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FINITE MUTATION CLASSES OF COLOURED QUIVERS

BY

HERMUND ANDRÉ TORKILDSEN (Trondheim)

Abstract. We show that the mutation class of a coloured quiver arising from an m-cluster tilting object associated with a finite-dimensional hereditary algebra H, is finite if and only if H is of finite or tame representation type, or it has at most two simples. This generalizes a result known for cluster categories.

Introduction. Mutation of skew-symmetric matrices, or equivalently quiver mutation, is central in the topic of cluster algebras [FZ]. Quiver mutation induces an equivalence relation on the set of quivers. The *mutation class* of a quiver Q is the set of quivers mutation equivalent to Q. In [BR] the authors show that the mutation class of an acyclic quiver Q is finite if and only if the underlying graph of Q is either Dynkin, extended Dynkin or has at most two vertices. In [BT] mutation on coloured quivers was defined. It is natural to ask when the mutation classes of coloured quivers are finite, and in this paper we will prove the result analogous to the main theorem in [BR].

Let $H = kQ$ be a finite-dimensional hereditary algebra over an algebraically closed field k, with Q a quiver with n vertices. The cluster category was defined in [BMRRT], and independently in [CCS] in the A_n case. Consider the bounded derived category $\mathcal{D}^b(H)$ of $\operatorname{mod} H$. Then the *cluster category* is defined to be the orbit category $\mathcal{C}_H = \mathcal{D}^b(H)/\tau^{-1}[1]$, where τ is the AR-translation and [1] is the shift functor. As a generalization we can consider the *m-cluster categories* defined by $\mathcal{C}_H^m = \mathcal{D}^b(H)/\tau^{-1}[m]$. These categories have been investigated by several authors: see for example [BT, IY, K, T, W, ZZ, Z]. They are triangulated [K], Krull–Schmidt, $(m+1)$-Calabi–Yau and have an AR-translate $\tau = [m]$. Up to isomorphism, the indecomposable objects are of the form $X[i]$, with $0 \leq i < m$, where X is an indecomposable H-module, and of the form $P[m]$, where P is a projective H-module.

An *m-cluster tilting object* is an object T in $\mathcal{C}_H^m$ with the property that X is in $\operatorname{add} T$ if and only if $\operatorname{Ext}^i_{\mathcal{C}_H^m}(T, X) = 0$ for all $i \in \{1, \dots, m\}$. It was shown

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in [W, ZZ] that a *maximal m-rigid object*, i.e. one which has the property that $X \in \operatorname{add} T$ if and only if $\operatorname{Ext}^i_{\mathcal{C}^m_H}(T \oplus X, T \oplus X) = 0$ for all $i \in \{1, \ldots, m\}$, is an m-cluster tilting object. We also know that an m-cluster tilting object always has n non-isomorphic indecomposable summands [Z].

An *almost complete m-cluster tilting object* $\bar{T}$ is an object with $n-1$ non-isomorphic indecomposable direct summands such that $\operatorname{Ext}^i_{\mathcal{C}^m_H}(\bar{T}, \bar{T}) = 0$ for $i \in \{1, \ldots, m\}$. From [W, ZZ] we know that any almost complete m-cluster tilting object has exactly $m+1$ complements, i.e. there exist $m+1$ non-isomorphic indecomposable objects T' such that $\bar{T} \oplus T'$ is m-cluster tilting. We denote by $T_k^{(c)}$, where $c \in \{0, 1, \ldots, m\}$, the complements of $\bar{T}$. In [IY] it is shown that the complements are connected by $m+1$ exchange triangles, $T_k^{(c)} \to B_k^{(c)} \to T_k^{(c+1)} \to$, where $B_k^{(c)}$ are in $\operatorname{add} \bar{T}$.

An *m-cluster tilted algebra* is an algebra of the form $\operatorname{End}_{\mathcal{C}^m_H}(T)$, where T is an m-cluster tilting object in $\mathcal{C}^m_H$.

1. Coloured quiver mutation. For $m = 1$ there is a well-known procedure for the exchange of indecomposable direct summands of a cluster tilting object. Given an almost complete cluster tilting object, there are exactly two complements, and the corresponding quivers are given by quiver mutation. For $m \geq 1$, the procedure is more complicated. An almost complete m-cluster tilting object has exactly $m+1$ complements, and the quiver does not give enough information to keep track of the exchange. Buan and Thomas defined a class of coloured quivers in [BT] in order to deal with this.

To an m-cluster tilting object T, they associate a coloured quiver $Q = Q_T$ with arrows of colours chosen from the set $\{0, 1, \ldots, m\}$. For each indecomposable summand T_i of T there is a vertex in Q_T. If T_i and T_j are two summands, there are r arrows from i to j of colour c, where r is the multiplicity of T_j in $B_i^{(c)}$. Buan and Thomas show that such quivers have no loops, all arrows from i to j are of the same colour, and if there are r arrows from i to j of colour c, then there are r arrows from j to i of colour $m - c$. Let j be a vertex in Q. The *mutation of Q at vertex j* is a quiver $\mu_j(Q)$ obtained as follows:

(1) For each pair of arrows $i \xrightarrow{(c)} j \xrightarrow{(0)} k$ where $i \neq k$ and $c \in \{0, 1, \ldots, m\}$, add an arrow from i to k of colour c and an arrow from k to i of colour $m - c$.
(2) If there exist arrows of different colours from a vertex i to a vertex k, cancel the same number of arrows of each colour until there are only arrows of the same colour from i to k.
(3) Add one to the colour of all arrows that go into j, and subtract one from the colour of all arrows going out of j.

Let $T = \bigoplus_{i=1}^{n} T_i$ be an m-cluster tilting object in $\mathcal{C}_H^m$, and let $T' = T/T_j \oplus T_j^{(1)}$ be an m-cluster tilting object where there is an exchange triangle $T_j \to B_j^{(0)} \to T_j^{(1)} \to$. Then it was shown in [BT] that $Q_{T'} = \mu_j(Q_T)$.

The quiver obtained from $Q = Q_T$ by removing all arrows of colour $c \neq 0$ is the Gabriel quiver of $\mathrm{End}_{\mathcal{C}_H^m}(T)$. We denote it by Q_G. Any m-cluster tilting object can be reached from any other m-cluster tilting object via iterated mutation [ZZ, BT], hence all quivers of m-cluster tilted algebras are given by repeated mutation of Q.

Let Q_G be an acyclic quiver and Q the coloured quiver obtained from Q_G by adding the necessary arrows of colour m, i.e. if there are r arrows from i to j of colour 0, add r arrows from j to i of colour m. Then the quivers arising from m-cluster tilting objects are exactly the quivers mutation equivalent to Q.

The proof of the following proposition is left to the reader.

PROPOSITION 1.1. *If T is an m-cluster tilting object, then Q_T is isomorphic to $Q_{T[k]}$ for all k. In particular Q_T is isomorphic to $Q_{\tau T}$.*

2. Finiteness of the number of non-isomorphic m-cluster tilted algebras. Let Q be a finite quiver without oriented cycles. Then the path algebra $H = kQ$ is of finite representation type if and only if the underlying graph of Q is Dynkin, and H is tame if and only if Q is extended Dynkin. Objects in $\operatorname{mod} H$, when H is of infinite type, are either preprojective, preinjective or regular (see [ASS, SS1, SS2]).

From [W] we know that if T is m-cluster tilting in $\mathcal{C}_H^m$, then it is induced from a tilting object in $\operatorname{mod} H_0 \vee \operatorname{mod} H_0[1] \vee \cdots \vee \operatorname{mod} H_0[m-1]$, where H_0 is derived equivalent to H. If H is finite or tame, then it was shown in [BR] that for any indecomposable projective H-module P, there are only a finite number of indecomposable objects X with $\mathrm{Ext}^1_{\mathcal{C}_H^1}(X, P) = 0$.

LEMMA 2.1. *Let $P[i]$ be a shift of an indecomposable projective H-module, where H is of finite or tame representation type. Then there is only a finite set of objects X in $\mathcal{C}_H^m$ with $\mathrm{Ext}^k_{\mathcal{C}_H^m}(X, P[i]) = 0$ for all $k \in \{1, \ldots, m\}$.*

Proof. We can assume that an m-cluster tilting object is induced from a tilting object in $\operatorname{mod} H \vee \operatorname{mod} H[1] \vee \cdots \vee \operatorname{mod} H[m-1]$. It follows from [BR] that there are only a finite number of indecomposable objects X lying inside $\operatorname{mod} H[i]$ with $\mathrm{Ext}^1_{\mathcal{C}_H^m}(X, P[i]) = 0$ for all i. We have $\mathrm{Ext}^{j+1}_{\mathcal{C}_H^m}(X, P) = \mathrm{Ext}^1_{\mathcal{C}_H^m}(X, P[j])$, so there are only finitely many indecomposable objects X in $\operatorname{mod} H[j]$ such that $\mathrm{Ext}^{j+1}_{\mathcal{C}_H^m}(X, P) = 0$. Consequently, there are only a finite number of indecomposable objects X such that $\mathrm{Ext}^k_{\mathcal{C}_H^m}(X, P) = 0$ for all $k \in \{1, \ldots, m\}$. ∎

It is known from [BKL] that in the tame case, a collection of one or more tubes is triangulated.

Proposition 2.2. *Let H be of tame representation type and $X \to Y \to Z \to$ be a triangle in $\mathcal{C}_H^m$, where two of the terms are shifts of regular modules. Then all terms are shifts of regular modules.*

Proposition 2.3. *Let H be of tame representation type, and let T be an m-cluster tilting object in $\mathcal{C}_H^m$. Then T has, up to τ, at least one direct summand which is a shift of a projective or an injective.*

Proof. We show that there are no m-cluster tilting objects in $\mathcal{C}_H^m$ with only shifts of regular H-modules as direct summands. Suppose for a contradiction that such a T exists. If all direct summands of T are of the same degree, we already have a contradiction, since a tilting module has at least one direct summand which is preprojective or preinjective (see [R]).

Assume T' is a direct summand of degree $k \leq m$. Let $\bar{T} = T/T'$. Then the complements of $\bar{T}$ are connected by $m+1$ exchange triangles,

$$M_{i+1} \to X_i \to M_i \to,$$

where $i \in \{0, 1, \ldots, m\}$ and $X_i \in \operatorname{add} \bar{T}$. The direct summands of X_i are by assumption shifts of regular modules. Since T' is a complement of $\bar{T}$, it is equal to M_j for some j. Hence M_i is a shift of a regular module for all i by Proposition 2.2, since T' is a shift of a regular. So all m-cluster tilting objects that can be reached from T by a finite number of mutations have only regular direct summands. This is a contradiction, because all m-cluster tilting objects can be reached from T by a finite number of mutations, and a tilting module in H induces an m-cluster tilting object in $\mathcal{C}_H^m$ with at least one direct summand preprojective or preinjective. ∎

We have the following easily observed fact.

Lemma 2.4. *If a path in the quiver of an m-cluster tilted algebra goes through two oriented cycles, then it is zero.*

The main theorem in this paper generalizes the main theorem in [BR].

Theorem 2.5. *Let k be an algebraically closed field and Q a connected finite quiver without oriented cycles. The following conditions are equivalent for $H = kQ$:*

1. *There are only a finite number of basic m-cluster tilted algebras associated with H, up to isomorphism.*
2. *There are only a finite number of Gabriel quivers occurring for m-cluster tilted algebras associated with H, up to isomorphism.*
3. *H is of finite or tame representation type, or has at most two non-isomorphic simple modules.*

(4) *There are only a finite number of τ-orbits of m-cluster tilting objects associated with H.*
(5) *There are only a finite number of coloured quivers occurring for m-cluster tilting objects associated with H, up to isomorphism.*
(6) *The mutation class of a coloured quiver arising from an m-cluster tilting object associated with H, is finite.*

Proof. (1)⇒(2) and (4)⇒(5) are clear.

(2)⇒(3). Suppose there are only a finite number of quivers occurring for m-cluster tilted algebras associated with H, and let u be the maximal number of arrows between two vertices in the quiver. Then by Lemma 2.4, for any two indecomposable summands T_1 and T_2 of an m-cluster tilting object T, we have $\dim \mathrm{Hom}_{\mathcal{C}_H^m}(T_1, T_2) < u^{2n}$, where n is the number of simple H-modules. Then it follows from [BR, Lemma 3.4] that H is not wild with more than three simples.

(3)⇒(4). If H is of finite representation type, this is clear.

Suppose H has at most two non-isomorphic simple modules. If there is only one simple module we have $H \simeq k$, so we can assume there are two simples. If R is a regular indecomposable H-module, then it is known (see for example [R]) that $\mathrm{Ext}^1_{\mathcal{C}_H^m}(R, R) \neq 0$. Then also $\mathrm{Ext}^1_{\mathcal{C}_H^m}(R[i], R[i]) \neq 0$ for any $i \in \{1, \ldots, m-1\}$. Up to τ in $\mathcal{C}_H^m$ we can assume that an m-cluster tilting object has a direct summand which is a shift of a projective H-module, say $P[j]$. Then $P[j]$ has $m+1$ indecomposable complements, so there are only a finite number of m-cluster tilting objects up to τ, since there are only a finite number of choices for $P[j]$.

Suppose H is tame. By Proposition 2.3, an m-cluster tilting object has at least one direct summand which is a shift of a projective or an injective, and hence up to τ we can assume it has an indecomposable direct summand which is a shift of a projective. From Lemma 2.1 there are only a finite number of m-cluster tilting objects with a shift of an indecomposable projective H-module as a direct summand.

(5)⇒(6) is clear, since mutation of m-cluster tilting objects corresponds to mutation of coloured quivers. We find that (4)⇒(1) by using Lemma 1.1. (6)⇒(2) is trivial, and so we are done. ∎

Corollary 2.6. *A coloured quiver Q corresponding to an m-cluster tilting object has finite mutation class if and only if Q is mutation equivalent to a quiver Q', with arrows only of colour* 0 *and* m, *and where Q'_G has underlying graph Dynkin or extended Dynkin, or Q has at most two vertices.*

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Hermund André Torkildsen
Department of Mathematical Sciences
Norwegian University of Science and Technology
7491 Trondheim, Norway
E-mail: hermunda@math.ntnu.no

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