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ALMOST HERMITIAN SURFACES WITH VANISHING TRICERRI–VANHECKE BOCHNER CURVATURE TENSOR

BY

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Abstract. We study the curvature properties of almost Hermitian surfaces with vanishing Bochner curvature tensor as defined by Tricerri and Vanhecke. Local structure theorems for such almost Hermitian surfaces are provided, and several examples related to these theorems are given.

1. Introduction. In almost Hermitian geometry, it is both natural and interesting to discuss the relationship between the almost Hermitian structure and given curvature conditions. The Bochner curvature tensor B was defined by Bochner as a formal analogy of the Weyl conformal curvature tensor [4]. Bochner Kähler manifolds, which are Kähler manifolds with vanishing Bochner curvature tensor, have been studied by Kamishima [12], Bryant [5] and many other authors [7, 8, 20, 28]. Tricerri and Vanhecke [30] studied the decomposition of the space of all curvature tensors on a Hermitian vector space from the view-point of unitary representation theory and defined a Bochner type curvature tensor $B(R)$ for any almost Hermitian manifold $M = (M, J, g)$ by considering the induced decomposition of the Weyl component. We call $B(R)$ the Tricerri–Vanhecke (briefly, TV) Bochner curvature tensor. Further, we define a TV Bochner flat almost Hermitian manifold as an almost Hermitian manifold with vanishing TV Bochner curvature tensor. By an almost Hermitian surface, we mean a four-dimensional almost Hermitian manifold. We remark that the tensor field $B(R)$ is invariant under locally conformal changes of the Riemannian metric g and coincides with the Bochner tensor if M is Kähler. So, we may easily observe that every conformally flat almost Hermitian manifold and every Hermitian manifold which is locally conformally equivalent to a Bochner Kähler manifold are TV Bochner flat.

Tricerri and Vanhecke also defined a generalized complex space form as a generalization of complex space forms and proved that a generalized

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complex space form is characterized as an Einstein and weakly $*$-Einstein TV Bochner flat almost Hermitian manifold. Further, they proved that any $2n(\geq 6)$-dimensional generalized complex space form reduces to a complex space form. Olszak [24] studied four-dimensional generalized complex space forms and gave a positive answer to a question posed by Tricerri and Vanhecke ([30, p. 389]). Lemence also discussed four-dimensional generalized complex space forms and proved that a 4-dimensional generalized complex space form is a space of constant sectional curvature or a globally conformal Kähler manifold ([18, Theorem A]) which improves upon Olszak's result ([24, Theorem 2]).

On the other hand, Euh, Park and Sekigawa [9] classified the local structures of a TV Bochner flat nearly Kähler manifold. As a result, we see that a TV Bochner flat nearly Kähler manifold is Bochner Kähler or conformally flat. In the same paper, they also proved that a 6-dimensional TV Bochner flat almost Hermitian manifold of type $W_1 + W_4$ in the Gray–Hervella classification [11] is either globally conformally equivalent to a strict nearly Kähler manifold of positive constant sectional curvature or locally conformally equivalent to a Bochner Kähler manifold. As far as we know, there is no example of a TV Bochner flat almost Hermitian manifold which is neither conformally flat nor locally conformally equivalent to a Bochner Kähler manifold. Based on these observations, the following question naturally arises:

QUESTION. *Is every TV Bochner flat almost Hermitian manifold conformally flat or locally conformally equivalent to a Bochner Kähler manifold?*

In [13], Kamishima studied Bochner flat locally conformal Kähler manifolds. In this paper, we shall study the above question for almost Hermitian surfaces. The paper is organized as follows. In §2 and §3, we prepare several fundamental concepts and formulas necessary for the arguments of the present paper. In §4 and §5, we discuss TV Bochner flat almost Kähler surfaces and TV Bochner flat Hermitian surfaces, respectively, and give some partial answers to the Question under some additional conditions.

2. Preliminaries. Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold and Ω the Kähler form of M defined by $\Omega(X,Y) = g(JX,Y)$ for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. We denote by ∇ and R the Levi-Civita connection and the curvature tensor of (M, J, g) defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. Further, we denote by ρ, ρ^*, τ and τ^* the Ricci tensor, the Ricci $*$-tensor, the scalar curvature and the $*$-scalar curvature defined respectively by

$$\rho(X,Y) = \operatorname{tr}(Z \mapsto R(Z,X)Y), \qquad \rho^*(X,Y) = \operatorname{tr}(Z \mapsto R(X,JZ)JY),$$
$$\tau = \operatorname{tr} Q, \qquad \tau^* = \operatorname{tr} Q^*,$$

where Q and Q^* are the Ricci operator and the Ricci $*$-operator defined by $g(QX,Y) = \rho(X,Y)$ and $g(Q^*X,Y) = \rho^*(X,Y)$ for $X, Y \in \mathfrak{X}(M)$, respectively. We may easily check that $\rho^*(X,Y) = \rho^*(JY, JX)$ for all $X, Y \in \mathfrak{X}(M)$, and $\rho^* = \rho$ if M is a Kähler manifold. An almost Hermitian manifold M is called a *weakly $*$-Einstein* manifold if $\rho^* = \frac{\tau^*}{2n} g$ on M; it is called a *$*$-Einstein* manifold if τ^* is constant. We denote by N the Nijenhuis tensor of J defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

for $X, Y \in \mathfrak{X}(M)$. It is well-known that the almost complex structure J of M is integrable if and only if N vanishes identically on M [21]. We now recall the following conditions on the curvature tensor of an almost Hermitian manifold $M = (M, J, g)$:

$$(G1) \quad R(X,Y,Z,U) = R(X,Y,JZ,JU),$$
$$(G2) \quad R(X,Y,Z,U) - R(JX,JY,Z,U) = R(JX,Y,JZ,U) + R(JX,Y,Z,JU),$$
$$(G3) \quad R(X,Y,Z,U) = R(JX,JY,JZ,JU)$$

for $X, Y, Z, U \in \mathfrak{X}(M)$ [10]. Then we easily observe that $(G1) \Rightarrow (G2) \Rightarrow (G3)$. We note that if an almost Hermitian manifold M satisfies $(G3)$, then the Ricci tensor ρ and the Ricci $*$-tensor ρ^* are both J-invariant (and hence, in particular, ρ^* is symmetric).

An almost Hermitian manifold is called an *almost Kähler manifold* if the Kähler form Ω is closed (equivalently, $\underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X J)Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\underset{X,Y,Z}{\mathfrak{S}}$ denotes the cyclic sum over X, Y and Z, namely, (M, Ω) is a symplectic manifold). An almost Hermitian manifold is called a *Hermitian manifold* if the almost complex structure J is integrable. In particular, four-dimensional almost Kähler manifolds and four-dimensional Hermitian manifolds are called *almost Kähler surfaces* and *Hermitian surfaces* respectively. We now recall the definition of the *TV Bochner curvature tensor* $B(R)$ of a $2n$-dimensional almost Hermitian manifold $M = (M, J, g)$ [9, 30]:

$$
\begin{aligned}
B(R) = R &- \frac{1}{4(n+2)(n-2)}\, g \triangle \rho + \frac{2n-3}{4(n-1)(n-2)}\, g \owedge \rho \\
&- \frac{1}{4(n+2)(n-2)}\, g \triangle (\rho J) + \frac{1}{4(n-1)(n-2)}\, g \owedge (\rho J) \\
&+ \frac{2n^2-5}{4(n+1)(n+2)(n-2)}\, g \triangle \rho^* - \frac{2n-1}{4(n+1)(n-2)}\, g \owedge \rho^* \\
&+ \frac{3}{4(n+1)(n+2)(n-2)}\, g \triangle (\rho^* J) - \frac{3}{4(n+1)(n-2)}\, g \owedge (\rho^* J) \\
&+ \frac{3n\tau - (2n^2-3n+4)\tau^*}{16(n+1)(n+2)(n-1)(n-2)}\, g \triangle g - \frac{\tau - \tau^*}{8(n-1)(n-2)}\, g \owedge g
\end{aligned}
$$

for $n \geq 3$, and

$$
\begin{aligned}
B(R) = R &+ \frac{1}{2}\, g \owedge \rho + \frac{1}{12} \big\{ g \triangle \rho^* - g \owedge \rho^* - g \triangle (\rho^* J) \\
&+ g \owedge (\rho^* J) \big\} + \frac{3\tau^* - \tau}{96} g \triangle g - \frac{\tau + \tau^*}{16}\, g \owedge g
\end{aligned} \tag{2.1}
$$

for $n = 2$, where for any $(0,2)$-tensors a and b, we set

$$
\begin{aligned}
&(a \owedge b)(x,y,z,w) \\
&\quad = a(x,z)b(y,w) - a(x,w)b(y,z) + b(x,z)a(y,w) - b(x,w)a(y,z), \\
&\bar{a}(x,y) = a(x,Jy)
\end{aligned}
$$

for $x, y, z, w \in T_pM$, $p \in M$, and

$$
a \triangle b = a \owedge b + \bar{a} \owedge \bar{b} + 2\bar{a} \otimes \bar{b} + 2\bar{b} \otimes \bar{a}.
$$

Further, the Weyl curvature tensor is given by

$$
W = R + \frac{1}{2n-2}\, g \owedge \rho - \frac{\tau}{2(2n-1)(2n-2)}\, g \owedge g. \tag{2.2}
$$

Let $\{e_i\}$ be an orthonormal basis of T_pM at any point $p \in M$. In this paper, we shall adopt the following notational convention:

$$
\begin{gathered}
R_{ijkl} = g(R(e_i,e_j)e_k,e_l), \\
R_{\bar{i}jkl} = g(R(Je_i,e_j)e_k,e_l), \\
\cdots\cdots \\
R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i,Je_j)Je_k,Je_l), \\
\rho_{ij} = \rho(e_i,e_j), \quad \ldots, \quad \rho_{\bar{i}\bar{j}} = \rho(Je_i,Je_j), \\
\rho^*_{ij} = \rho^*(e_i,e_j), \quad \ldots, \quad \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i,Je_j), \\
J_{ij} = g(Je_i,e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i}J)e_j,e_k), \\
N_{ijk} = g(N(e_i,e_j),e_k),
\end{gathered}
$$

and so on, where the Latin indices run over the range $1, \ldots, 2n$. We set

$$G = \sum_{i,j} (\rho^*_{ij} - \rho^*_{ji})^2. \tag{2.3}$$

It is evident that $G = 0$ on M if and only if the Ricci $*$-tensor ρ^* is symmetric. Further, we may note that the symmetry of the Ricci $*$-tensor and the equality $(2n-1)\tau^* - \tau = 0$ are both preserved by any conformal change of the metric.

Now, we define 2-forms ϕ and ψ on M respectively by

$$\begin{aligned} \phi(X, Y) &= \mathrm{tr}(Z \mapsto J(\nabla_X J)(\nabla_Y J)Z), \\ \psi(X, Y) &= \mathrm{tr}(Z \mapsto R(X, Y)JZ) \end{aligned} \tag{2.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Then it is well-known that the first Chern form γ of M is given by

$$8\pi\gamma = -\phi + 2\psi, \tag{2.5}$$

and the 2-form γ represents the first Chern class $c_1(M)$ of M in the second de Rham cohomology group.

In the rest of this section, we review some fundamental equalities on an almost Hermitian surface $M = (M, J, g)$. In addition to the usual identities on an almost Hermitian manifold of arbitrary dimension, the following specific identities hold [11, 14]:

$$d\Omega = \omega \wedge \Omega \quad \text{(and hence, } d\omega \wedge \Omega = 0\text{)}, \tag{2.6}$$

where ω is the Lee form of M defined by $\omega = -\delta\Omega \circ J$,

$$\begin{aligned} 2g((\nabla_X J)Y, Z) &= \omega(Y)\Omega(Z, X) + \omega(Z)\Omega(X, Y) + \omega(JY)g(X, Z) \\ &\quad - \omega(JZ)g(X, Y) + g(N(Y, Z), JX), \end{aligned} \tag{2.7}$$

$$\rho^*(X, Y) + \rho^*(Y, X) - \{\rho(X, Y) + \rho(JX, JY)\} = \frac{\tau^* - \tau}{2} g(X, Y) \tag{2.8}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

It is well-known that the vector bundle $\Lambda^2 M$ of 2-forms on M decomposes as

$$\Lambda^2 M = \mathbb{R}\Omega \oplus \Lambda^0_{1,1} M \oplus LM,$$

where $\Lambda^0_{1,1} M$ and LM are the vector bundles of the primitive J-invariant and J-skew-invariant 2-forms on M, respectively. Further, we see that $\Lambda^2_+ M = \mathbb{R}\Omega \oplus LM$ and $\Lambda^2_- M = \Lambda^0_{1,1} M$ hold with respect to the orientation defined by the volume form $dv = \frac{1}{2}\Omega^2$. Since $g((\nabla_Z J)JX, JY) = -g((\nabla_Z J)X, Y)$ on M, we see that

$$\nabla\Omega = \alpha \otimes \Phi + \beta \otimes J\Phi \tag{2.9}$$

for some local 1-forms α and β, where $\{\Phi, J\Phi\}$ is a local basis for LM.

From (2.3), (2.4), (2.5), (2.8) and (2.9) we have the following integral formula on a compact almost Hermitian surface [27]:

$$c_1(M)^2 = \frac{1}{16\pi^2}\int_M \left\{ \frac{\tau^{*2}}{2} + \frac{1}{2}G + \frac{1}{2}|\rho - \rho\circ J|^2 - 2\left|\rho - \frac{\tau}{4}g\right|^2 + \frac{\tau^*}{2}\sum(\nabla_i J_{jk})\nabla_{\bar{i}}J_{\bar{j}\bar{k}} - \sum \rho^*_{\bar{j}i}(\nabla_i J_{ab})\nabla_{\bar{j}}J_{\bar{a}\bar{b}} \right\} dv, \tag{2.10}$$

where $(\rho\circ J)(X,Y) = \rho(JX,JY)$ for $X,Y \in \mathfrak{X}(M)$.

3. Some formulas on TV Bochner flat almost Hermitian surfaces. In this section, we shall discuss TV Bochner flat almost Hermitian surfaces and give some fundamental formulas for these surfaces. Let $M = (M,J,g)$ be a TV Bochner flat almost Hermitian surface. Then, by (2.1), the curvature tensor R of M can be expressed explicitly by

$$\begin{aligned} R(X,Y,Z,U) &= \frac{1}{2}\{g(X,U)\rho(Y,Z) + g(Y,Z)\rho(X,U) \\ &\quad - g(X,Z)\rho(Y,U) - g(Y,U)\rho(X,Z)\} \\ &\quad + \frac{1}{12}\{2g(X,JY)(\rho^*(U,JZ) - \rho^*(JZ,U)) \\ &\quad + 2g(Z,JU)(\rho^*(Y,JX) - \rho^*(JX,Y)) \\ &\quad + g(X,JZ)(\rho^*(U,JY) - \rho^*(JY,U)) \\ &\quad + g(Y,JU)(\rho^*(Z,JX) - \rho^*(JX,Z)) \\ &\quad + g(X,JU)(\rho^*(Y,JZ) - \rho^*(JZ,Y)) \\ &\quad + g(Y,JZ)(\rho^*(X,JU) - \rho^*(JU,X))\} \\ &\quad + \frac{3\tau^* - \tau}{48}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \\ &\quad - 2g(X,JY)g(Z,JU) - g(X,JZ)g(Y,JU) \\ &\quad + g(Y,JZ)g(X,JU)\} \\ &\quad - \frac{\tau + \tau^*}{8}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\} \end{aligned} \tag{3.1}$$

for $X,Y,Z,U \in \mathfrak{X}(M)$. On the other hand, from (2.2), the Weyl curvature tensor W is given by

$$
\begin{aligned}
(3.2)\qquad W(X,Y,Z,U) = R(X,Y,Z,U) &- \frac{1}{2}\{g(X,U)\rho(Y,Z) + g(Y,Z)\rho(X,U) \\
&\quad - g(X,Z)\rho(Y,U) - g(Y,U)\rho(X,Z)\} \\
&+ \frac{\tau}{6}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\}
\end{aligned}
$$

for $X,Y,Z,U \in \mathfrak{X}(M)$. From (3.1) and (3.2), the Weyl curvature tensor W is also expressed by

$$
\begin{aligned}
(3.3)\qquad & W(X,Y,Z,U) \\
&= \frac{\tau - 3\tau^*}{24}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\} \\
&+ \frac{1}{12}\{2g(X,JY)(\rho^*(U,JZ) - \rho^*(JZ,U)) \\
&\quad + 2g(Z,JU)(\rho^*(Y,JX) - \rho^*(JX,Y)) \\
&\quad + g(X,JZ)(\rho^*(U,JY) - \rho^*(JY,U)) \\
&\quad + g(Y,JU)(\rho^*(Z,JX) - \rho^*(JX,Z)) \\
&\quad + g(X,JU)(\rho^*(Y,JZ) - \rho^*(JZ,Y)) \\
&\quad + g(Y,JZ)(\rho^*(X,JU) - \rho^*(JU,X))\} \\
&+ \frac{3\tau^* - \tau}{48}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \\
&\quad - 2g(X,JY)g(Z,JU) - g(X,JZ)g(Y,JU) \\
&\quad + g(Y,JZ)g(X,JU)\}
\end{aligned}
$$

for $X,Y,Z,U \in \mathfrak{X}(M)$. First, from (3.1), we have the following theorem.

Theorem 3.1 ([9]). *Let $M = (M,J,g)$ be a TV Bochner flat almost Hermitian surface. Then the curvature tensor R satisfies the curvature identity*

$$
\begin{aligned}
(3.4)\qquad & R(X,Y,Z,U) - R(JX,JY,Z,U) \\
&\qquad - R(X,Y,JZ,JU) + R(JX,JY,JZ,JU) \\
&= R(X,JY,Z,JU) + R(X,JY,JZ,U) + R(JX,Y,JZ,U) + R(JX,Y,Z,JU)
\end{aligned}
$$

for $X,Y,Z,U \in \mathfrak{X}(M)$.

From Theorem 3.1, we immediately have the following result.

Corollary 3.2. *Let $M = (M, J, g)$ be a TV Bochner flat almost Hermitian surface satisfying the condition $(G3)$. Then M satisfies the condition $(G2)$.*

Remark 3.3. It is known that the curvature tensor of any Hermitian manifold satisfies the curvature identity (3.4) [10]. However, the converse is not true in general. In fact, Tricerri and Vanhecke [29] gave an example of a locally flat almost Hermitian surface which is not Hermitian.

The $(0,3)$-tensor field $C = (C_{ijk})$ defined by $C = \operatorname{div} W$ is called the *Cotton–York tensor*. From (3.2), we obtain immediately

$$C_{jkl} = \sum_i \nabla_i W_{ijkl} = \frac{1}{2}(\nabla_l \rho_{jk} - \nabla_k \rho_{jl}) - \frac{1}{12}\{(\nabla_l \tau) g_{jk} - (\nabla_k \tau) g_{jl}\}. \tag{3.5}$$

Let C_Z ($Z \in \mathfrak{X}(M)$) be the 2-form defined by the Cotton–York tensor $C_Z(X, Y) = C(Z, X, Y)$ for $X, Y \in \mathfrak{X}(M)$ [1]. From (3.5), we see immediately that if M is Ricci parallel, the Cotton–York tensor vanishes on M. We denote by C_Z^+ the self-dual part of C_Z. Further $C^+ = 0$ will mean that $C_Z^+ = 0$ for any $Z \in \mathfrak{X}(M)$. If $C^+ = 0$, then

$$\sum_{j,k} C_{ijk} N_{jkl} = 0, \qquad \sum_{j,k} C_{ijk} J_{jk} = 0. \tag{3.6}$$

From (2.7) and (3.3), by direct calculations, we have the following.

Theorem 3.4. *Let $M = (M, J, g)$ be a TV Bochner flat almost Hermitian surface. If the Ricci $*$-tensor ρ^* is symmetric, then*

$$\begin{aligned} 2C_{jik} = \frac{1}{24}\{&-\nabla_k(3\tau^* - \tau) g_{ji} + \nabla_i(3\tau^* - \tau) g_{jk} + 2\nabla_{\bar{j}}(3\tau^* - \tau) J_{ik} \\ &+ \nabla_{\bar{i}}(3\tau^* - \tau) J_{jk} - \nabla_{\bar{k}}(3\tau^* - \tau) J_{ji}\} \\ &+ \frac{3\tau^* - \tau}{24}\Big(3\omega_{\bar{j}} J_{ik} + \frac{3}{2}\omega_{\bar{i}} J_{jk} - \frac{3}{2}\omega_{\bar{k}} J_{ji} + \frac{3}{2}\omega_i g_{jk} - \frac{3}{2}\omega_k g_{ji} \\ &- N_{ikj} + \frac{1}{2} N_{jik} - \frac{1}{2} N_{jki}\Big). \end{aligned} \tag{3.7}$$

From (3.3), we have the following theorem.

Theorem 3.5 ([17]). *Let $M = (M, J, g)$ be a TV Bochner flat almost Hermitian surface. Then M is self-dual. Further, M is conformally flat if and only if ρ^* is symmetric and $3\tau^* - \tau = 0$ on M.*

We now recall the integral formulas representing the first Pontryagin number $p_1(M)$ and the Euler number $\chi(M)$ of a compact TV Bochner flat almost Hermitian surface $M = (M, J, g)$ [17].

Let $M = (M, J, g)$ be a compact TV Bochner flat almost Hermitian surface. Taking account of (2.3), the first Pontryagin number is given by

$$p_1(M) = \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{12} + G \right\} dv. \tag{3.8}$$

The Euler number χ is given by

$$\chi(M) = \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{24} - 2\left|\rho - \frac{\tau}{4}g\right|^2 + \frac{\tau^2}{6} + \frac{1}{2}G \right\} dv. \tag{3.9}$$

From (3.8) and (3.9), by Wu's theorem [31], the first Chern number is given by

$$\begin{aligned} c_1(M)^2 &= p_1(M) + 2\chi(M) \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{6} - 4\left|\rho - \frac{\tau}{4}g\right|^2 + \frac{\tau^2}{3} + 2G \right\} dv. \end{aligned} \tag{3.10}$$

From (2.10) and (3.10), we have

$$\begin{aligned} \int_M \Big\{ G + \frac{(\tau^* - \tau)^2}{2} - |\rho - \rho \circ J|^2 \\ - \tau^* \sum (\nabla_i J_{jk}) \nabla_{\bar{i}} J_{\bar{j}\bar{k}} + 2 \sum \rho^*_{\bar{j}i} (\nabla_i J_{ab}) \nabla_{\bar{j}} J_{a\bar{b}} \Big\} dv = 0. \end{aligned} \tag{3.11}$$

Based on (3.9), (3.10) and Theorem 3.5 we immediately have the following results.

THEOREM 3.6. *Let $M = (M, J, g)$ be a compact TV Bochner flat almost Hermitian surface. Then $c_1(M)^2 \geq 2\chi(M)$, with equality holding if and only if M is conformally flat.*

Now we comment on the geometric meaning of the function $3\tau^* + \tau$ on an almost Hermitian manifold. In [25], Sato proved that if $M = (M, J, g)$ is a $2n$-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature c, then $3\tau^* + \tau = 4n(n + 1)c$ on M. On the other hand, Koda proved that a self-dual Einstein almost Hermitian surface is a space of pointwise constant holomorphic sectional curvature ([15, Theorem A]). Thus, from the result of Koda and Theorem 3.5, it follows that any 4-dimensional generalized complex space form is an almost Hermitian surface of pointwise constant holomorphic sectional curvature $c = (3\tau^* + \tau)/24$. The following formula yields a geometric meaning of the function $\tau^* - \tau$ (or $\tau - \tau^*$) on an almost Hermitian surface $M = (M, J, g)$ [27]:

$$\tau - \tau^* = 2\delta\omega + |\omega|^2 - \frac{1}{8}|N|^2. \tag{3.12}$$

Next, we assume that $M = (M, J, g)$ is a compact TV Bochner flat almost Hermitian surface satisfying the condition $c_1(M)^2 \leq 3\chi(M)$ and $(3\tau^* + \tau)(\tau^* - \tau) \geq 0$ on M. Then, by (3.9) and (3.10), we get

$$(3.13)\qquad 0 \geq \int_M \left\{ \frac{G}{2} + \frac{(3\tau^* - \tau)^2}{24} + 2\left|\rho - \frac{\tau}{4}g\right|^2 - \frac{\tau^2}{6} \right\} dv$$
$$= \int_M \left\{ \frac{G}{2} + \frac{1}{8}(3\tau^* + \tau)(\tau^* - \tau) + 2\left|\rho - \frac{\tau}{4}g\right|^2 \right\} dv.$$

Thus, from (3.13) and the hypothesis, we have

$$(3.14)\qquad G = 0, \quad \rho = \frac{\tau}{4}g \quad \text{and} \quad (3\tau^* + \tau)(\tau^* - \tau) = 0$$

on M. Thus, taking account of (2.8) and (3.14), we see that M is Einstein and weakly $*$-Einstein, and hence a generalized complex space form [30]. Therefore, by the result of Lemence ([18, Theorem B]), we have the following theorem.

THEOREM 3.7. *Let $M = (M, J, g)$ be a compact TV Bochner flat almost Hermitian surface satisfying the condition $c_1(M)^2 \leq 3\chi(M)$. If $(3\tau^* + \tau)(\tau^* - \tau) \geq 0$ on M, then M is a compact locally flat non-Hermitian almost Hermitian surface or locally a compact complex space form.*

COROLLARY 3.8. *Let $M = (M, J, g)$ be a compact Bochner Kähler surface satisfying the condition $c_1(M)^2 \leq 3\chi(M)$. Then M is locally a complex space form (and hence $c_1(M)^2 = 3\chi(M)$).*

REMARK 3.9. The above result is obtained from the classification of compact Bochner Kähler manifolds established by Kamishima [12].

4. TV Bochner flat almost Kähler surfaces. First, we recall several fundamental formulas on an almost Kähler surface $M = (M, J, g)$:

$$(4.1)\qquad \nabla_{\bar{i}} J_{\bar{j}k} = -\nabla_i J_{jk},$$

$$(4.2)\qquad R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\sum_a (\nabla_a J_{ij})\nabla_a J_{kl}.$$

Then (4.2) leads to

$$(4.3)\qquad \rho^*_{ij} + \rho^*_{ji} - \rho_{ij} - \rho_{\bar{i}\bar{j}} = \sum_{a,b} (\nabla_a J_{ib})\nabla_a J_{jb}.$$

Thus from (2.8) and (4.3), we have

$$(4.4)\qquad \sum_{a,b} (\nabla_a J_{bi})\nabla_a J_{bj} = \frac{\tau^* - \tau}{2} g_{ij}.$$

From (4.3), we have

$$|\nabla J|^2 = 2(\tau^* - \tau). \tag{4.5}$$

On the other hand, it is well-known (see [32]) that the Nijenhuis tensor N of an almost Kähler manifold is expressed by

$$N_{ijk} = -2\nabla_{\bar{k}} J_{ij}. \tag{4.6}$$

Since $d\Omega = 0$ (and hence $\omega = 0$), taking account of (4.6) we may also obtain the formula (4.5) directly from (3.12).

The following is the arrangement of Kowalski's example ([16, Example III.53]) by Apostolov et al. [1]. We also refer to [22].

Example 4.1. Let $M = \mathbb{R}_4^+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 > 0\}$ and define the global basis $\{e_i\}$ by

$$e_1 = -x_1 \frac{\partial}{\partial x_1}, \quad e_2 = -\frac{1}{x_1}\frac{\partial}{\partial x_2},$$
$$e_3 = \sqrt{x_1}x_2 \frac{\partial}{\partial x_4} + \sqrt{x_1}\frac{\partial}{\partial x_3}, \quad e_4 = -\frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_4}.$$

Further, we define an almost Hermitian structure on M by $Je_1 = e_2$, $Je_2 = -e_1$, $Je_3 = e_4$, $Je_4 = -e_3$, and $g(e_i, e_j) = \delta_{ij}$ $(1 \le i, j \le 4)$. We denote the dual basis of $\{e_i\}$ by $\{e^i\}$. Then

$$e^1 = -\frac{1}{x_1}dx_1, \quad e^2 = -x_1 dx_2, \quad e^3 = \frac{1}{\sqrt{x_1}}dx_3, \quad e^4 = \sqrt{x_1}x_2 dx_3 - \sqrt{x_1}dx_4.$$

We see that the Kähler form of the almost Hermitian structure (J, g) is given by

$$\Omega = e^1 \wedge e^2 + e^3 \wedge e^4 = dx_1 \wedge dx_2 - dx_3 \wedge dx_4. \tag{4.7}$$

Hence $d\Omega = 0$, and so (M, J, g) is an almost Kähler surface. By direct calculation, we have

$$\begin{aligned} &R_{1212} = 1, && R_{1234} = -\tfrac{1}{2}, && R_{1313} = \tfrac{1}{4}, \\ &R_{1324} = -\tfrac{1}{4}, && R_{1414} = \tfrac{1}{4}, && R_{1423} = \tfrac{1}{4}, \\ &R_{2323} = \tfrac{1}{4}, && R_{2424} = \tfrac{1}{4}, && R_{3434} = -\tfrac{1}{2}, \end{aligned} \tag{4.8}$$

the other coefficients being zero. From (4.8), we have further

$$Qe_1 = -\tfrac{3}{2}e_1, \quad Qe_2 = -\tfrac{3}{2}e_2, \quad Qe_3 = Qe_4 = 0. \tag{4.9}$$

$$Q^*e_1 = -\tfrac{1}{2}e_1, \quad Q^*e_2 = -\tfrac{1}{2}e_2, \quad Q^*e_3 = e_3, \quad Q^*e_4 = e_4. \tag{4.10}$$

Then, from (4.9) and (4.10), we have immediately

$$\tau = -3, \quad \tau^* = 1. \tag{4.11}$$

From (4.5) and (4.11), we see that (M, J, g) is a strictly almost Kähler surface.

From (4.8), we may easily check that the almost Kähler surface (M, J, g) in Example 4.1 satisfies the condition $(G2)$. Apostolov et al. [1] proved that a strictly almost Kähler surface satisfying the condition $(G2)$ is locally homothetic to Kowalski's example. His example is a 4-dimensional Riemannian symmetric space of order 3 (known also as a Riemannian 3-symmetric space). It is well-known that a Riemannian 3-symmetric space gives rise to a quasi-Kähler manifold with respect to the canonical almost complex structure associated with the Riemannian 3-symmetric structure and also that every 4-dimensional quasi-Kähler manifold is necessarily an almost Kähler manifold (that is, an almost Kähler surface). Consequently, we have the following theorem.

THEOREM 4.2. *Let $M = (M, J, g)$ be a TV Bochner flat almost Kähler surface satisfying the condition $(G3)$. Then M is Bochner Kähler.*

Proof. First, from the hypothesis, by taking account of Corollary 3.2, we see that M satisfies the condition $(G2)$. We now suppose that M is strictly almost Kähler. Then, by the observation concerning Example 4.1, we see that M is locally homothetic to Kowalski's example. However, from (3.1), (4.8)–(4.11), we may easily check that M is not TV Bochner flat. This is a contradiction. Therefore, M is a Kähler surface (and hence a Bochner Kähler surface). ∎

From Theorem 4.2, we immediately have the following result which is a generalization of the result by Matsumoto and Tanno [19] and Derdziński [8] in dimension four.

COROLLARY 4.3. *Let $M = (M, J, g)$ be a TV Bochner flat almost Kähler surface satisfying the condition $(G3)$. If the scalar curvature τ of M is constant, then M is locally one of the following:*

(1) *M is a complex space form of complex dimension 2,*
(2) *M is locally a product of two oriented surfaces of different constant Gaussian curvatures K and $-K$ $(K > 0)$.*

EXAMPLE 4.4 ([23]). We set $(M, g) = \mathbb{H}^3(-1) \times \mathbb{R}$, where $\mathbb{H}^3(-1)$ is a 3-dimensional real hyperbolic space of constant sectional curvature -1 and $\mathbb{R}$ is the real line. Let

$$e_1 = x_1 \frac{\partial}{\partial x_1}, \qquad e_2 = x_1 \frac{\partial}{\partial x_2}, \qquad e_3 = x_1 \frac{\partial}{\partial x_3}, \qquad e_4 = \frac{\partial}{\partial x_4}.$$

on $M = \mathbb{R}^4_+ = \mathbb{R}^3_+ \times \mathbb{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 > 0\}$ and define an almost Hermitian structure (J, g) on M by $g(e_i, e_j) = \delta_{ij}$ and $Je_i = \sum_{j=1}^{4} J_{ij} e_j$, where

$$(J_{ij}) = \begin{pmatrix} 0 & \cos x_4 & \sin x_4 & 0 \\ -\cos x_4 & 0 & 0 & -\sin x_4 \\ -\sin x_4 & 0 & 0 & \cos x_4 \\ 0 & \sin x_4 & -\cos x_4 & 0 \end{pmatrix}.$$

We denote by $\{e^i\}_{i=1}^4$ the dual basis of $\{e_i\}$. Then the Kähler form Ω is given by

$$\begin{aligned}\Omega &= J_{12}e^1 \wedge e^2 + J_{13}e^1 \wedge e^3 + J_{14}e^1 \wedge e^4 \\ &\quad + J_{23}e^2 \wedge e^3 + J_{24}e^2 \wedge e^4 + J_{34}e^3 \wedge e^4 \\ &= \frac{1}{x_1^2}\cos x_4 dx_1 \wedge dx_2 + \frac{1}{x_1^2}\sin x_4 dx_1 \wedge dx_3 \\ &\quad - \frac{1}{x_1}\sin x_4 dx_2 \wedge dx_4 + \frac{1}{x_1}\cos x_4 dx_3 \wedge dx_4.\end{aligned}$$

Thus, we have $d\Omega = 0$, and hence (M, J, g) is an almost Kähler manifold.

We may easily check that Example 4.4 is a locally symmetric, conformally flat, TV Bochner flat, strictly almost Kähler surface with constant scalar curvature $\tau = -6$ and constant $*$-scalar curvature $\tau^* = -2$. We may also check that the Ricci tensor ρ of (M, J, g) is not J-invariant, and hence (M, J, g) does not satisfy the condition $(G3)$.

Remark 4.5. Catalano et al. [6] also gave an example of a conformally flat strictly almost Kähler surface with non-constant scalar curvature.

From Example 4.4, we see that the assumption of Theorem 4.2 (that a TV Bochner flat almost Kähler surface $M = (M, J, g)$ satisfies the condition $(G3)$) is essential. Furthermore, in Corollary 4.3, the assumption that the scalar curvature is constant cannot be removed. In fact, the following example by Tachibana and Liu [28] illustrates this situation.

Example 4.6. Let $M = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 < 1\}$ and $f = \sin^{-1} t$, $t = z_1\bar{z}_1 + z_2\bar{z}_2$, be the Kähler potential. Then we may check that the corresponding Kähler surface is Bochner flat and the scalar curvature τ is given by $\tau = -24t/\sqrt{1 - t^2}$.

Next, we shall prove the following theorem.

Theorem 4.7. *Let $M = (M, J, g)$ be a TV Bochner flat almost Kähler surface with the symmetric Ricci $*$-tensor ρ^*. If $C^+ = 0$, then M is conformally flat or Bochner Kähler.*

Proof. Since $\omega = 0$, by (4.6), the equality (3.7) reduces to

$$2C_{jik} = \frac{1}{24}\{-\nabla_k(3\tau^* - \tau)g_{ji} + \nabla_i(3\tau^* - \tau)g_{jk} + 2\nabla_{\bar{j}}(3\tau^* - \tau)J_{ik} + \nabla_{\bar{i}}(3\tau^* - \tau)J_{jk} - \nabla_{\bar{k}}(3\tau^* - \tau)J_{ji}\} + \frac{3\tau^* - \tau}{8}\nabla_{\bar{j}}J_{ik}. \tag{4.12}$$

Thus, from (3.6), (4.1), (4.5), and (4.12), we get

$$0 = 2\sum_{i,k} C_{jik}J_{ik} = \frac{1}{3}\nabla_{\bar{j}}(3\tau^* - \tau) \tag{4.13}$$

and

$$0 = 2\sum_{i,j,k} C_{jik}N_{ikj} = -4\sum_{i,j,k} C_{jik}\nabla_{\bar{j}}J_{ik} = -(3\tau^* - \tau)(\tau^* - \tau). \tag{4.14}$$

Thus, from (4.13), $3\tau^* - \tau$ is constant on M. Now, from (4.14), we see easily that $3\tau^* - \tau = 0$ on M or $3\tau^* - \tau \neq 0$ and $\tau^* - \tau = 0$ on M. Therefore, by Theorem 3.5, M is conformally flat or Bochner Kähler. ∎

Further, we shall prove the following theorem.

THEOREM 4.8. *Let $M = (M, J, g)$ be a compact TV Bochner flat almost Kähler surface satisfying the condition $c_1(M)^2 \leq 3\chi(M)$. If $C^+ = 0$ and $\tau \geq 0$ on M, then M is locally a Kähler surface of non-negative constant holomorphic sectional curvature.*

Proof. First, since $C^+ = 0$, from (3.5), (3.6), (4.1), and (4.6), we get

$$\sum(\nabla_k\rho_{ji} - \nabla_i\rho_{jk})J_{li}\nabla_jJ_{kl} = -\sum C_{jik}N_{ikj} = 0. \tag{4.15}$$

Now, we set

$$A \equiv \sum \rho_{ij}(\nabla_iJ_{kl})\nabla_jJ_{kl}.$$

Thus, taking account of (2.8), (4.15) and Green's Theorem, we have

$$\begin{aligned}
\int_M A\,dv &= \int_M \sum \rho_{ij}(\nabla_iJ_{kl})\nabla_jJ_{kl}\,dv \\
&= -\int_M \sum \rho_{ij}(\nabla_kJ_{li})\nabla_jJ_{kl}\,dv - \int_M \sum \rho_{ij}(\nabla_lJ_{ik})\nabla_jJ_{kl}\,dv \\
&= -2\int_M \sum \rho_{ij}(\nabla_kJ_{li})\nabla_jJ_{kl}\,dv \\
&= 2\int_M \sum(\nabla_k\rho_{ij})J_{li}\nabla_jJ_{kl}\,dv + 2\int_M \sum \rho_{ij}J_{li}\nabla_k\nabla_jJ_{kl}\,dv
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \sum \{(\nabla_k \rho_{ji} - \nabla_i \rho_{jk}) J_{li} \nabla_j J_{kl}\} \, dv \\
&\quad -2 \int_M \sum \rho_{ij} J_{li} (R_{kjkt} J_{tl} + R_{kjlt} J_{kt}) \, dv \\
&= -2 \int_M \sum {\rho_{ij}}^2 \, dv + \int_M \sum \rho_{ij} (\rho^*_{ij} + \rho^*_{ji}) \, dv \\
&= -\frac{1}{2} \int_M |\rho - \rho \circ J|^2 \, dv + \int_M \frac{\tau}{2} (\tau^* - \tau) \, dv,
\end{aligned}
$$

and hence

$$
\int_M \left\{ A + \frac{1}{2} |\rho - \rho \circ J|^2 \right\} dv = \int_M \frac{\tau}{2} (\tau^* - \tau) \, dv. \tag{4.16}
$$

On the other hand, from (2.8), (4.4) and (4.5) we see easily that the equality (3.11) reduces to

$$
\int_M \left\{ A + \frac{1}{2} |\rho - \rho \circ J|^2 \right\} dv = \int_M \left\{ \frac{1}{2} G + \frac{\tau^* - \tau}{4} (3\tau^* + \tau) \right\} dv. \tag{4.17}
$$

Thus, from (4.16) and (4.17), we have

$$
\int_M \left\{ \frac{1}{2} G + \frac{\tau^* - \tau}{4} (3\tau^* - \tau) \right\} dv = 0. \tag{4.18}
$$

Therefore, from the hypothesis of the theorem, (3.13) and (4.18), we have

$$
0 \geq \int_M \left\{ \frac{G}{4} + \frac{\tau}{4} (\tau^* - \tau) + 2 \left| \rho - \frac{\tau}{4} g \right|^2 \right\} dv \geq 0,
$$

and hence

$$
G = 0, \qquad \rho = \frac{\tau}{4} g \quad \text{and} \quad \tau(\tau^* - \tau) = 0 \tag{4.19}
$$

on M. Thus, from (4.19), we see that M is an Einstein and weakly $*$-Einstein manifold, and hence a generalized complex space form of dimension four. Therefore, from the results of [18, Theorem A] and [26], the assertion of the theorem follows immediately. ∎

From Theorem 3.7, Remark 3.9 and (4.5), we immediately deduce the following theorem.

THEOREM 4.9. *Let $M = (M, J, g)$ be a compact TV Bochner flat almost Kähler surface with pointwise non-negative constant holomorphic sectional curvature. If M satisfies the condition $c_1(M)^2 \leq 3\chi(M)$, then M is locally a complex space form of non-negative constant holomorphic sectional curvature.*

5. TV Bochner flat Hermitian surfaces. Let $M = (M, J, g)$ be a Hermitian surface. Then, from (2.7), we have

$$
\begin{aligned}
(5.1)\quad & R_{ijkl} - R_{ij\bar{k}\bar{l}} \\
&= \frac{1}{2}\Big\{ g_{jl}\Big(\nabla_i\omega_k + \frac{1}{2}\omega_i\omega_k - \frac{1}{2}|\omega|^2 g_{ik}\Big) - g_{il}\Big(\nabla_j\omega_k + \frac{1}{2}\omega_i\omega_k - \frac{1}{2}|\omega|^2 g_{jk}\Big) \\
&\quad + J_{kj}\Big(\nabla_i\omega_{\bar{l}} + \frac{1}{2}\omega_i\omega_{\bar{l}} - \frac{1}{2}|\omega|^2 J_{li}\Big) - J_{ki}\Big(\nabla_j\omega_{\bar{l}} + \frac{1}{2}\omega_j\omega_{\bar{l}} - \frac{1}{2}|\omega|^2 J_{lj}\Big) \\
&\quad - J_{lj}\Big(\nabla_i\omega_{\bar{k}} + \frac{1}{2}\omega_i\omega_{\bar{k}}\Big) + J_{li}\Big(\nabla_j\omega_{\bar{k}} + \frac{1}{2}\omega_j\omega_{\bar{k}}\Big) \\
&\quad - g_{jk}\Big(\nabla_i\omega_l + \frac{1}{2}\omega_i\omega_l\Big) + g_{ik}\Big(\nabla_j\omega_l + \frac{1}{2}\omega_j\omega_l\Big)\Big\}.
\end{aligned}
$$

From (5.1), we have

$$
\begin{aligned}
(5.2)\quad & \rho_{jk} - \rho^*_{jk} \\
&= \frac{1}{2}\Big\{ -\nabla_j\omega_k + \nabla_{\bar{j}}\omega_{\bar{k}} - \frac{1}{2}\omega_j\omega_k + \frac{1}{2}\omega_{\bar{j}}\omega_{\bar{k}} + \frac{1}{2}(2\delta\omega + |\omega|^2)g_{jk}\Big\}.
\end{aligned}
$$

From (5.2), we obtain

$$
(5.3)\qquad \tau - \tau^* = 2\delta\omega + |\omega|^2
$$

and

$$
(5.4)\qquad \nabla_l\omega_k - \nabla_k\omega_l - \nabla_{\bar{l}}\omega_{\bar{k}} + \nabla_{\bar{k}}\omega_{\bar{l}} + 2(\rho^*_{kl} - \rho^*_{lk}) = 0
$$

Here, since $N = 0$, we can also obtain the formula (5.3) from (3.12).

We assume that ρ^* is symmetric. Since $\sum_j \nabla_j\omega_{\bar{j}} = 0$ by (2.6), from (5.4) we see that $(d\omega)^+ = 0$, where $(d\omega)^+$ denotes the self-dual part of the 2-form $d\omega$. Thus, we have the following theorem [3].

THEOREM 5.1. *Let $M = (M, J, g)$ be a Hermitian surface. Then ρ^* is symmetric if and only if $(d\omega)^+ = 0$ on M.*

Now, let $M = (M, J, g)$ be a TV Bochner flat Hermitian surface with symmetric Ricci $*$-tensor ρ^* (equivalently, with the Lee form ω satisfying $d\omega^+ = 0$). Then the equality (3.7) reduces to

$$
\begin{aligned}
(5.5)\quad 2C_{jik} = \frac{1}{24}\{ & -\nabla_k(3\tau^* - \tau)g_{ji} + \nabla_i(3\tau^* - \tau)g_{jk} \\
& + 2\nabla_{\bar{j}}(3\tau^* - \tau)J_{ik} + \nabla_{\bar{i}}(3\tau^* - \tau)J_{jk} - \nabla_{\bar{k}}(3\tau^* - \tau)J_{ji}\} \\
& + \frac{3\tau^* - \tau}{24}\Big(3\omega_{\bar{j}}J_{ik} + \frac{3}{2}\omega_{\bar{i}}J_{jk} - \frac{3}{2}\omega_{\bar{k}}J_{ji} + \frac{3}{2}\omega_i g_{jk} - \frac{3}{2}\omega_k g_{ji}\Big).
\end{aligned}
$$

We assume that $C^+ = 0$. Then, from (3.6) and (5.5), we have

$$0 = 2\sum_{i,k} C_{jik}J_{ik} = \frac{1}{3}\nabla_{\bar{j}}(3\tau^* - \tau) + \frac{1}{2}(3\tau^* - \tau)\omega_{\bar{j}},$$

and hence

$$d(3\tau^* - \tau) = -\frac{3}{2}(3\tau^* - \tau)\omega. \tag{5.6}$$

Thus, from (5.6), we see immediately that the function $3\tau^* - \tau$ vanishes either everywhere or nowhere on M. If it vanishes everywhere, M becomes conformally flat. Now, assume that $3\tau^* - \tau$ vanishes nowhere on M. Then, taking the exterior derivative of (5.6), we obtain $d\omega = 0$, and hence M is a locally conformal Kähler surface. Now, we define the Riemannian metric $\widetilde{g} = (3\tau^* - \tau)^{2/3} g$ and denote by $\widetilde{\Omega}$ the Kähler form of the almost Hermitian structure $(J, \widetilde{g})$. Then

$$\widetilde{\Omega} = (3\tau^* - \tau)^{2/3}\Omega. \tag{5.7}$$

Taking the exterior derivative of (5.7) and making use of (2.6) and (5.6), we obtain

$$\begin{aligned} d\widetilde{\Omega} &= (d(3\tau^* - \tau)^{2/3}) \wedge \Omega + (3\tau^* - \tau)^{2/3}\omega \wedge \Omega \\ &= \frac{1}{(3\tau^* - \tau)^{1/3}}\left\{\frac{2}{3}\ d(3\tau^* - \tau) + (3\tau^* - \tau)\omega\right\} \wedge \Omega = 0. \end{aligned} \tag{5.8}$$

Thus, $(M, J, \widetilde{g})$ is a Kähler surface, and hence (M, J, g) is a globally conformal Kähler surface. This yields the following generalization of a result by Lemence ([18, Theorem A]).

THEOREM 5.2. *Let $M = (M, J, g)$ be a TV Bochner flat Hermitian surface with symmetric Ricci $*$-tensor ρ^*. If $C^+ = 0$ on M, then M is conformally flat or conformal to a Bochner Kähler surface.*

Theorem 5.2 and (5.6) have the following immediate corollary.

COROLLARY 5.3. *Let $M = (M, J, g)$ be a TV Bochner flat Hermitian surface with symmetric Ricci $*$-tensor ρ^*. If $C^+ = 0$ and $3\tau^* - \tau$ is constant, then M is conformally flat or Bochner Kähler.*

Now, we give a concrete example of a 4-dimensional generalized complex space form illustrating Corollary 5.3 which is obtained by applying a result of Olszak ([24, Theorem 1]) to Example 4.6.

Let $M = (M, J, g)$ be the Bochner Kähler surface given by Example 4.6. We set $\widetilde{M} = M \setminus \{(0, 0)\}$ and define the Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ by $\widetilde{g} = e^{-\frac{1}{2}\log\tau^2} g$, where τ $(= -24t/\sqrt{1 - t^2})$ is the scalar curvature of M. Here, we denote the restriction of the given Hermitian structure (J, g) to $\widetilde{M}$ by using the same letter. Then, from the result of Olszak ([24, Theorem 1]), we see that $\widetilde{M} = (\widetilde{M}, J, \widetilde{g})$ is a generalized complex space form. By straightforward calculation, we have easily

$$\widetilde{\tau} = 0, \quad \widetilde{\tau}^* = \frac{2}{3}\tau^3, \tag{5.9}$$

where $\widetilde{\tau}$ and $\widetilde{\tau}^*$ are the scalar curvature and the $*$-scalar curvature of $\widetilde{M}$, respectively. From (5.9), we see immediately that $3\widetilde{\tau}^* - \widetilde{\tau} = 2\tau^3$ is not constant and vanishes nowhere on $\widetilde{M}$. Further, we see that $\widetilde{M}$ is not conformally equivalent to any 4-dimensional complex space form. In fact, if there exists a 4-dimensional complex space form which is conformally equivalent to $\widetilde{M}$, it is homothetic to the Bochner Kähler surface $(\widetilde{M}, J, g)$, and hence the scalar curvature of $(\widetilde{M}, J, g)$, which is nothing but the restriction of the scalar curvature τ of M to $\widetilde{M}$, must be constant. But this is a contradiction.

We recall the following result of Apostolov et al. [2].

Theorem 5.4. *Any compact self-dual Hermitian surface $M = (M, J, g)$ which is not conformally flat is conformally equivalent either to $\mathbb{C}P^2$ with the Fubini–Study metric or to a compact quotient of the unit ball in $\mathbb{C}^2$ with the Bergman metric.*

Since a TV Bochner flat almost Hermitian surface is self-dual by Theorem 3.5, Theorem 5.4 provides an answer to the Question in §1 for any compact TV Bochner flat Hermitian surface.

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