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HOLOMORPHICALLY PSEUDOSYMMETRIC KÄHLER METRICS ON $\mathbb{CP}^n$

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Abstract. The aim of this paper is to present examples of holomorphically pseudosymmetric Kähler metrics on the complex projective spaces $\mathbb{CP}^n$, where $n \geq 2$.

1. Introduction. The aim of this paper is to give new examples of holomorphically pseudosymmetric Kähler metrics on complex projective spaces. Holomorphically pseudosymmetric Kähler manifolds were defined by Z. Olszak [O-1] in 1989 and studied in [D], [H], [Y]. A Kähler manifold (M, g, J) is called *holomorphically pseudosymmetric* (HP Kähler) if its curvature tensor R satisfies the condition $R.R = \phi\Pi.R$, where R, Π act as derivations of the tensor algebra,

$$\Pi(U,V)X = \tfrac{1}{4}(g(V,X)U - g(U,X)V + g(JV,X)JU \\ - g(JU,X)JV - 2g(JU,V)JX)$$

is the Kähler type curvature tensor of constant holomorphic sectional curvature, and $\phi \in C^\infty(M)$ is a smooth function. A Riemannian manifold (M, g) is called *semisymmetric* if $R.R = 0$ (see [Sz-1], [Sz-2]). Until recently, examples of compact, holomorphically pseudosymmetric but not semisymmetric Kähler manifolds were not known. The first examples of compact, simply connected HP Kähler manifolds which are not semisymmetric were given by the author in [J-2]. These manifolds turned out to be *QCH Kähler manifolds*, i.e., Kähler manifolds admitting a smooth, two-dimensional, J-invariant distribution $\mathcal{D}$ such that the holomorphic curvature $K(\pi) = R(X, JX, JX, X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X,X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is the orthogonal projection of X onto $\mathcal{D}$. In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|)$$

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where $\phi(x,t) = a(x) + b(x)t^2 + c(x)t^4$ and a, b, c are smooth functions on M. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type (see [G-M-1], [G-M-2], [J-1]). In the present paper we construct QCH Kähler metrics on the manifolds $\mathbb{CP}^n - \{p_0\}$, where $p_0 \in \mathbb{CP}^n$, and show that these metrics extend smoothly to HP Kähler metrics on $\mathbb{CP}^n$.

2. QCH Kähler manifolds. We set $h = g \circ (p_{\mathcal{D}} \times p_{\mathcal{D}})$, where $p_{\mathcal{D}}$ is the orthogonal projection onto $\mathcal{D}$. We denote by $\Omega = g(J\cdot,\cdot)$ the Kähler form of (M, g, J), and by ω the Kähler form of $\mathcal{D}$, i.e. $\omega(X,Y) = h(JX,Y)$. We now recall some results of Ganchev and Mihova [G-M-1]. Let $R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$ and write

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

We shall identify $(1,3)$ tensors with $(0,4)$ tensors in this way. If R is the curvature tensor of a QCH Kähler manifold (M, g, J), then

$$R = a\Pi + b\Phi + c\Psi,$$

where $a, b, c \in C^\infty(M)$ and Π is the standard Kähler tensor of constant holomorphic curvature,

$$\begin{aligned}\Pi(X,Y,Z,U) = \tfrac{1}{4}(&g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\\ &+ g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)),\end{aligned}$$

the tensor Φ is defined as follows:

$$\begin{aligned}\Phi(X,Y,Z,U) = \tfrac{1}{8}(&g(Y,Z)h(X,U) - g(X,Z)h(Y,U)\\ &+ g(X,U)h(Y,Z) - g(Y,U)h(X,Z) + g(JY,Z)\omega(X,U)\\ &- g(JX,Z)\omega(Y,U) + g(JX,U)\omega(Y,Z) - g(JY,U)\omega(X,Z)\\ &- 2g(JX,Y)\omega(Z,U) - 2g(JZ,U)\omega(X,Y)),\end{aligned}$$

and

$$\Psi(X,Y,Z,U) = -\omega(X,Y)\omega(Z,U) = -(\omega\otimes\omega)(X,Y,Z,U).$$

3. HP metrics on complex projective spaces. Let

$$\phi : \mathbb{CP}^{n+1} - \{[0,\dots,0,1]\} \to \mathbb{CP}^n$$

be the holomorphic mapping defined as follows:

$$\begin{aligned}&\mathbb{CP}^{n+1} - \{[0,\dots,0,1]\} \ni [z_0, z_1, \dots, z_n]\\ &\qquad\qquad \to \phi([z_0, z_1, \dots, z_n]) = [z_0, z_1, \dots, z_{n-1}] \in \mathbb{CP}^n.\end{aligned}$$

We shall show that ϕ is the projection onto the base of a holomorphic line bundle whose total space is $H = \mathbb{CP}^{n+1} - \{[0,\dots,0,1]\}$. Consider the mapping $\phi_i : H_{|U_i} \to U_i \times \mathbb{C}$ where $U_i = \{[z_0, z_1, \dots, z_{n-1}] \in \mathbb{CP}^n : z_i \neq 0\}$,

$H_{|U_i} = \phi^{-1}(U_i)$, defined by

$$\phi_i([z_0, z_1, \dots, z_n]) = ([z_0, z_1, \dots, z_{n-1}], z_n/z_i).$$

Then $\phi_i \circ \phi_j^{-1}(x, z) = \phi_i([z_0, z_1, \dots, z_n])$, where $x = [z_0, z_1, \dots, z_{n-1}]$ and $z_n = z_j z$. Moreover $\phi_i([z_0, z_1, \dots, z_n]) = (x, z_n/z_i) = (x, z_j z/z_i) = (x, z z_j/z_i)$. It follows that the transition functions for our bundle are $\phi_{ij}(x) = z_j/z_i$. Consequently, the bundle H is isomorphic to the line hyperplane bundle over $\mathbb{CP}^n$. Hence $\mathbb{CP}^{n+1}$ arises from the line hyperplane bundle over $\mathbb{CP}^n$ by adding a point to its total space. We have $c_1(\mathbb{CP}^{n-1}) = n\alpha$, where $\alpha \in H^2(\mathbb{CP}^{n-1}, \mathbb{Z})$ is an indivisible integral class. Let $p : P \to \mathbb{CP}^{n-1}$ be a circle bundle over $\mathbb{CP}^{n-1}$ classified by α. Then P is a principal S^1 bundle. Let θ be a connection form of P. Then $\left[\frac{d\theta}{2\pi}\right] = p^*\alpha$ in $H^2(P, \mathbb{R})$. It follows that $\mathbb{CP}^n$, where $n \geq 2$, can be described as the quotient of the product $[0, L] \times P$ by the equivalence relation in which $\{0\} \times P$ is identified with $[0, \dots, 0, 1]$, and two points $(L, s), (L, q)$ are related if $p(s) = p(q)$. Let $g_{\mathbb{CP}^{n-1}}$ be the Fubini–Study metric on $\mathbb{CP}^{n-1}$ and let θ be the standard connection form on P. The metric

$$g = dt^2 + f(t)^2 \theta \otimes \theta + h(t)^2 g_{\mathbb{CP}^{n-1}}$$

on the product $[0, L] \times P$ extends to a smooth metric on $\mathbb{CP}^n$ if the smooth functions f, h on $[0, L]$ are also positive on $(0, L)$ as well as odd at 0 and satisfy $f'(0) = h'(0) = 1$ while f is odd at L with $f'(L) = -1$, and h is even at L with $h(L) \neq 0$. This metric is Kähler if $f = hh'$, and also admits a holomorphic Killing vector field with a Kähler–Ricci potential h^2 (see [J-1], [D-M-1], [D-M-2]).

THEOREM 3.1. *Consider an analytic real function Q on $\mathbb{R}$, which is positive on $[0, 1)$, even at 0 and such that $Q(0) = 1$, $Q'(0) = 0$, $Q(1) = 0$, $Q'(1) = -2$. Let a function h satisfy the equation $h' = \sqrt{Q(h)}$ and $h'' = \frac{1}{2}Q'(h)$, $h(0) = 0$, $h'(0) = 1$. Then, for $n \geq 2$,*

$$g = dt^2 + (h'(t)h(t))^2 \theta \otimes \theta + h(t)^2 g_{\mathbb{CP}^{n-1}}$$

extends to a smooth, Kähler metric on $\mathbb{CP}^n$, which is a QCH Kähler metric on $\mathbb{CP}^n - \{[0, \dots, 0, 1]\}$.

Proof. We shall show that h is odd at 0. It suffices to show that $h^{(2k)}(0) = 0$ for every $k \in \mathbb{N}$. For $k = 0, 1$ this equality is true. Assume that it holds for $l < k$. Note that $h^{(3)} = \frac{1}{2}Q''(h)h'$. Consequently,

$$2h^{(2k)}(0) = (Q''(h)h')^{(2k-3)}(0).$$

We first show that $\frac{d^l}{dt^l}(Q(h))(0) = 0$ for all odd $l < 2k$. This holds for $k = 1$, since $\frac{d}{dt}(Q(h))(0) = Q'(0)h'(0) = 0$. Next

$$\frac{d^l}{dt^l}(Q(h))(0) = \frac{d^{l-1}}{dt^{l-1}}(Q'(h)h') = \sum_{p=0}^{l-1} C_{l-1}^p \frac{d^p}{dt^p}(Q'(h))(0)\left(\frac{d^{l-1-p}}{dt^{l-1-p}}h\right)(0) = 0.$$

Hence

$$2h^{(2k)}(0) = \sum_{l=0}^{2k-3} C_{2k-3}^l \frac{d^l}{dt^l}(Q^{(2)}(h))(0)h^{(2k-2-l)}(0) = 0,$$

where $C_{2k-3}^l = \frac{(2k-3)!}{l!(2k-3-l)!}$, since for l odd we have $\frac{d^l}{dt^l}(Q^{(2)}(h))(0) = 0$, and for l even we have $h^{(2k-2-l)}(0) = 0$ by induction assumption. Hence if Q is an analytic function which is positive on $[0,1)$, even at 0 and such that

(1) $$Q(0) = 1, \quad Q'(0) = 0, \quad Q(1) = 0, \quad Q'(1) = -2$$

and h satisfies $h'' = \frac{1}{2}Q'(h), h(0) = 0, h'(0) = 1$ then $h' = \sqrt{Q(h)}$, $L = \int_0^1 dh/\sqrt{Q(h)}$ and the metric

$$g = dt^2 + (h'(t)h(t))^2\theta \otimes \theta + h(t)^2 g_{\mathbb{CP}^{n-1}}$$

is a Kähler metric on $\mathbb{CP}^n$, which is a QCH metric on $\mathbb{CP}^n - \{[0,\dots,0,1]\}$ (see [J-1]; note that we write $h = r\sqrt{n}$ and $s = \frac{2}{n}$ since in our case $k = 1$). Hence this metric is a HP metric on a dense, open subset of $\mathbb{CP}^n$, which means that it is a HP metric on the whole of $\mathbb{CP}^n$. In fact $R.R = \phi \mathit{\Pi}.R$, where $\phi = -4h''/h = -2Q'(h)/h$ (see [J-2]). Note that in [J-2] there is a sign mistake and the formula for $a + \frac{b}{2}$ should read

$$a + \frac{b}{2} = 4\left(\frac{(r')^2}{r^2} - \frac{f'r'}{fr}\right) = -4\frac{r''}{r}.$$

However the fact that $a + b/2$ changes sign in the case considered in [J-2] remains true. The function $\phi = a + b/2$ depends only on t and extends smoothly to the whole of $\mathbb{CP}^n$. We have

(2) $$\phi([0,\dots,0,1]) = -\lim_{h\to 0}\frac{2Q'(h)}{h} = -2Q''(0). \ \blacksquare$$

Let us consider as an example the family of polynomials

$$Q_\alpha(t) = 1 + (\alpha - 1)t^2 - 2\alpha t^4 + \alpha t^6.$$

Note that for $\alpha > -4$ every polynomial Q_α is positive on $[0,1)$ and satisfies conditions (1). Let $\alpha > -4$ and h_α be a solution of the problem

$$h_\alpha'' = \frac{1}{2}Q_\alpha'(h_\alpha), \quad h_\alpha(0) = 0, \quad h_\alpha'(0) = 1.$$

Then we obtain a family of HP metrics $g_\alpha = dt^2 + (h_\alpha'(t)h_\alpha(t))^2\theta \otimes \theta + h_\alpha(t)^2 g_{\mathbb{CP}^{n-1}}$ on $\mathbb{CP}^n$. Note that for $\alpha = 0$ we get $h_0(t) = \sin t$ and we obtain the standard symmetric metric on $\mathbb{CP}^n$ (see [P, p. 17]). We also have $\phi_\alpha = -2Q_\alpha'(h_a)/h_\alpha = 4(-(\alpha-1)+4\alpha h_\alpha^2 - 3\alpha h_\alpha^4)$. If $\alpha \in (-3,1)$ then $\phi_\alpha > 0$ and consequently we get examples of compact HP Kähler manifolds with

$\phi > 0$. If $\alpha \in \{-3, 1\}$ then $\phi_\alpha \geq 0$. Z. Olszak [O-2] proved that a compact HP Kähler manifold which has $\phi \geq 0$ and constant scalar curvature must be locally symmetric. Our examples show that the assumption of constant scalar curvature in Olszak's theorem is necessary.

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