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ELASTICITY OF $A+XB[X]$ WHEN $A\subset B$ IS A MINIMAL EXTENSION OF INTEGRAL DOMAINS

BY

AHMED AYACHE (Sana'a) and HANEN MONCEUR (Sfax)

Abstract. We investigate the elasticity of atomic domains of the form $\Re = A + XB[X]$, where X is an indeterminate, A is a local domain that is not a field, and $A\subset B$ is a minimal extension of integral domains. We provide the exact value of the elasticity of $\Re$ in all cases depending the position of the maximal ideals of B. Then we investigate when such domains are half-factorial domains.

Introduction. We begin by recalling some basic definitions. An integral domain R satisfies the *ascending chain condition on principal ideals* (ACCP) if any ascending chain of principal ideals of R terminates. Classical classes of domains satisfying ACCP are Dedekind, Krull and Noetherian domains. These domains are atomic. Following P. M. Cohn [10], we say that an integral domain R is *atomic* if each nonzero nonunit of R is a product of irreducible elements of R. It is well-known that if R satisfies ACCP, then R is atomic, but the converse is false. The first counter-example is due to A. Grams [17, Example 2.1]. If R is a unique factorization domain (UFD), then any two factorizations of a nonzero nonunit of R into the product of irreducible elements have the same length. However, this need not be true for an arbitrary atomic domain. Following A. Zaks [21], we say that an atomic domain is a *half-factorial* domain (HFD) if whenever $x_1\cdots x_m = y_1\cdots y_n$ with each $x_i, y_j\in A$ irreducible, then $m = n$. Examples of HFDs are UFDs, and more generally any Krull domain R with class group such that $|Cl(R)|\le 2$ [22, Theorem 1.4]. Recently, there has been much activity on HFD's, other factorization properties weaker than unique factorization [4], [5], [6], [9], [12] and on invariants that measure different lengths of factorizations. In order to measure how far an atomic domain R is from being an HFD, we define the *elasticity* of R to be

$$\rho(R) = \sup\{m/n : x_1\cdots x_m = y_1\cdots y_n \text{ for } x_i, y_j\in R \text{ irreducible}\}$$

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if R is not a field, and $\rho(R) = 1$ if R is a field. Thus $1 \leq \rho(R) \leq \infty$, and $\rho(R) = 1$ if and only if R is an HFD. The elasticity of an integral domain was first introduced by R. J. Valenza [20] for Dedekind domains with finite divisor class group. Subsequently $\rho(R)$ has received considerable attention [1], [2], [3].

Throughout this paper, A is a local integral domain with maximal ideal m that is not a field and $A \subset B$ is a minimal extension of integral domains, i.e. B is an integral domain containing A such that there is no proper intermediate ring between A and B. Two cases may happen [11], [18]: Either A is integrally closed in B, or $A \subset B$ is an integral extension. Let $\Re = A + XB[X]$. In [19], the authors have studied the transfer of some properties from A and B to $\Re$ because, in fact, this type of construction is useful in order to get examples of domains which satisfy or do not satisfy assigned factorization properties. Our main purpose here is to determine the elasticity of $\Re$, so the present work is a continuation of the investigation developed in [15] and [16].

We first prove that if A is atomic and integrally closed in B, then A is a rank one discrete valuation domain with quotient field B. In this case we show that $\Re$ is never an atomic domain. So, we will focus on the case where $A \subset B$ is a (minimal) integral extension. Under such conditions, we can provide the exact value of $\rho(\Re)$. We find that $\rho(\Re) = \infty$ when B has two maximal ideals M and N and satisfies ACCP; $\rho(\Re) = \rho(B[X])$ if B is local with maximal ideal $M = m$; and finally $3/2 \leq \rho(\Re) \leq 3\rho(B[X])$ if B is local with maximal ideal $M \neq m$ and satisfies ACCP. In particular, if B is a local UFD with maximal ideal $M \neq m$, we get $\rho(\Re) = 3/2$.

Consequently, we conclude that $\Re$ is a HFD if and only if $A \subset B$ is a minimal integral extension, $B[X]$ is a HFD and B is local with maximal ideal m. Finally, to illustrate all different cases of our study, we produce some explicit examples.

If R is an integral domain, $U(R)$ will denote its group of units. For any undefined terminology or notation, see [13].

Elasticity of $A + XB[X]$

LEMMA 1. *If A is atomic and integrally closed in B, then A is a rank one discrete valuation domain with quotient field B.*

Proof. According to [8, Theorem 1.2], there is a prime ideal P of A such that $PA_P = P$, $B = A_P$ and A/P is a rank one-valuation domain with quotient field B/P. Let p be an element of P. If $p \neq 0$, then p is a nonzero nonunit of A. We claim that p is not irreducible in A. Indeed, pick an element $\alpha \in m \setminus P$. Then $p = \alpha(p/\alpha)$, where α is not a unit of A and p/α is not a unit of A since $p/\alpha \in PA_P = P \subset m$. Now, as A is atomic, p can be written as a product $p_1 \cdots p_n$ of irreducible elements of A. One of the p_i, say p_1, necessarily belongs to P. But this implies, as above, that p_1 is not irreducible in A, a contradiction. Thus $P = (0)$, and hence A is a rank one-valuation domain with quotient field B. Finally, it is known that an atomic nontrivial valuation domain is a discrete valuation domain. ∎

THEOREM 2. *If A is integrally closed in B, then $\Re$ is never atomic.*

Proof. Suppose, by way of contradiction, that $\Re$ is atomic. Since A and $\Re$ have the same units, A is also atomic. But in light of Lemma 1, we conclude that A is a discrete valuation domain with quotient field B. Let $\alpha \in m$, $\alpha \neq 0$. Then $X = (X/\alpha)\alpha$ is not a product of irreducible elements of $\Re$, a contradiction. ∎

Now, suppose that $A \subset B$ is a (minimal) integral extension of rings. From [19, Proposition 3.2], we know that $m = (A : B) = \{x \in A : xB \subseteq A\}$ is the conductor of A in B, and there are at most two maximal ideals of B lying over m [14, Corollary 2.2]. On the other hand, the extension $\Re \subset B[X]$ is integral, therefore, if B satisfies ACCP, then $B[X]$ satisfies ACCP [17, p. 321]. This implies that $\Re$ satisfies ACCP [17, Proposition 2.1], so $\Re$ is atomic.

THEOREM 3. *Assume that $A \subset B$ is an integral extension such that B satisfies ACCP. If there are two maximal ideals M and N of B lying over m, then the elasticity of $\Re$ is infinite.*

Proof. Let $x \in M$ and $y \in N$ be such that $x + y = 1$. Then $x \in M \setminus N$, $y \in N \setminus M$ and

$$xy \in MN = M \cap N = M \cap A = N \cap A = m.$$

If u is a unit of B, then $ux^n \notin A$ for all $n \geq 1$. Indeed, if $ux^n \in A$ for some $n \geq 1$, then $ux^n \in A \cap M = m$. It follows that $ux^n \in N$, so $x \in N$, a contradiction. Likewise, we can show that $uy^n \notin A$ for all $n \geq 1$.

Consider the polynomials $r_n = X(X + x)^n$ and $s_n = X(X + y)^n$. Then r_n and s_n are irreducible [15, Lemma 3.1]. Moreover, we have the following factorizations in $\Re$:

$$r_n s_n = X^2(X^2 + X + xy)^n.$$

We deduce that $\rho(\Re) \geq (2 + n)/2$. ∎

Now, we will treat the case where B is local with maximal ideal M. We distinguish two cases: $M = m$ and $M \neq m$. To study the first case, we recall two useful conditions that are introduced by N. Gonzalez [15, Definitions 1.3 & 2.3] to control the irreducible polynomials of $\Re$ of order 0 or 1.

We say that $A \subset B$ satisfies *condition* (C_1^*) if, for each $a \in A$, $a = xy$ $(x, y \in B)$ implies $a = x'y'$, where $x', y' \in A$, $x = ux'$, $y = vy'$ and $uv = 1$ $(u, v$ are units of $B)$.

We say that $A \subset B$ satisfies *condition* (C_2^*) if each element of B is associated in B to an element of A, i.e., for each $b \in B$, there exists a unit u of B such that $ub \in A$.

It is shown that these two conditions are independent [15, Examples 1.10(1) & 2.8]. However, if $A \subset B$ satisfies condition (C_2^*) and $(A : B)$ is a maximal ideal of A, then $A \subset B$ satisfies condition (C_1^*) [15, Lemma 2.9]. This fact will play an important role in the following result.

Theorem 4. *Assume that $A \subset B$ is an integral extension such that $B[X]$ is atomic. If B is local with maximal ideal $m = M$, then $\rho(\Re) = \rho(B[X])$.*

Proof. We will prove that $A \subset B$ satisfies condition (C_2^*). Let $b \in B \setminus A$. Then b is integral over A, so b is a root of an equation of the form

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0,$$

where $a_0, a_1, \ldots, a_{n-1} \in A$ and $n > 1$. We may suppose that this integrality equation has minimal degree for b over A. We have

$$b(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1) = -a_0 \in A.$$

If $a_0 \in m$, then $u = b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1 \in M = m$. It follows that

$$b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_2 b + (a_1 - u) = 0.$$

But this contradicts the choice of n. Thus $a_0 \notin m$. We deduce that u is a unit of B and $ub \in A$. Therefore, $A \subset B$ satisfies (C_2^*). As $m = (A : B)$ is a maximal ideal of A, the extension $A \subset B$ satisfies condition (C_1^*) [15, Lemma 2.9]. It follows that $\Re \subset B[X]$ satisfies condition (C_1^*) [15, Proposition 2.6]. Hence, $\Re$ is atomic and $\rho(\Re) = \rho(B[X])$ [15, Proposition 2.7]. ∎

The most interesting case turns out to be when $M \neq m$. In this special case, we have $M = \sqrt{m}$. Let $x \in M \setminus m$. Then $x^r \in m$ for some integer $r \geq 2$. Set $\zeta = x^{r-1}$. Then $\zeta \in M \setminus m$ and $\zeta^2 \in m$. As $\zeta \in B \setminus A$, we have $B = A[\zeta] = A + A\zeta$. We keep this notation for the remainder of this paper.

Theorem 5. *Assume that $A \subset B$ is an integral extension such that B satisfies ACCP. If B is local with maximal ideal $M \neq m$, then* $3/2 \leq \rho(\Re) \leq 3\rho(B[X])$.

Proof. ζX is irreducible in $\Re$. Indeed, suppose that $\zeta X = a(bX)$, where $a \in A$ and $b \in B$. Then $\zeta = ab$. If $a \in m$, then $\zeta \in m$, a contradiction, so $a \in A \setminus m = U(A)$. We have the following factorizations in $\Re$:

$$(\zeta X)(\zeta X) = \zeta^2 X^2$$

with two irreducible factors on the left and at least three irreducible factors on the right. Thus $\rho(\Re) \geq 3/2$.

To get an upper bound on $\rho(\Re)$, we distinguish four types of polynomials in $B[X]$:

- Type α: $a_0 + a_1\zeta + X\varphi(X)$, where $\varphi(X) \in B[X]$, $a_0 \in m$ and $a_1 \in m$.
- Type β: $a_0 + a_1\zeta + X\varphi(X)$, where $\varphi(X) \in B[X]$, $a_0 \in U(A)$ and $a_1 \in m$.
- Type γ: $a_0 + a_1\zeta + X\varphi(X)$, where $\varphi(X) \in B[X]$, $a_0 \in m$ and $a_1 \in U(A)$.
- Type δ: $a_0 + a_1\zeta + X\varphi(X)$, where $\varphi(X) \in B[X]$, $a_0 \in U(A)$ and $a_1 \in U(A)$.

Note that the polynomials of types α and β are in $\Re$ whereas those of type γ and δ are in $B[X]$. Furthermore, a polynomial of type δ is associated (in $B[X]$) to a polynomial of type β: indeed, if $g = a_0 + a_1\zeta + X\varphi(X)$ is of type δ, then $u = a_0 - a_1\zeta \in U(B)$ and $(a_0)^2 - (a_1)^2\zeta^2 \in U(A)$, so

$$(a_0 - a_1\zeta)g(X) = (a_0)^2 - (a_1)^2\zeta^2 + X(a_0 - a_1\zeta)\varphi(X)$$

is of type β. Therefore, in any factorization of a polynomial of $B[X]$ into irreducible polynomials, we may consider only polynomials of type α, β and γ. The type of their products depends only on the product of their constant terms in $B = A + A\zeta$:

$\times$	α	β	γ
α	α	α	α
β	α	β	γ
γ	α	γ	α

Let f be an irreducible polynomial in $\Re$. We factor it in $B[X]$: since $B[X]$ is atomic, we write

$$f = \alpha^r\beta^s\gamma^t$$

to indicate that the numbers of irreducible factors in $B[X]$ of type α, β and γ are, respectively, r, s and t.

(1) Suppose that $r \neq 0$. Note that the product of a polynomial of type α by any other polynomial of $B[X]$ is also of type α. Since $r \neq 0$, f necessarily has a factor of type α, which will be denoted by α^*. We claim that no product of other factors can be in $\Re$. Towards a contradiction, suppose that such a product g belongs to $\Re$ and let h be the product of the remaining factors. Then $f = \alpha^* gh$. As $f = g(\alpha^* h)$, and the polynomial $\alpha^* h$ is in $\Re$ because of type α, we obtain a contradiction, since f is irreducible in $\Re$. Consequently, the hypothesis $r \neq 0$ implies $r = 1$, $s = 0$ and $t \leq 1$. In this situation, the longest factorizations are of the form $f = \alpha\gamma$.

(2) Suppose that $r = 0$. We have $f = \beta^s\gamma^t$. Then either $s \neq 0$, so $s = 1$ and $t = 0$; or $s = 0$ and $2 \leq t \leq 3$. In this situation, the longest factorizations are of the form $f = \gamma^3$.

In conclusion, the longest factorizations of f are always of length 3. In view of [15, Lemma 2.1], we obtain $\rho(\Re) \leq 3\rho(B[X])$. ∎

In the case where B is a UFD, we are able to provide the exact value of $\rho(\Re)$.

THEOREM 6. *Assume that $A \subset B$ is an integral extension such that B is a UFD. If B is local with maximal ideal $M \neq m$, then $\rho(\Re) = 3/2$.*

The proof breaks into two lemmas:

LEMMA 7. *Assume that $A \subset B$ is an integral extension such that B is local with maximal ideal $M \neq m$ and $B[X]$ is atomic. Then every polynomial $f \in \Re$ of type β that is prime in $B[X]$ is also prime in $\Re$.*

Proof. We will show that the ideal $f\Re$ generated by f is a prime ideal of $\Re$. Let g and h be two polynomials of $\Re$ such that $gh \in f\Re$. As $fB[X]$ is a prime ideal of $B[X]$, we may assume that $g \in fB[X]$. Thus, there is a polynomial $k \in B[X]$ such that $g = fk$. As f is of type β, the multiplication table of types shows that the polynomials g and k have the same type. Hence $k \in \Re$ and $g \in f\Re$. ∎

The second lemma makes use of a length function. In fact, length functions are frequently used to determine upper and lower bounds for the elasticity of an atomic domain. Let us recall the definition of a length function on an atomic domain R. A function $\varphi : R - \{0\} \to \mathbb{N}$ is called a *length function* on R if it satisfies the following two conditions:

(i) $\varphi(xy) = \varphi(x) + \varphi(y)$ for every $x, y \in R - \{0\}$.
(ii) $\varphi(x) = 0$ if and only if $x \in U(R)$.

For a length function φ on R, we set

$$M^* = M^*(R, \varphi) = \sup\{\varphi(x) : x \in R \text{ is irreducible but not prime}\},$$
$$m^* = m^*(R, \varphi) = \inf\{\varphi(x) : x \in R \text{ is irreducible but not prime}\},$$

with the convention $M^* = m^* = 1$ if R is a UFD. According to [1, Theorem 2.1], we have $1 \leq \rho(R) \leq M^*/m^*$. This double inequality has often been used to compute the elasticity of some polynomial rings [5, Lemma 2.3], [6, Lemma 2.3], [7, Theorems 4.3 & 4.4], [15, Theorem 3.11].

Finally, if T is a subring of R and φ is a length function on R, then the restriction of φ on T is a length function to T if and only if $U(R) \cap T = U(T)$ [1, Example 1].

Now, we will define a length function using the classification of polynomials of $B[X]$. More precisely, if f is a nonzero polynomial of $B[X]$, we can decompose f into irreducible factors of $B[X]$ of type α, β and γ, namely $f = \alpha^r \beta^s \gamma^t$, where r, s and t indicate, respectively, the number of irreducible factors of type α, β and γ. We can verify easily that the function $\varphi : B[X] - \{0\} \to \mathbb{N}$ defined by

(i) $\varphi(f) = 2r + s + t$,
(ii) $\varphi(u) = 0$ if and only if $u \in U(B)$,

is a length function on $B[X]$.

Since $U(B[X]) \cap \Re = U(B) \cap A = U(A) = U(\Re)$, the restriction φ' of φ to $\Re$ is also a length function on $\Re$. We can then consider the notations $M^* = M^*(\Re, \varphi')$ and $m^* = m^*(\Re, \varphi')$, as defined above. With these notations and hypotheses, we obtain the following.

Lemma 8. *Assume that $A \subset B$ is an integral extension such that B is a local UFD with maximal ideal $M \neq m$. Then $M^* \leq 3$ and $m^* = 2$.*

Proof. As shown in the proof of Theorem 5, if f is an irreducible polynomial in $\Re$, then f can be factorized in $B[X]$ as $f = \alpha^r \beta^s \gamma^t$. Moreover, we have the following possibilities:

- $r = 1$, $s = 0$ and $t \leq 1$,
- $r = 0$, $s = 1$ and $t = 0$,
- $r = 0$, $s = 0$ and $2 \leq t \leq 3$.

Hence, in any case, $\varphi'(f) = 2r + s + t \leq 3$. Thus $M^* \leq 3$.

Now, we will show that $m^* = 2$. The polynomial X is irreducible in $\Re$ of type α. But X is not prime in $\Re$ since X divides $\zeta^3 X = \zeta^2(\zeta X)$ while X divides neither ζ^2 nor ζX. Thus, $\varphi'(X) = 2$ and $m^* \leq 2$. For the reverse inequality, let f be an irreducible polynomial of $\Re$ such that $\varphi'(\Re) = 1$. Then f is necessarily of type β. As f is also irreducible in $B[X]$, f is a prime element of $B[X]$ since $B[X]$ is a UFD. From Lemma 7, f is prime in $\Re$. It follows that $m^* \geq 2$. ∎

Finally, by application of Theorem 5, Lemma 8 and the inequality $\rho(R) \leq M^*/m^*$, we deduce Theorem 6. ∎

From our study, we derive the following interesting characterization:

Corollary 9. *Let $A \subset B$ be a minimal extension of rings such A is local with maximal ideal m. Then $\Re$ is a HFD if and only if the following three conditions hold:*

(i) *$A \subset B$ is an integral extension.*
(ii) *$B[X]$ is a HFD.*
(iii) *B is local with maximal ideal m.*

We now present some examples to illustrate different cases of our study.

Example 10. Let $B = K[[Y]]$ be the ring of power series in one indeterminate Y over a field K and $A = K + Y^2K[[Y]]$. Then A is a local ring with integral closure B. We have $B/Y^2B \simeq K[Y]/(Y^2)$ and $A/Y^2B \simeq K$. If t is the coset $Y + (Y^2)$ of B/Y^2B, then B/Y^2B is a 2-dimensional vector space over K with basis $\{1, t\}$, where $t^2 = 0$. Thus $K \subset B/Y^2B$ is a minimal extension. It follows that $A \subset B$ is also a minimal integral extension. Set $\Re = A + XB[X] = K + Y^2K[[Y]] + XK[[Y]][X]$. Then $\rho(\Re) = 3/2$, by Theorem 6.

Example 11. Let $K \subset L$ be an extension of fields such that $[L : K]$ is a prime number and Y an indeterminate over K. Let $B = L[Y]_{(Y)} = L + (Y)L[Y]_{(Y)}$ and $A = K + (Y)L[Y]_{(Y)}$. As $K \subset L$ is a minimal extension, it follows that $A \subset B$ is also a minimal integral extension. Set $\Re = A + XB[X] = K + (Y)L[Y]_{(Y)} + XL[Y]_{(Y)}[X]$. Then $\rho(\Re) = \rho(B[X]) = 1$, by Theorem 4, so $\Re$ is a HFD.

Example 12. Let K be a field and Y an indeterminate over K. Consider two valuation domains: $V = K[Y]_{(Y)} = K + M_1$, where $M_1 = (Y)K[Y]_{(Y)}$, and $W = K[Y]_{(Y+1)} = K + N_1$, where $N_1 = (Y + 1)K[Y]_{(Y+1)}$. Then V and W are incomparable (for instance, if $V \subseteq W$, then $N_1 \subseteq M_1$, so $1 = (Y + 1) - Y \in M_1$, a contradiction). Set $B = V \cap W$. Then B is a PID with two maximal ideals $M = M_1 \cap B$ and $N = N_1 \cap B$ such that $B/M \simeq K$ and $B/N \simeq K$. Set $m = M \cap N$ and $A = K + m$. Then A is local with maximal ideal m. As $B/m \simeq K \times K$ is a 2-dimensional vector space over K, it follows that $A/m \subset B/m$ is a minimal extension. Hence $A \subset B$ is also a minimal integral extension. Set $\Re = A + XB[X] = K + M \cap N + X(V \cap W)[X]$. Then $\rho(\Re) = \infty$, by Theorem 3.

Example 13. Let $B = K[[Y]]$ be the ring of power series in one indeterminate Y over a field K and $A = K[[Y^2, Y^3]]$ the subring of B consisting of those power series with zero Y-term. Then A is a local ring with maximal ideal $m = Y^2A + Y^3A$. We claim that $A \subset B$ is a minimal extension. Indeed, if C is an intermediate ring between A and B, and $\delta = a_0 + a_1Y + a_2Y^2 + \cdots \in C \setminus A$, then $a_1 \neq 0$ and $Y = (1/a_1)(\delta - a_0 - a_2Y^2 - \cdots) \in C$, so $C = B$. Set $\Re = A + XB[X] = K[[Y^2, Y^3]] + XK[[Y]][X]$. Then $\rho(\Re) = 3/2$, by Theorem 6.

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Ahmed Ayache
Department of Mathematics
Faculty of Science
University of Sana'a
P.O. Box 12460, Sana'a, Yemen
E-mail: aayache@yahoo.com

Hanen Monceur
Department of Mathematics
Faculty of Science
University of Sfax
P.O. Box 1171, 3000 Sfax, Tunisia
E-mail: hanen.monceur@yahoo.fr

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