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FRACTIONAL HARDY INEQUALITY WITH A REMAINDER TERM

BY

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Abstract. We prove a Hardy inequality for the fractional Laplacian on the interval with the optimal constant and additional lower order term. As a consequence, we also obtain a fractional Hardy inequality with the best constant and an extra lower order term for general domains, following the method of M. Loss and C. Sloane [J. Funct. Anal. 259 (2010)].

1. Main result and discussion. Recently Loss and Sloane [16] have proved the following fractional Hardy inequality:

$$\frac{1}{2}\int_{D\times D}\frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy \ge \kappa_{n,\alpha}\int_D \frac{u(x)^2}{\operatorname{dist}(x,D^c)^\alpha}\,dx, \qquad u\in C_c(D), \tag{1.1}$$

for convex domains $D\subset\mathbb{R}^n$ and $1<\alpha<2$. Here

$$\kappa_{n,\alpha} = \pi^{(n-1)/2}\frac{\Gamma\big(\frac{1+\alpha}{2}\big)}{\Gamma\big(\frac{n+\alpha}{2}\big)}\,\frac{B\big(\frac{1+\alpha}{2},\frac{2-\alpha}{2}\big)-2^\alpha}{\alpha 2^\alpha} \tag{1.2}$$

is the optimal constant, B is the Euler beta function, and $C_c(D)$ denotes the class of all continuous functions $u\colon\mathbb{R}^n\to\mathbb{R}$ with compact support in D. Inequality (1.1) with the optimal constant was earlier obtained for half-spaces and $\mathbb{R}^n\setminus\{0\}$ (see [9, 11, 5, 10]). In this note we will prove the following strengthening of (1.1) for the interval.

THEOREM 1.1. *Let $1<\alpha<2$ and $-\infty<a<b<\infty$. For every $u\in C_c(a,b)$,*

$$\frac{1}{2}\int_a^b\!\!\int_a^b\frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy \ge \kappa_{1,\alpha}\int_a^b u(x)^2\Big(\frac{1}{x-a}+\frac{1}{b-x}\Big)^{\alpha}dx + \frac{4-2^{3-\alpha}}{\alpha(b-a)}\int_a^b u(x)^2\Big(\frac{1}{x-a}+\frac{1}{b-x}\Big)^{\alpha-1}dx, \tag{1.3}$$

and $\kappa_{1,\alpha}$ cannot be replaced by a larger constant in (1.3).

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For an open set $D \subset \mathbb{R}^n$ we consider the quadratic form

$$\mathcal{E}(u) = \frac{1}{2} \int_{D\times D} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\, dx\, dy, \qquad u \in C_c(D).$$

The method developed by Loss and Sloane in [16] and Theorem 1.1 yield a fractional Hardy inequality with a *remainder* for general domains, stated as Theorem 1.2 below. In the statement we use the following notation from [16, 7]. Let D be bounded. For a direction $w = (w_1, w_2, \ldots, w_n) \in S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ and $x \in D$ we define $d_{w,D}(x) = \min\{|t| : x + tw \notin D\}$, $\delta_{w,D}(x) = \sup\{|t| : x + tw \in D\}$ and

(1.4)
$$\frac{1}{M_\alpha(x)^\alpha} = \frac{\int_{S^{n-1}} \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^\alpha dw}{\int_{S^{n-1}} |w_n|^\alpha\, dw} = \frac{\int_{S^{n-1}} \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^\alpha dw}{2\kappa_{n,\alpha}/\kappa_{1,\alpha}}.$$

THEOREM 1.2. *Let $1 < \alpha < 2$ and let $D \subset \mathbb{R}^n$ be a bounded domain. Then*

(1.5)
$$\mathcal{E}(u) \ge \kappa_{n,\alpha} \int_D \frac{u(x)^2}{M_\alpha(x)^\alpha}\, dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D} \int_D \frac{u(x)^2}{M_{\alpha-1}(x)^{\alpha-1}}\, dx, \qquad u \in C_c(D),$$

where $\lambda_{n,\alpha} = \pi^{(n-1)/2}\Gamma\left(\frac{\alpha}{2}\right)(4 - 2^{3-\alpha})/\left(\alpha\Gamma\left(\frac{n+\alpha-1}{2}\right)\right)$. In particular, if D is a bounded and convex domain, then

(1.6)
$$\mathcal{E}(u) \ge \kappa_{n,\alpha} \int_D \frac{u(x)^2}{\operatorname{dist}(x, D^c)^\alpha}\, dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D} \int_D \frac{u(x)^2}{\operatorname{dist}(x, D^c)^{\alpha-1}}\, dx, \quad u \in C_c(D).$$

$\kappa_{n,\alpha}$ cannot be replaced by a larger constant in (1.5) *and* (1.6).

Theorem 1.2 is a strengthening of [16, Theorem 1.1]. The main new ingredient is the remainder with smaller singularity at the boundary of D in (1.5) and (1.6), when D is bounded. We note that for cones (e.g., $\mathbb{R}^n \setminus \{0\}$) the remainder vanishes. Indeed, we consider the dilations of u and see that the homogeneities of $\mathcal{E}(u)$ and $\int_D \frac{u(x)^2}{\operatorname{dist}(x,D^c)^\alpha}\, dx$ are the same, but different from that of $\int_D \frac{u(x)^2}{\operatorname{dist}(x,D^c)^{\alpha-1}}\, dx$.

As a consequence of Theorem 1.2 we obtain the following estimate for the first eigenvalue λ_1 of the regional fractional Laplacian for D [13]:

$$\lambda_1 \ge \frac{\Gamma((n+\alpha)/2)}{2^{-\alpha}\pi^{n/2}|\Gamma(-\alpha/2)|} \left(\frac{\kappa_{n,\alpha}}{\left(\frac{1}{2}\operatorname{diam} D\right)^\alpha} + \frac{\lambda_{n,\alpha}}{\left(\frac{1}{2}\right)^{\alpha-1}(\operatorname{diam} D)^\alpha} \right)$$

(see also [15] or [14] for other applications of Hardy inequalities).

We denote

$$Lu(x) = \lim_{\varepsilon\to 0^+} \int\limits_{(-1,1)\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{1+\alpha}}\,dy,$$

which equals, up to a multiplicative constant, the regional fractional Laplacian for D [13]. To prove Theorem 1.1 we calculate Lw for the function $w(x) = (1-x^2)^{(\alpha-1)/2}$ (see Lemma 2.1), and the result follows from the ideas of [1, 8] (see also [10, Proposition 2.3] or Lemma 2.2 below). The calculation of Lw uses the Kelvin transform. For a discussion of the Kelvin transform and the fractional Laplacian we refer the reader to [6] and [4].

An explicit formula for $Lu_p(x)$, where $u_p(x) = (1-x^2)^p$, may be deduced from [12] for $p = \alpha/2$, and from [2] for $p = (\alpha-2)/2$ (see the remarks after the proof of Lemma 2.1).

Finally, the symmetric bilinear form obtained from $\mathcal{E}$ by polarisation is up to a multiplicative constant the Dirichlet form of the censored stable process in $D = (-1,1)$ (see [3]). The following result is a close counterpart of Lemma 2.3 and Theorem 1.1 stated for the Dirichlet form of the killed stable process [3] and it turns out to have a remarkably simple form.

COROLLARY 1.3. *Let $0<\alpha<2$ and $w(x) = (1-x^2)^{(\alpha-1)/2}$. For every $u \in C_c(-1,1)$,*

(1.7)

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{u(x)}{w(x)}-\frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\,dx\,dy \\ &\quad + \frac{B\big(\frac{1+\alpha}{2},\frac{2-\alpha}{2}\big)}{\alpha}\int_{-1}^{1} u(x)^2(1-x^2)^{-\alpha}\,dx \\ &\geq \frac{B\big(\frac{1+\alpha}{2},\frac{2-\alpha}{2}\big)}{\alpha 2^\alpha}\int_{-1}^{1} u(x)^2\left(\frac{1}{x+1}+\frac{1}{1-x}\right)^{\alpha} dx.\end{aligned}$$

$\frac{B(\frac{1+\alpha}{2},\frac{2-\alpha}{2})}{\alpha 2^\alpha}$ cannot be replaced by a larger constant in (1.7).

2. Proofs. We start by calculating the regional fractional Laplacian for power functions.

LEMMA 2.1. *Let $p>-1$ and $u_p(x) = (1-x^2)^p$. For $0<\alpha<2$ we have*

$$\begin{aligned}Lu_p(x) = \frac{(1-x^2)^{p-\alpha}}{\alpha}\big((1-x)^\alpha + (1+x)^\alpha \\ - (2p+2-\alpha)B(p+1, 1-\alpha/2) + \alpha I(p)\big),\end{aligned} \tag{2.1}$$

where

$$I(p) = \text{p.v.} \int_{-1}^{1} \frac{(1-tx)^{\alpha-1-2p}-1}{|t|^{1+\alpha}}(1-t^2)^p\,dt,$$

and p.v. *means the Cauchy principal value. We have* $I\big(\frac{\alpha}{2}\big) = \frac{2}{\alpha}B\big(1+\frac{\alpha}{2}, 1-\frac{\alpha}{2}\big)\big(1-(1-x^2)^{\alpha/2}\big)$, $I(\frac{\alpha-1}{2}) = I\big(\frac{\alpha-2}{2}\big) = 0$, *and if* $1 < \alpha < 2$ *then* $I\big(\frac{\alpha-3}{2}\big) = x^2 B\big(\frac{\alpha-1}{2}, 1-\frac{\alpha}{2}\big)$.

Proof. By changing variable $t = y^2$ and integrating by parts, we have

$$\begin{aligned}
Lu_p(0) &= 2\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{(1-y^2)^p-1}{y^{1+\alpha}}\,dy\\
&= 2\lim_{\varepsilon\to 0^+}\bigg(\frac{1}{2}\int_{\varepsilon^2}^{1}(1-t)^p t^{-1-\alpha/2}[(1-t)+t]\,dt - \int_{\varepsilon}^{1} y^{-1-\alpha}\,dy\bigg)\\
&= 2\lim_{\varepsilon\to 0^+}\bigg(\frac{1}{\alpha}(1-\varepsilon^2)^{p+1}\varepsilon^{-\alpha} - \frac{p+1}{\alpha}\int_{\varepsilon^2}^{1}(1-t)^p t^{-\alpha/2}\,dt\\
&\qquad + \frac{1}{2}\int_{\varepsilon^2}^{1}(1-t)^p t^{-\alpha/2}\,dt + \frac{1}{\alpha} - \frac{\varepsilon^{-\alpha}}{\alpha}\bigg).
\end{aligned}$$

It is easy to see that

$$\lim_{\varepsilon\to 0^+}\bigg(\frac{1}{\alpha}(1-\varepsilon^2)^{p+1}\varepsilon^{-\alpha} - \frac{\varepsilon^{-\alpha}}{\alpha}\bigg) = \lim_{\varepsilon\to 0^+}\frac{\varepsilon^{2-\alpha}}{\alpha}\,\frac{(1-\varepsilon^2)^{p+1}-1}{\varepsilon^2} = 0.$$

Therefore

$$Lu_p(0) = \frac{2}{\alpha}[1-(p+1-\alpha/2)B(p+1, 1-\alpha/2)].$$

For $x_0 \in (-1,1)$ we have

$$Lu_p(x_0) = \text{p.v.}\int_{-1}^{1}\frac{(1-y^2)^p-(1-x_0^2)^p}{|y-x_0|^{1+\alpha}}\,dy.$$

We change the variable in the following way:

$$t = \varphi(y) = \frac{x_0-y}{1-x_0 y}, \qquad y = \varphi(t),$$

$$\varphi'(y) = \frac{x_0^2-1}{(1-x_0y)^2}, \qquad y - x_0 = \frac{t(1-x_0^2)}{tx_0-1}, \qquad 1-y^2 = \frac{(1-x_0^2)(1-t^2)}{(tx_0-1)^2}.$$

The principal value integral transforms as follows:

(2.2)

$$\begin{aligned}Lu_p(x_0) &= (1-x_0^2)^{p-\alpha}\,\text{p.v.}\int_{-1}^{1}\frac{(1-t^2)^p-(1-tx_0)^{2p}}{|t|^{1+\alpha}}\,(1-tx_0)^{\alpha-1-2p}\,dt\\ &= (1-x_0^2)^{p-\alpha}\Big[Lu_p(0)-\text{p.v.}\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{|t|^{1+\alpha}}\,dt\\ &\quad+\text{p.v.}\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1-2p}-1}{|t|^{1+\alpha}}(1-t^2)^p\,dt\Big].\end{aligned}$$

We consider the integral in (2.2),

$$I:=\text{p.v.}\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{|t|^{1+\alpha}}\,dt=\lim_{\varepsilon\to0^+}(J_\varepsilon(x_0)+J_\varepsilon(-x_0)),$$

where

$$\begin{aligned}J_\varepsilon(x_0) &= \int_{\varepsilon}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{t^{1+\alpha}}\,dt=\int_{\varepsilon}^{1}\left(\frac{1}{t}-x_0\right)^{\alpha-1}\frac{dt}{t^2}-\frac{\varepsilon^{-\alpha}-1}{\alpha}\\ &= \frac{1}{\alpha}\left(\frac{1}{\varepsilon}-x_0\right)^{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}-\frac{\varepsilon^{-\alpha}-1}{\alpha}\\ &= \frac{1}{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}+\frac{(1-\varepsilon x_0)^{\alpha}-1}{\alpha\varepsilon^{\alpha}}.\end{aligned}$$

By the l'Hôpital rule we find that

$$I=\frac{2}{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}-\frac{1}{\alpha}(1+x_0)^{\alpha},$$

and the first part of the lemma is proved.

We have

$$\begin{aligned}I(\alpha/2) &= \text{p.v.}\int_{-1}^{1}\frac{(1-tx)^{-1}-1}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}\,dt\\ &= \int_{-1}^{1}\frac{\sum_{k=2}^{\infty}(tx)^k}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}\,dt=2\int_{0}^{1}\frac{\sum_{k=1}^{\infty}(tx)^{2k}}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}\,dt\\ &= \sum_{k=1}^{\infty}B(k-\alpha/2,1+\alpha/2)x^{2k}\\ &= \Gamma(1+\alpha/2)\Gamma(-\alpha/2)\left(\sum_{k=0}^{\infty}\frac{x^{2k}\Gamma(k-\alpha/2)}{\Gamma(-\alpha/2)k!}-1\right)\\ &= \frac{2B(1+\alpha/2,1-\alpha/2)}{\alpha}(1-(1-x^2)^{\alpha/2}).\end{aligned}$$

Calculating $I(p)$ for $p = (\alpha-1)/2$, $p = (\alpha-2)/2$ and $p = (\alpha-3)/2$ is easy and will be omitted. ■

We will apply Lemma 2.1 only to $p = (\alpha-1)/2$. The fractional Laplacian applied to $u_{\alpha/2}$ extended to be zero on $\mathbb{R}\setminus(-1,1)$ was calculated by using the Fourier transform and hypergeometric function in [12]. From those calculations we may confirm our formula for $Lu_{\alpha/2}$, and consequently for $I(\alpha/2)$. Also the value of $Lu_{(\alpha-2)/2}$ can be calculated from known results. Namely,

$$u_{(\alpha-2)/2}(x) = \frac{1}{2}(K(x,-1)+K(x,1)) \quad \text{for } |x|<1,$$

where $K(x,Q) = (1-x^2)^{\alpha/2}/|x-Q|$ is the Martin kernel for the interval [2, (3.36)]. Hence $u_{(\alpha-2)/2}(x)$ extended to be zero on $\mathbb{R}\setminus(-1,1)$ annihilates on $(-1,1)$ the fractional Laplacian (see [4, Chapter 3] and [3, (3.14)]).

The next lemma may be considered a special case of Proposition 2.3 of [10] (see also [8]). For the reader's convenience we give an elementary proof following [5].

Lemma 2.2. *Let $D\subset\mathbb{R}^n$ be an open set. For every $u\in C_c(D)$ and any strictly positive function $w\in C^2(D)\cap L^1(D,(1+|x|)^{-n-\alpha}\,dx)$, we have*

$$\mathcal{E}(u) = \int_D u(x)^2\,\frac{-Lw(x)}{w(x)}\,dx + \frac{1}{2}\int_D\int_D \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{n+\alpha}}\,dx\,dy.$$

Proof. We have

$$(u(x)-u(y))^2 + u(x)^2\,\frac{w(y)-w(x)}{w(x)} + u(y)^2\,\frac{w(x)-w(y)}{w(y)} = \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 w(x)w(y).$$

We integrate against $1_{\{|x-y|>\varepsilon\}}|x-y|^{-n-\alpha}\,dx\,dy$, and let $\varepsilon\to 0$. We can use Taylor's expansion for w and the compactness of the support of u to justify an application of the Lebesgue dominated convergence theorem. ■

We next state a result analogous to the ground state representation obtained for half-spaces and $\mathbb{R}^n\setminus\{0\}$ by Frank and Seiringer [10, 11] (we return to considering $D=(-1,1)$ and $n=1$).

Lemma 2.3. *Let $0<\alpha<2$. Let $w(x) = (1-x^2)^{(\alpha-1)/2}$. For every $u\in C_c(-1,1)$,*

$$\mathcal{E}(u) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\,dx\,dy$$
$$+ 2^\alpha \kappa_{1,\alpha}\int_{-1}^{1} u(x)^2(1-x^2)^{-\alpha}\,dx$$
$$+ \frac{1}{\alpha}\int_{-1}^{1} u(x)^2[2^\alpha - (1+x)^\alpha - (1-x)^\alpha](1-x^2)^{-\alpha}\,dx.$$

Proof of Lemma 2.3. The result follows immediately from Lemma 2.2 applied to $w(x) = (1-x^2)^{(\alpha-1)/2}$ and Lemma 2.1 with $p = (\alpha-1)/2$. ■

Proof of Theorem 1.1. By scaling we may and do assume that $a = -1$ and $b = 1$. By Lemma 2.3 it is enough to verify that

$$2^\alpha - (1+x)^\alpha - (1-x)^\alpha \geq (2^\alpha - 2)(1-x^2), \qquad 1 \leq \alpha \leq 2,\ 0 \leq x \leq 1. \tag{2.3}$$

Substituting $u = x^2$, it suffices to prove that

$$g(u) = (2^\alpha - 2)u - (1-\sqrt{u})^\alpha - (1+\sqrt{u})^\alpha + 2$$

is concave, or

$$g'(u) = 2^\alpha - 2 + \frac{\alpha}{2\sqrt{u}}((1-\sqrt{u})^{\alpha-1} - (1+\sqrt{u})^{\alpha-1})$$

is decreasing. We substitute $u = t^2$ and observe that

$$\frac{(1-t)^{\alpha-1} - (1+t)^{\alpha-1}}{t} = \frac{h(t) - h(0)}{t},$$

where $h(t) = (1-t)^{\alpha-1} - (1+t)^{\alpha-1}$. Since h is concave, the function $t \mapsto (h(t) - h(0))/t$ is decreasing, and so too is g'. This proves (2.3) and (1.3).

The fact that $\kappa_{1,\alpha}$ in (1.3) is optimal follows from [16]. ■

The constant $2^\alpha - 2$ in (2.3) is the largest possible (consider $x = 0$). However it is not clear if the constant $(4 - 2^{3-\alpha})/(\alpha(b-a))$ is optimal in (1.3).

Proof of Theorem 1.2. The proof is analogous to the proof of Theorem 1.1 in [16], but instead of applying [16, Corollary 2.3] we use Theorem 1.1. For the reader's convenience we repeat part of the argument of Loss and Sloane. We denote by $\mathcal{L}_w$ the $(n-1)$-dimensional Lebesgue measure on the plane $x \cdot w = 0$. By [16, Lemma 2.4 and Corollary 2.3], writing

$$\iiint = \int_{S^{n-1}} dw \int_{\{x\,:\,x\cdot w=0\}} d\mathcal{L}_w(x) \int_{x+sw\in D} ds,$$

we find that

$$\begin{aligned}
\mathcal{E}(u) &= \frac{1}{4}\iiint \int_{x+tw\in D} dt\, \frac{|u(x+sw)-u(x+tw)|^2}{|s-t|^{1+\alpha}} \\
&\geq \kappa_{1,\alpha}\,\frac{1}{2}\iiint u(x+sw)^2\left[\frac{1}{d_w(x+sw)}+\frac{1}{\delta_w(x+sw)}\right]^{\alpha} \\
&\quad + \frac{4-2^{3-\alpha}}{2\alpha}\iiint u(x+sw)^2\left[\frac{1}{d_w(x+sw)}+\frac{1}{\delta_w(x+sw)}\right]^{\alpha-1} \\
&\quad \times \frac{1}{d_w(x+sw)+\delta_w(x+sw)} \\
&= \kappa_{1,\alpha}\,\frac{1}{2}\int_{S^{n-1}} dw \int_D u(x)^2\left[\frac{1}{d_{w,D}(x)}+\frac{1}{\delta_{w,D}(x)}\right]^{\alpha} dx \\
&\quad + \frac{4-2^{3-\alpha}}{2\alpha}\int_{S^{n-1}} dw \int_D u(x)^2\left[\frac{1}{d_{w,D}(x)}+\frac{1}{\delta_{w,D}(x)}\right]^{\alpha-1} \\
&\quad \times \frac{1}{d_{w,D}(x)+\delta_{w,D}(x)}\, dx \\
&\geq \kappa_{n,\alpha}\int_D \frac{u(x)^2}{M_\alpha(x)^\alpha}\, dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D}\int_D \frac{u(x)^2}{M_{\alpha-1}(x)^{\alpha-1}}\, dx.
\end{aligned}$$

In the last line we have used [16, (7)], which is valid for any $\alpha > 0$, hence also for $\alpha-1$ in place of α. This proves (1.5).

Inequality (1.6) follows from [16, (9)], which is also valid for any $\alpha > 0$. ∎

Proof of Corollary 1.3. The equality follows from Lemma 2.3 and the following formula, where we take $D=(-1,1)$ in the definition of $\mathcal{E}(u)$:

(2.4)

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\, dx\, dy &= \mathcal{E}(u) + \int_{-1}^{1} u(x)^2\, \frac{(1+x)^{-\alpha}+(1-x)^{-\alpha}}{\alpha}\, dx \\
&= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{u(x)}{w(x)}-\frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\, dx\, dy \\
&\quad + \frac{2^\alpha(\kappa_{1,\alpha}\alpha+1)}{\alpha}\int_{-1}^{1} u(x)^2(1-x^2)^{-\alpha}\, dx. \ \blacksquare
\end{aligned}$$

The sharpness of the constant $\frac{B\left(\frac{1+\alpha}{2},\frac{2-\alpha}{2}\right)}{\alpha 2^\alpha}$ in (1.7) follows from [16].

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