, \ m \in \mathbb{Z}.\]

The aim of this section is to prove the following theorem playing a prominent role in the proof of Theorem 1.1.

**Theorem 2.2.** Let \(B\) be a tilted algebra \(\text{End}_H(T)\) of Dynkin type, \(\Delta_T\) the canonical section of \(\Gamma_B\) given by the images \(\text{Hom}_H(T,I)\) of indecomposable injective \(H\)-modules \(I\) via the functor \(\text{Hom}_H(T,-)\): \(\text{mod} \, H \rightarrow \text{mod} \, \widehat{B}\), and \(M_T\) the direct sum of indecomposable \(B\)-modules lying on \(\Delta_T\). Moreover, let \(\varphi\) be a strictly positive automorphism of \(\widehat{B}\), \(A = \widehat{B}/(\varphi)\), and \(F^\varphi_\lambda\): \(\text{mod} \, \widehat{B} \rightarrow \text{mod} \, A\) the associated push-down functor. The following statements are equivalent:

(i) \(F^\varphi_\lambda(M_T)\) is not the middle of a short chain in \(\text{mod} \, A\).

(ii) \(\varphi = \rho \nu^2_B\) for a positive automorphism \(\rho\) of \(\widehat{B}\).

**Proof.** It follows from Theorem 2.1 that \(\widehat{B}\) is a locally representation-finite locally bounded category \cite{8}, that is, for any indecomposable module \(N\) in \(\text{mod} \, \widehat{B}\) the number of objects \(x\) in \(\widehat{B}\) with \(N(x) \neq 0\) is finite. Then, applying \cite{8} Theorem 3.6], the push-down functor \(F^\varphi_\lambda\): \(\text{mod} \, \widehat{B} \rightarrow \text{mod} \, A\) is a Galois covering of module categories preserving almost split sequences. In particular, for any indecomposable modules \(X\) and \(Y\) in \(\text{mod} \, \widehat{B}\), \(F^\varphi_\lambda(X)\) and \(F^\varphi_\lambda(Y)\) are indecomposable modules in \(\text{mod} \, A\), and \(F^\varphi_\lambda\) induces \(K\)-linear
isomorphisms
\[ \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\hat{B}}(X, \varphi^r Y) \xrightarrow{\sim} \text{Hom}_A(F_{\hat{X}}^\varphi(X), F_{\hat{X}}^\varphi(Y)), \]
\[ \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\hat{B}}(\varphi^r X, Y) \xrightarrow{\sim} \text{Hom}_A(F_{\hat{X}}^\varphi(X), F_{\hat{X}}^\varphi(Y)). \]

Here, \( \varphi^r X \) and \( \varphi^r Y \) denote the shifts of \( X \) and \( Y \) by the automorphism of mod \( \hat{B} \) induced by \( \varphi^r \).

Assume that \( F_{\hat{X}}^\varphi(M_T) \) is the middle of a short chain in mod \( A \). Then there is an indecomposable non-projective module \( N \) in mod \( A \), indecomposable direct summands \( U \) and \( V \) of \( F_{\hat{X}}^\varphi(M_T) \), and non-zero homomorphisms \( N \to U \) and \( V \to \tau_A N \). Therefore, there exist indecomposable direct summands \( X \) and \( Y \) of \( M_T \), an indecomposable non-projective module \( Z \) in mod \( \hat{B} \), and non-zero homomorphisms \( Y \to \tau_B Z \) and \( Z \to \varphi^r X \) in mod \( \hat{B} \) with \( r \geq 1 \) such that \( F_{\hat{X}}^\varphi(X) = F_{\hat{X}}^\varphi(\varphi^r X) = U \), \( F_{\hat{X}}^\varphi(Y) = V \), and \( F_{\hat{X}}^\varphi(Z) = N \). Observe that for modules \( L, L' \) in mod \( \hat{B} \), \( \text{Hom}_{\hat{B}}(L, L') \neq 0 \) implies that supp(\( L \)) and supp(\( L' \)) have a common object. Since supp(\( M_T \)) = \( B = B_0 \) and \( Y \) is a direct summand of \( M_T \), we conclude that supp(\( Y \)) is contained in \( B \).

Similarly, \( \varphi^r X \) is a direct summand of \( \varphi^r M_T \) and supp(\( \varphi^r M_T \)) = \( \varphi^r B \), and so supp(\( \varphi^r X \)) is contained in \( \varphi^r B \). Applying now Theorem 2.1, we infer that supp(\( \tau_B Z \)) is contained in \( B \) or one of the full subcategories \( S_{i_1}^+ \cdots S_{i_n}^+ \) for some \( p \in \{1, \ldots, n-1\} \) and the corresponding reflection sequence \( i_1, \ldots, i_n \) of sinks of \( Q_B \). Note that \( B = \nu_B^{-1}(\nu_B(B)) = \nu_B^{-1}(S_{i_1}^+ \cdots S_{i_n}^+) \). Then it follows that supp(\( Z \)) is contained in \( S_{i_1}^+ \cdots S_{i_n}^+ B \) or in \( S_{i_1}^+ \cdots S_{i_n}^+ = \nu_B(B) \) (if \( p = n-1 \)). Hence \( \text{Hom}_{\hat{B}}(Z, \varphi^r X) \neq 0 \) forces that supp(\( \varphi^r X \)) is contained in a full subcategory of \( \hat{B} \) of one of the forms \( S_{i_1}^+ \cdots S_{i_n}^+ B \) for \( r \in \{1, \ldots, n\} \), or \( \nu_B(S_{i_1}^+ \cdots S_{i_n}^+) \) for \( q \in \{1, \ldots, n-1\} \). This shows that supp(\( \varphi^r X \)) = \( \varphi^r(\text{supp}(X)) \) is contained in the full subcategory \( T_{i_1}^+ \cdots T_{i_n}^+ \) of \( \hat{B} \) given by the objects of \( B \) and \( \nu_B(B) \). Summing up, we have proved that if \( \varphi = \rho \nu_B^2 \) for a positive automorphism \( \rho \) of \( \hat{B} \), then \( F_{\hat{X}}^\varphi(M_T) \) is not the middle of a short chain in mod \( A \). Therefore, (ii) implies (i).

Assume now that \( \varphi \) is not of the form \( \rho \nu_B^2 \) for a positive automorphism \( \rho \) of \( \hat{B} \). Then \( \varphi B \) is a full subcategory of \( T_{i_1}^+ \cdots T_{i_n}^+ \) of \( \hat{B} \) given by the objects of \( B \) and \( \nu_B(B) \). Take an indecomposable direct summand \( X \) of \( M_T \). Then \( \varphi X \) is an indecomposable direct summand \( \varphi M_T \), and so supp(\( \varphi X \)) is a full subcategory of supp(\( \varphi M_T \)) = \( \varphi(\text{supp} M_T) = \varphi B \). Thus supp(\( \varphi X \)) is a full subcategory of \( T_{i_1}^+ \cdots T_{i_n}^+ B \). We have two cases to consider.

Assume first that supp(\( \varphi X \)) contains an object \( j \) which is not in \( B \). Then \( j = \nu_B(i) \) for some object \( i \) of \( B \). Take the indecomposable projective-
injective \( \hat{B} \)-module \( P_{\hat{B}}(j) \) at \( j \). Clearly, we have \( \text{Hom}_{\hat{B}}(P_{\hat{B}}(j), \varphi X) \neq 0 \). In fact, since \( X \) is not a projective-injective \( \hat{B} \)-module, \( \varphi X \) is not a projective-injective \( \hat{B} \)-module, and hence \( \text{Hom}_{\hat{B}}(P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j), \varphi X) \neq 0 \). Clearly then \( \text{Hom}_{\hat{B}}(P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j), \varphi MT) \neq 0 \). Observe also that we have in \( \text{mod } B \) a canonical almost split sequence

\[
0 \to \text{rad } P_{\hat{B}}(j) \to (\text{rad } P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j)) \oplus P_{\hat{B}}(j) \to P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j) \to 0,
\]

and then \( \text{rad } P_{\hat{B}}(j) = \tau_{\hat{B}}(P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j)) \). Since \( j = \nu_{\hat{B}}(i) \) for some vertex \( i \) of \( Q_B \), we conclude that \( \text{soc } P_{\hat{B}}(j) \) is the simple \( \hat{B} \)-module \( S_{\hat{B}}(i) \) at \( i \), and consequently \( \text{Hom}_{\hat{B}}(MT, \text{rad } P_{\hat{B}}(j)) \neq 0 \). This shows that \( F_{\chi}^\varphi(M_T) = F_{\chi}^\varphi(\varphi MT) \) is the middle of a short chain

\[
F_{\chi}^\varphi(P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j)) \to F_{\chi}^\varphi(M_T) \to \tau_A F_{\chi}^\varphi(P_{\hat{B}}(j)/\text{soc } P_{\hat{B}}(j))
\]

since \( \tau_A F_{\chi}^\varphi(L) \cong F_{\chi}^\varphi(\tau_{\hat{B}}L) \) for any indecomposable non-projective module \( L \) in \( \text{mod } \hat{B} \).

Assume now that \( \text{supp}(\varphi X) \) is contained in \( B \). Since \( \varphi \) is a strictly positive automorphism of \( \hat{B} \), the support \( \text{supp}(\tau_{\hat{B}}(\varphi X)) \) of \( \tau_{\hat{B}}(\varphi X) \) is also contained in \( B \). Clearly, \( \varphi X \) is an indecomposable \( \hat{B} \)-module which is a successor of an indecomposable direct summand of \( M_T \), because \( X \) is an indecomposable direct summand of \( M_T \). Moreover, every indecomposable module in \( \text{mod } B \) is cogenerated or generated by \( M_T \). Hence \( \text{Hom}_{\hat{B}}(M_T, \tau_{\hat{B}}(\varphi X)) = \text{Hom}_{B}(M_T, \tau_{\hat{B}}(\varphi X)) \neq 0 \). This shows that \( F_{\chi}^\varphi(M_T) \) is the middle of a short chain in \( \text{mod } A \) of the form

\[
F_{\chi}^\varphi(X) \to F_{\chi}^\varphi(M_T) \to \tau_A F_{\chi}^\varphi(X)
\]

because \( F_{\chi}^\varphi(X) \) is an indecomposable direct summand of \( F_{\chi}^\varphi(M_T) \) and \( F_{\chi}^\varphi(\tau_{\hat{B}}(\varphi X)) \cong \tau_A F_{\chi}^\varphi(\varphi X) \cong \tau_A F_{\chi}^\varphi(X) \). Therefore, (i) implies (ii).

3. Self-injective algebras with deforming ideals. In this section we present criteria for self-injective algebras to be orbit algebras of the repetitive categories of algebras with respect to infinite cyclic automorphism groups, playing a fundamental role in the proof of the main theorem.

Let \( A \) be a self-injective algebra. For a subset \( X \) of \( A \), we may consider the left annihilator \( l_A(X) = \{ a \in A \mid ax = 0 \} \) of \( X \) in \( A \) and the right annihilator \( r_A(X) = \{ a \in A \mid xa = 0 \} \) of \( X \) in \( A \). Then by a theorem due to Nakayama (see [30, Theorem IV.6.10]) the annihilator operation \( l_A \) induces a Galois correspondence from the lattice of right ideals of \( A \) to the lattice of left ideals of \( A \), and \( r_A \) is the inverse Galois correspondence to \( l_A \). Let \( I \) be an ideal of \( A \), \( B = A/I \), and \( e \) an idempotent of \( A \) such that \( e + I \) is the identity of \( B \). We may assume that \( 1_A = e_1 + \cdots + e_r \) with \( e_1, \ldots, e_r \) pairwise orthogonal primitive idempotents of \( A \), \( e = e_1 + \cdots + e_n \) for some \( n \leq r \), and \( \{ e_i \mid 1 \leq i \leq n \} \) is the set of all idempotents in \( \{ e_i \mid 1 \leq i \leq r \} \) which
are not in $I$. Then such an idempotent $e$ is uniquely determined by $I$ up to an inner automorphism of $A$, and is called a residual identity of $B = A/I$. Observe also that $B \cong eAe/eIe$.

We have the following lemma from [27, Lemma 5.1].

**Lemma 3.1.** Let $A$ be a self-injective algebra, $I$ an ideal of $A$, and $e$ an idempotent of $A$ such that $l_A(I) = Ie$ or $r_A(I) = eI$. Then $e$ is a residual identity of $A/I$.

We also recall the following proposition proved in [23, Proposition 2.3].

**Proposition 3.2.** Let $A$ be a self-injective algebra, $I$ an ideal of $A$, $B = A/I$, $e$ a residual identity of $B$, and assume that $IeI = 0$. The following conditions are equivalent:

(i) $Ie$ is an injective cogenerator in $\text{mod } B$.
(ii) $eI$ is an injective cogenerator in $\text{mod } B^{\text{op}}$.
(iii) $l_A(I) = Ie$.
(iv) $r_A(I) = eI$.

Moreover, under these equivalent conditions, we have $\text{soc } A \subseteq I$ and $l_{eAe}(I) = eIe = r_{eAe}(I)$.

The following theorem proved in [25, Theorem 3.8] (sufficiency part) and [27, Theorem 5.3] (necessity part) will be fundamental for our considerations.

**Theorem 3.3.** Let $A$ be a self-injective algebra. The following conditions are equivalent:

(i) $A$ is isomorphic to an orbit algebra $\hat{B}/(\varphi \nu \hat{B})$, where $B$ is an algebra and $\varphi$ is a positive automorphism of $\hat{B}$.
(ii) There is an ideal $I$ of $A$ and an idempotent $e$ of $A$ such that

1. $r_A(I) = eI$;
2. the canonical algebra epimorphism $eAe \to eAe/eIe$ is a retraction.

Moreover, in this case, $B$ is isomorphic to $A/I$.

Let $A$ be an algebra, $I$ an ideal of $A$, and $e$ a residual identity of $A/I$. Following [23], $I$ is said to be a deforming ideal of $A$ if the following conditions are satisfied:

(D1) $l_{eAe}(I) = eIe = r_{eAe}(I)$;
(D2) the valued quiver $Q_{A/I}$ of $A/I$ is acyclic.

Assume $I$ is a deforming ideal of $A$. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and $I$ can be considered as an $(eAe/eIe)$-$(eAe/eIe)$-bimodule. Denote by $A[I]$ the direct sum of $K$-vector spaces.
(eAe/eIe) ⊕ I with the multiplication

\[(b, x) \cdot (c, y) = (bc, by + xc + xy)\]

for \(b, c \in eAe/eIe\) and \(x, y \in I\). Then \(A[I]\) is a \(K\)-algebra with the identity \((e + eIe, 1_A - e)\), and, by identifying \(x \in I\) with \((0, x) \in A[I]\), we may consider \(I\) as an ideal of \(A[I]\). Observe that \(e = (e + eIe, 0)\) is a residual identity of \(A[I]/I = eAe/eIe \sim A/I\), \(eA[I]e = (eAe/eIe) ⊕ eIe\) and the canonical algebra epimorphism \(eA[I]e \to eA[I]e/eIe\) is a retraction.

The following properties of the algebra \(A[I]\) were established in [23, Theorem 4.1] and [24, Theorem 3].

**Theorem 3.4.** Let \(A\) be a self-injective algebra and \(I\) a deforming ideal of \(A\). The following statements hold.

(i) \(A[I]\) is a self-injective algebra with the same Nakayama permutation as \(A\) and \(I\) is a deforming ideal of \(A[I]\).

(ii) \(A\) and \(A[I]\) are socle equivalent.

(iii) \(A\) and \(A[I]\) are stably equivalent.

We note that if \(A\) is a self-injective algebra, \(I\) an ideal of \(A\), \(B = A/I\), \(e\) an idempotent of \(A\) such that \(r_A(I) = eI\), and the valued quiver \(Q_B\) of \(B\) is acyclic, then by Lemma 3.1 and Proposition 3.2 \(I\) is a deforming ideal of \(A\) and \(e\) is a residual identity of \(B\).

The following theorem proved in [25, Theorem 4.1] shows the importance of the algebras \(A[I]\).

**Theorem 3.5.** Let \(A\) be a self-injective algebra, \(I\) an ideal of \(A\), \(B = A/I\) and \(e\) an idempotent of \(A\). Assume that \(r_A(I) = eI\) and \(Q_B\) is acyclic. Then \(A[I]\) is isomorphic to the orbit algebra \(\hat{B}/(\varphi_{\nu_{\hat{B}}})\) for some positive automorphism \(\varphi\) of \(\hat{B}\).

We point out that there are self-injective algebras \(A\) with deforming ideals \(I\) such that the algebras \(A\) and \(A[I]\) are not isomorphic (see [25, Example 4.2]).

The following result proved in [26, Proposition 3.2] describes a situation when the algebras \(A\) and \(A[I]\) are isomorphic.

**Theorem 3.6.** Let \(A\) be a self-injective algebra with a deforming ideal \(I, B = A/I, e\) a residual identity of \(B\) and \(\nu\) the Nakayama permutation of \(A\). Assume that \(IeI = 0\) and \(e_i \neq e_{\nu(i)}\), for any primitive summand \(e_i\) of \(e\). Then the algebras \(A\) and \(A[I]\) are isomorphic. In particular, \(A\) is isomorphic to the orbit algebra \(\hat{B}/(\varphi_{\nu_{\hat{B}}})\) for some positive automorphism \(\varphi\) of \(\hat{B}\).

4. **Proof of Theorem 1.1.** Let \(A\) be a non-simple, finite-dimensional, basic, indecomposable, self-injective \(K\)-algebra over a field \(K\).
Assume \( \text{mod } A \) admits a pure sectional module \( M \) which is not the middle of a short chain. We will show first that \( A \) is socle equivalent to the self-injective orbit algebra \( \hat{B}/(\varphi_\nu_B) \), where \( B \) is a tilted algebra of the form \( B = \text{End}_H(T) \) for a hereditary algebra \( H \) of Dynkin type and a tilting module \( T \) in \( \text{mod } H \) without indecomposable projective direct summands, and \( \varphi \) is a positive automorphism of \( B \). Let \( \Delta \) be the full-valued subquiver of the stable Auslander–Reiten quiver \( \Gamma_A^s \) of \( M \). We recall that then \( \Gamma_A^s \cong \mathbb{Z}\Delta/G \) for an infinite cyclic group \( G \) of automorphisms of the translation quiver \( \mathbb{Z}\Delta \), and \( \Delta \) is a Dynkin quiver whose underlying graph is the Dynkin type \( \Delta(A) \) of \( A \). Let \( I = r_A(M) \) and \( B = A/I \). Then \( M \) is a faithful, hence sincere, right \( B \)-module which is not the middle of a short chain in \( \text{mod } B \), because \( M \) is not the middle of a short chain in \( \text{mod } A \) (see [13, Proposition 2.3]). So \( B \) is a tilted algebra, by the main result of [12]. Further, \( H = \text{End}_A(M) = \text{End}_B(M) \) is the hereditary algebra, by [13, Corollary 1.2]. Clearly, \( H \) is then a hereditary algebra of Dynkin type with \( Q_H = \Delta^{\text{op}} \). Observe also that \( M \) is a faithful \( B \)-module with \( \text{Hom}_B(M, \tau_B M) = 0 \), and hence \( \text{pd}_B(M) \leq 1 \) and \( \text{Ext}^1_B(M, M) \cong \text{DHom}_B(M, \tau_B(M)) = 0 \) (see [1, Lemma VIII.5.1 and Theorem IV.2.13]). Therefore, \( M \) is a partial tilting \( B \)-module. Since the rank of \( K_0(B) \) coincides with the number of indecomposable direct summands of \( M \), we conclude that \( M \) is a tilting \( B \)-module. Hence, by the Brenner–Butler theorem [1, Theorem VI.3.8], \( M \) is a tilting module in \( \text{mod } H^{\text{op}} \), \( T = D(M) \) is a tilting module in \( \text{mod } H \), \( B \cong \text{End}_H(T) \), and \( M \) is isomorphic to the right \( B \)-module \( \text{Hom}_H(T, D(H)) \). In particular, we conclude that the indecomposable direct summands of \( M \) form the canonical section \( \Delta_T = \Delta \) of the connecting component \( C_T = \Gamma_B \). Moreover, since \( M \) is a pure sectional module in \( \text{mod } A \), we find that no indecomposable injective \( B \)-module is a direct summand of \( M \), or equivalently, the indecomposable direct summands of \( \tau_B^{-1}M \) form another section \( \tau_B^{-1}\Delta_T \) of \( C_T = \Gamma_B \). Finally, we note that \( T \) is a splitting tilting module in \( \text{mod } H \), since \( H \) is a hereditary algebra [1, Corollary VI.5.7]. Then, invoking the description of the indecomposable injective modules in \( \text{mod } B \), given in [1, Proposition VI.5.8], and \( M \cong \text{Hom}_H(T, D(H)) \), we conclude that \( T \) has no indecomposable projective direct summand.

Let \( e_1, \ldots, e_r \) be a set of pairwise orthogonal, primitive idempotents of \( A \) such that \( 1_A = e_1 + \cdots + e_r \) and that \( e = e_1 + \cdots + e_n \), for some \( n \leq r \), is a residual identity of \( B \). We claim that \( I \) is a deforming ideal of \( A \) satisfying \( IeI = 0 \). Observe that the valued quiver \( Q_B \) of \( B = A/I \) is acyclic, because \( B \) is a tilted algebra. Therefore, by Proposition [3.2] it remains to show that \( r_A(I) = eI \).

Denote by \( J \) the trace ideal of \( M \) in \( A \), that is, the ideal of \( A \) generated by the images of all homomorphisms from \( M \) to \( A \) in \( \text{mod } A \), and by \( J' \)
the trace ideal of the left $A$-module $D(M)$ in $A$. Observe that $I$ is the left annihilator of $D(M)$ in $A$.

**Lemma 4.1.** We have $J \cup J' \subseteq I$.

**Proof.** First we show that $J \subseteq I$. By definition, there exists an epimorphism $\varphi: M^r \to J$ for some integer $r \geq 1$. Suppose that there exists a homomorphism $f: A \to M$ in mod $A$ with $f(J) \neq 0$. Since $M$ has no projective-injective indecomposable direct summands, the homomorphism $f$ factors through $A/\text{soc} A$. Hence we have in mod $A$ a sequence of homomorphisms

$$
M^r \xrightarrow{\varphi} J \xrightarrow{\omega} A \xrightarrow{\pi} A/\text{soc} A \xrightarrow{g} M
$$

with $g \pi \omega \varphi \neq 0$, where $\omega: J \to A$ is the canonical inclusion homomorphism, $\pi: A \to A/\text{soc} A$ is the canonical epimorphism, and $f = g \pi$. Observe that $g \pi \omega \varphi$ factors through a module from $\text{add}(\tau^{-1}_A M)$, and consequently $\text{Hom}_A(\tau^{-1}_A M, M) \neq 0$. This is a contradiction because $M$ is not the middle of a short chain in mod $A$. Hence we conclude

$$
J \subseteq \bigcap_{f: A \to M} \ker f = I.
$$

Suppose now that there is a homomorphism $f': A \to D(M)$ in mod $A^{\text{op}}$ such that $f'(J') \neq 0$. Then $f'$ factors through $A/\text{soc} A$, because $D(M)$ has no projective-injective indecomposable direct summands. Moreover, we have in mod $A^{\text{op}}$ an epimorphism $\varphi': D(M)^s \to J'$ for some integer $s \geq 1$. Hence we obtain in mod $A^{\text{op}}$ a sequence of homomorphisms

$$
D(M)^s \xrightarrow{\varphi'} J' \xrightarrow{\omega'} A \xrightarrow{\pi} A/\text{soc} A \xrightarrow{g'} D(M)
$$

with $g' \pi \omega' \varphi' \neq 0$, where $\omega': J' \to A$ is the canonical inclusion homomorphism and $f' = g' \pi$. Observe also that $g' \pi \omega' \varphi'$ factors through a module from $\text{add}(\tau^{-1}_{A^{\text{op}}} M)$, and consequently $\text{Hom}_{A^{\text{op}}}(\tau^{-1}_{A^{\text{op}}} D(M), D(M)) \neq 0$. Since $\tau^{-1}_{A^{\text{op}}} D(M) = \text{Tr} M = D(\tau_A M)$, we conclude that $\text{Hom}_A(M, \tau_A M) \neq 0$. This is again a contradiction, because $M$ is not the middle of a short chain in mod $A$. Therefore we obtain

$$
J' \subseteq \bigcap_{f': A \to D(M)} \ker f' = I.
$$

**Lemma 4.2.** We have $l_A(I) = J, r_A(I) = J'$ and $I = r_A(J) = l_A(J')$.

**Proof.** We prove the lemma only for $J$, the proof for $J'$ being dual. Since $J$ is a right $B$-module, we have $JI = 0$, and hence $I \subseteq r_A(J)$. In order to show the converse inclusion, take a monomorphism $u: M \to A^t$ for some integer $t \geq 1$, and let $u_i: M \to A$ be the composite of $u$ with the projection of $A^t_A$ on the $i$th component. Then there is a monomorphism $v: M \to \bigoplus_{i=1}^t \text{Im} u_i$ induced by $u$. Moreover, by definition of $J$, $\bigoplus_{i=1}^t \text{Im} u_i$ is contained in $\bigoplus_{i=1}^t J$. 

These steps show that $I$ is contained in $r_A(J)$ and $r_A(J)$ is contained in $I$. Thus $I = r_A(J)$.
This leads to the inclusions
\[ r_A(J) = r_A \left( \bigoplus_{i=1}^{t} J \right) \subseteq r_A(M) = I. \]
Hence \( I = r_A(J) \). Finally, applying a theorem by Nakayama (see [30, Theorem IV.6.10]), we obtain \( J = l_A r_A(J) = l_A(I) \).

**Lemma 4.3.** We have \( eIe = eJe = eJ'e \). In particular, \((eIe)^2 = 0\).

**Proof.** Since \( e \) is a residual identity of \( B = A/I \), we have \( B \cong eAe/eIe \). Thus \( M \) is a faithful right \( eAe/eIe \)-module and the direct sum of indecomposable modules forming a section of \( \Gamma_{eAe/eIe} \). Further, it follows from Lemma 4.1 that \( eJe = eJ \) is an ideal of \( eAe \) with \( eJe \subseteq eIe \). Consider the algebra \( B' = eAe/eJ \). Then \( M \) is a sincere right \( B' \)-module which is not the middle of a short chain in \( \text{mod} B' \), because \( B' \) is a factor algebra of \( B \) and \( M \) is not the middle of a short chain in \( \text{mod} B \) [15, Proposition 2.3]. Applying [15, Corollary 3.2] we conclude that \( M \) is a faithful \( B' \)-module. This implies that \( eIe/eJ = \Gamma_{B'}(M) = 0 \), and hence \( eIe = eJe \). In a similar way we show that \( eJe = eJ'e \). Finally, it follows from Lemma 4.2 that \((eIe)^2 = (eJe)(eJe) = eJJe = 0\).

We shall also use the following general lemma on almost split sequences over triangular matrix algebras (see [19, (2.5)], [23, Lemma 5.6]).

**Lemma 4.4.** Let \( R \) and \( S \) be algebras and \( N \) be an \((S, R)\)-bimodule. Let \( A = \left( \begin{array}{cc} S & N \\ 0 & R \end{array} \right) \) be the matrix algebra defined by the bimodule \( SN_R \). Then an almost split sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \text{mod} R \) is an almost split sequence in \( \text{mod} A \) if and only if \( \text{Hom}_R(N, X) = 0 \).

**Lemma 4.5.** Let \( f \) be a primitive idempotent in \( I \) such that \( fJ \neq fAe \). Then \( K = fAeAf + fJ + fAeAfAe + eAf + eIe \) is an ideal of \( F = (e + f)A(e + f) \), and \( N = fAe/fKe \) is a \( B \)-module such that \( \text{Hom}_B(N, M) = 0 \) and \( \text{Hom}_B(M, N) \neq 0 \).

**Proof.** It follows from Lemma 4.3 that \( fAeIe \subseteq fJ \). Then the fact that \( K \) is an ideal of \( F \) is a direct consequence of \( f \in I \). Observe also that \( fKe = fJ + fAeAfAe, fKf \subseteq \text{rad}(fAf), eKe = eIe \) and \( eKf = eAf \). We have \( N \neq 0 \). Indeed, if \( fAe = fKe \) then, since \( eAe \subseteq \text{rad}(eAe) \), we obtain \( fAe = fJ + fAe(\text{rad}(eAe)) \), and so \( fAe = fJ \) (Nakayama lemma, [30, Lemma I.3.3]), which contradicts our assumption. Further, \( B = eAe/eIe \) and \( (fAe)(eIe) = fAeJ \subseteq fJ \subseteq fKe \), and hence \( N \) is a \( B \)-module. Moreover, \( N \) is also a left module over \( S = fAf/fKf \) and \( A = F/K \) is isomorphic to the triangular matrix algebra \( \left( \begin{array}{cc} S & N \\ 0 & R \end{array} \right) \). Invoking now our assumption that \( M \) is a pure sectional module in \( \text{mod} A \), we conclude that, for any indecomposable direct summand \( X \) of \( M \), we have in \( \text{mod} B \) an almost split sequence \( 0 \to X \to Y \to Z \to 0 \) which is also an almost split sequence in
mod $A$, and so an almost split sequence in mod $A$. Applying Lemma $4.4$, we obtain $\text{Hom}_B(N, M) = 0$. On the other hand, since every indecomposable module in mod $B$ is either generated or cogenerated by $M$, we conclude that $\text{Hom}_B(M, N) \neq 0$. □

**Proposition 4.6.** We have $Ie = J$ and $eI = J'$.

**Proof.** This follows exactly as [23, Proposition 5.9] by applying Lemmas $4.1$, $4.2$, $4.3$, $4.5$. □

The following direct consequence of Lemma $4.2$ and Proposition $4.6$ completes the proof that $I$ is a deforming ideal of $A$ with $IeI = 0$.

**Corollary 4.7.** We have $r_A(I) = eI$ and $l_A(I) = Ie$.

Applying Theorems $3.4$ and $3.5$ we conclude that:

1. $A$ is socle equivalent to $A[I]$;
2. $A$ is stably equivalent to $A[I]$;
3. $A[I]$ is isomorphic to a self-injective orbit algebra $\hat{B}/(\varphi \nu_B)$ for some positive automorphism $\varphi$ of $\hat{B}$.

Since $A$ and $A[I]$ are socle equivalent, the quotient algebras $A/\text{soc} A$ and $A[I]/\text{soc} A[I]$ are isomorphic, and consequently there is a canonical isomorphism $\Phi: \text{mod}(A/\text{soc} A) \rightarrow \text{mod}(A[I]/\text{soc} A[I])$ of their module categories. Observe also that the indecomposable modules in mod$(A/\text{soc} A)$ (respectively, mod$(A[I]/\text{soc} A[I])$) are precisely the indecomposable non-projective modules in mod $A$ (respectively, mod $A[I]$). Further, for any non-projective indecomposable modules $L, N$ in mod $A$ and non-projective indecomposable modules $U, V$ in mod $A[I]$ we have the equalities of homomorphism spaces $\text{Hom}_A(L, N) = \text{Hom}_{A/\text{soc} A}(L, N)$ and $\text{Hom}_{A[I]}(U, V) = \text{Hom}_{A[I]/\text{soc} A[I]}(U, V)$. We also note that the Auslander–Reiten quiver $\Gamma_{A/\text{soc} A}$ of $A/\text{soc} A$ (respectively, $\Gamma_{A[I]/\text{soc} A[I]}$ of $A[I]/\text{soc} A[I]$) is obtained from $\Gamma_A$ (respectively, $\Gamma_{A[I]}$) by removing all indecomposable projective modules $P$, making their radicals $P$ injective modules and the socle factors $P/\text{soc} P$ projective modules, and keeping the indecomposable non-projective modules as well their Auslander–Reiten translations unchanged. Finally, the functor $\Phi$ induces a canonical isomorphism of the stable Auslander–Reiten quivers $\Gamma^s_A \sim \Gamma^s_{A[I]}$. Summing up, we conclude that the image $\Phi(M)$ of the pure sectional module in mod $A$ is a pure sectional module $M$ in mod $A[I]$ and is not the middle of a short chain. Applying Theorem $2.2$, we conclude that $\varphi \nu_B = \rho \nu^2_B$ for some positive automorphism $\rho$ of $\hat{B}$. Since, by Theorem $3.4$, the Nakayama permutations of $A$ and $A[I]$ are the same, an isomorphism $A[I] \cong \hat{B}/(\rho \nu^2_B)$ forces that $e_i \neq e_{\nu(i)}$ for any primitive direct summand $e_i$ of the common residual identity $e$ of $A/I \cong A[I]/I$. Applying now Theorem $3.6$, we conclude that the algebras $A$ and $A[I]$ are isomorphic. Therefore, $A$ is isomorphic to the
orbit algebra $\tilde{B}/(\rho v_B^2)$. This proves the implication $(i) \Rightarrow (ii)$ of Theorem 1.1. The converse implication $(ii) \Rightarrow (i)$ follows from Theorem 2.2.

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