C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 122 2011 NO. 1

POWERFUL AMICABLE NUMBERS

BY

PAUL POLLACK (Urbana, IL)

Abstract. Let $s(n) := \sum_{d|n,\, d<n} d$ denote the sum of the proper divisors of the natural number n. Two distinct positive integers n and m are said to form an *amicable pair* if $s(n) = m$ and $s(m) = n$; in this case, both n and m are called *amicable numbers.* The first example of an amicable pair, known already to the ancients, is $\{220, 284\}$. We do not know if there are infinitely many amicable pairs. In the opposite direction, Erdős showed in 1955 that the set of amicable numbers has asymptotic density zero.

Let $\ell \geq 1$. A natural number n is said to be ℓ-full (or ℓ-*powerful*) if p^ℓ divides n whenever the prime p divides n. As shown by Erdős and Szekeres in 1935, the number of ℓ-full $n \leq x$ is asymptotically $c_\ell x^{1/\ell}$, as $x \to \infty$. Here c_ℓ is a positive constant depending on ℓ.

We show that for each fixed ℓ, the set of amicable ℓ-full numbers has relative density zero within the set of ℓ-full numbers.

1. Introduction. Let $s(n) := \sum_{d|n,\, d<n} d$ be the sum of the proper divisors of the natural number n. Two distinct natural numbers n and m are said to form an *amicable pair* if $s(n) = m$ and $s(m) = n$; in this case, both n and m are called *amicable numbers*. The first amicable pair, 220 and 284, was known already to the Pythagorean school. Despite their long history, we still know very little about such pairs. For example, we know over 10 million examples [19] (cf. [11]), but we have no proof that there are infinitely many. In the opposite direction, Erdős [5] showed in 1955 that the set of amicable numbers has asymptotic density zero. The strongest result of this type is due to Pomerance [23].

Another unsolved problem concerns the existence of an amicable pair of opposite parity. It is easy to prove (see, e.g., [12]) that in such a pair, the odd member is a square and the even member is either a square or twice a square. Thus, it is natural to ask how often $s(n^2)$ is a perfect square or twice a square. Iannucci and Luca [18] have shown that the number of $n \leq x$ for which $s(n^2)$ is a square is bounded by $x/(\log x)^{3/2+o(1)}$, as $x \to \infty$, and their method yields the same estimate for how often $s(n^2)$ is twice a square. As a corollary, the set of n for which n^2 is amicable has asymptotic density zero.

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37.
Key words and phrases: amicable number, powerful number, sociable number, aliquot sequence.

DOI: 10.4064/cm122-1-10

Squares are an example of 2-full numbers. Here the natural number n is called *ℓ-full* if every prime dividing n appears to at least the ℓth power in the prime factorization of n. It is known (see [9]) that the number of ℓ-full numbers $n \le x$ is asymptotically $c_\ell x^{1/\ell}$, as $x \to \infty$, where c_ℓ is a positive constant depending on ℓ.

Our main result is the following:

THEOREM 1.1. *Fix $\ell \ge 1$. As $x \to \infty$, only $o(x^{1/\ell})$ of the ℓ-full natural numbers not exceeding x are amicable.*

Note that the case $\ell = 1$ is exactly the theorem of Erdős quoted in the opening paragraph.

Our method is entirely different from that of Iannucci and Luca. We adopt two strategies, depending on the parity of ℓ. If ℓ is odd, we show that an ℓ-full number n usually satisfies either $n < s(n) < s(s(n))$ or $n > s(n) > s(s(n))$. In neither case can n be amicable. The ideas here trace back to Erdős [5] (see also [7]) and Erdős–Granville–Pomerance–Spiro [8]. Suppose now that ℓ is even. We apply an elementary sieving argument to show that for most ℓ-full n, the number $s(n)$ has many primes appearing to the first power. Thus, $\sigma(s(n))$ is divisible by a large power of 2. On the other hand, if n is amicable, then $\sigma(s(n)) = \sigma(n)$. For most ℓ-full numbers n, nearly all of the primes dividing n show up to precisely the ℓth power; since ℓ is even, these prime powers have odd σ-value. Thus, for an ℓ-full number n, we expect that $\sigma(n)$ is only divisible by a small power of 2. So most ℓ-full numbers are not amicable.

Amicable numbers are a special class of *sociable numbers*. In the final section of this paper, we review what it means for a number to be sociable, and we prove the analogue of Theorem 1.1 for sociable numbers of a fixed odd order.

Notation. Throughout, the letters p and q are reserved for primes. We write $P(n)$ for the largest prime divisor of n, with the understanding that $P(1) = 1$. We say that n is *y-smooth* if $P(n) \le y$. We let $\omega(n) := \sum_{p \mid n} 1$ stand for the number of distinct prime divisors of n, and we write $\mathrm{rad}(n) := \prod_{p \mid n} p$ for the largest squarefree divisor of n. If d divides n and $\gcd(d, n/d) = 1$, we say that *d exactly divides n*, and we write $d \parallel n$. We write $\nu_p(n)$ for the largest integer satisfying $p^{\nu_p(n)} \mid n$ (the *p-adic order* of n). We use $\mathbf{1}_C$ for the indicator function of the set defined by the condition C; so, for example, $\mathbf{1}_{v \text{ squarefree}} = \mu^2(v)$.

If $\mathscr{S}_1$ and $\mathscr{S}_2$ are subsets of the natural numbers, by the *relative density* of $\mathscr{S}_1$ within $\mathscr{S}_2$ we mean the limit

$$\lim_{x \to \infty} \frac{\#\{n \le x : n \in \mathscr{S}_1\}}{\#\{n \le x : n \in \mathscr{S}_2\}},$$

if it exists. The upper and lower relative densities are defined analogously, with lim sup and lim inf replacing lim. We say that a property holds for almost all m if it holds for all m outside of a set of density zero (or relative density zero, depending on context). We use both the Landau–Bachmann O-notation and Vinogradov's $\ll$ notation; implied constants are absolute unless specified otherwise by subscripts. We write $\log_1 x = \max\{1, \log x\}$, and for $k > 1$, we let $\log_k$ denote the kth iterate of $\log_1$.

2. Preliminaries for the case of odd ℓ. Fix an odd natural number ℓ. We begin by proving the following theorem, in which the lower-bound half generalizes a theorem from [5] (see also [7]) and the upper-bound half generalizes [8, Theorem 5.1]. In both our notation and our general strategy, we follow [8] closely.

THEOREM 2.1. *For each fixed $\epsilon > 0$, all but $o(x^{1/\ell})$ of the ℓ-full numbers $n \le x$ satisfy*

$$\frac{s(n)}{n} - \epsilon \le \frac{s(s(n))}{s(n)} \le \frac{s(n)}{n} + \epsilon. \tag{1}$$

Rather than treating all ℓ-full numbers on an equal footing, we introduce a certain convenient partition: For each ℓ-full number n, we write n in the form dv^ℓ, where d is $(\ell+1)$-full and v is squarefree. Then most ℓ-full numbers correspond to small values of d. Indeed, the number of ℓ-full $n \le x$ for which $d > y$ (say) is bounded by

$$\sum_{\substack{d>y\\ d\,(\ell+1)\text{-full}}} \sum_{v \le (x/d)^{1/\ell}} 1 \le x^{1/\ell} \sum_{\substack{d>y\\ d\,(\ell+1)\text{-full}}} d^{-1/\ell} \ll_\ell x^{1/\ell} y^{-1/\ell(\ell+1)}.$$

It follows that to prove a result about almost all ℓ-full numbers, such as Theorem 1.1 or Theorem 2.1, we may assume that the $(\ell + 1)$-full part d of n is fixed. Our task then becomes showing that as $x \to \infty$, all but $o(x)$ of the squarefree $v \le x$ coprime to d are such that $n = dv^\ell$ satisfies the statement in question.

For the rest of this section, we assume the $(\ell + 1)$-full number d is fixed. Whenever we use v below, we always mean a squarefree integer coprime to d, whether this is stated explicitly or not.

LEMMA 2.2. *Fix a natural number T. For each value of v with $1 < v \le x$, write*

$$dv^\ell = n_1 n_2 \quad \text{and} \quad s(dv^\ell) = N_1 N_2,$$

where $P(n_1 N_1) \le T$ and every prime dividing $n_2 N_2$ exceeds T. Then, except for $o(x)$ (as $x \to \infty$) choices of v, we have $n_1 = N_1$.

Proof. At the cost of excluding $o(x)$ values of $v \le x$, we may assume that

$$n_1 \le d(\log_2 x)^{1/2}\Big(\prod_{p\le T} p\Big)^{-1}.$$

Indeed, in the opposite case, v^ℓ has a T-smooth divisor $e := n_1/(n_1, d)$ exceeding $(\log_2 x)^{1/2}(\prod_{p\le T} p)^{-1}$. For each such e, let e_1 be the minimal integer with $e \mid e_1^\ell$. Since $e \mid v^\ell$, we have that $e_1 \mid v$; moreover, $e_1 \ge e^{1/\ell}$. So the number of v that may arise is at most

$$\sum_{\substack{e\ T\text{-smooth} \\ e>(\log_2 x)^{1/2}(\prod_{p\le T} p)^{-1}}} \sum_{\substack{v\le x \\ e_1\mid v}} 1 \le x \sum_{\substack{e\ T\text{-smooth} \\ e>(\log_2 x)^{1/2}(\prod_{p\le T} p)^{-1}}} \frac{1}{e^{1/\ell}}. \tag{2}$$

Observing that

$$\sum_{e\,T\text{-smooth}} \frac{1}{e^{1/\ell}} = \prod_{p\le T}\left(1 + \frac{1}{p^{1/\ell}} + \frac{1}{p^{2/\ell}} + \cdots\right) < \infty,$$

we deduce that the right-hand side of (2) is $o(x)$, as $x \to \infty$.

Now we show that for all but $o(x)$ values of $v \le x$, the number $\sigma(dv^\ell)$ is divisible by every natural number not exceeding $(\log_2 x)^{2/3}$. Let m be a natural number. For all but $\ll x/(\log x)^{1/\varphi(m)}$ exceptional values of $v \le x$, one can find a prime $p \equiv -1 \pmod m$ for which $p \,\|\, v$ (see [22, Theorem 2]). If v is not exceptional, then

$$m \mid p + 1 = \sigma(p) \mid \sigma(v) \mid \sigma(v^\ell) \mid \sigma(d)\sigma(v^\ell) = \sigma(dv^\ell).$$

(Here the relation $\sigma(v) \mid \sigma(v^\ell)$ holds because v is squarefree and ℓ is odd, and the final equality holds since d is coprime to v.) Summing over $m \le (\log_2 x)^{2/3}$, we see that the number of v that we must exclude is

$$\begin{aligned} \ll \sum_{m\le(\log_2 x)^{2/3}} \frac{x}{(\log x)^{1/\varphi(m)}} &\le x(\log_2 x)^{2/3}(\log x)^{-1/(\log_2 x)^{2/3}} \\ &\le x(\log_2 x)^{2/3}\exp(-(\log_2 x)^{1/3}) = o(x), \end{aligned}$$

as $x \to \infty$.

If v is not excluded by the considerations of the first paragraph, then for large x,

$$n_1 \prod_{p\le T} p \le d(\log_2 x)^{1/2} \le (\log_2 x)^{2/3}.$$

Furthermore, if v is not excluded by the considerations of the second paragraph, we have $n_1 \prod_{p\le T} p \mid \sigma(dv^\ell)$. So, apart from $o(x)$ exceptional values of v, we have

$$s(dv^\ell) \equiv -dv^\ell \ \Big(\mathrm{mod}\ n_1 \prod_{p\le T} p\Big).$$

This congruence implies that n_1 divides $s(dv^\ell)$ and that $s(dv^\ell)/n_1$ is coprime to $\prod_{p\le T} p$. Hence, $s(dv^\ell)$ has T-smooth part exactly n_1, i.e., $n_1 = N_1$. ∎

Proof of the lower bound in Theorem 2.1. Fix $\delta > 0$. We will show that the number of $n \le x$ for which the left-hand inequality in (1) fails is smaller than $3\delta x$, once x is large.

First, we show that we can fix a number B so that $\sigma(dv^\ell)/dv^\ell \le B$, except for at most δx exceptional $v \le x$. This follows from a first moment argument: We have

$$\sum_{v\le x} \frac{\sigma(dv^\ell)}{dv^\ell} = \frac{\sigma(d)}{d} \sum_{v\le x} \sum_{e\mid v^\ell} \frac{1}{e} = \frac{\sigma(d)}{d} \sum_{e} \frac{1}{e} \sum_{\substack{v\le x\\ e\mid v^\ell}} 1$$
$$\le x\left(\frac{\sigma(d)}{d} \sum_{e} \frac{1}{e\cdot \mathrm{rad}(e)}\right),$$

where we use the fact that $\mathrm{rad}(e)$ divides v whenever e divides v^ℓ. Since

$$\sum_{e} \frac{1}{e\cdot \mathrm{rad}(e)} = \prod_{p}\left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) < \infty,$$

we obtain our claim with

$$B = \delta^{-1} \frac{\sigma(d)}{d} \sum_{e} \frac{1}{e\cdot \mathrm{rad}(e)}.$$

Next, we show that we can choose a fixed T so that, with n_2 defined as in Lemma 2.2, we have

$$\frac{\sigma(n_2)}{n_2} \le \exp(\epsilon/B)$$

except for at most δx exceptional $v \le x$. Again, we use a first moment argument. We may assume that $T > d$, so that any prime $p > T$ which divides dv^ℓ necessarily divides v. Then

$$\sum_{v\le x} \log \frac{\sigma(n_2)}{n_2} \le \sum_{v\le x} \sum_{\substack{p\mid v\\ p>T}} \log\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$
$$\le \sum_{v\le x} \sum_{\substack{p\mid v\\ p>T}} \frac{1}{p-1} \le x \sum_{p>T} \frac{1}{p(p-1)} < \frac{x}{T}.$$

Now choosing $T = \lceil B/(\delta\epsilon) \rceil$, we have our claim.

Lemma 2.2 gives that for large x, we have $n_1 = N_1$ except for at most δx values of $v \le x$.

If v is not in any of the exceptional classes defined above, then

$$\frac{\sigma(s(dv^\ell))}{s(dv^\ell)} = \frac{\sigma(N_1N_2)}{N_1N_2} \geq \frac{\sigma(N_1)}{N_1} = \frac{\sigma(n_1)}{n_1} = \frac{\sigma(dv^\ell)/dv^\ell}{\sigma(n_2)/n_2}$$
$$\geq \frac{\sigma(dv^\ell)}{dv^\ell}\exp\left(-\frac{\epsilon}{B}\right) \geq \frac{\sigma(dv^\ell)}{dv^\ell}\left(1-\frac{\epsilon}{B}\right) \geq \frac{\sigma(dv^\ell)}{dv^\ell} - \epsilon.$$

Subtracting 1, we obtain exactly the left-hand inequality in Theorem 2.1, with $n = dv^\ell$. Note that the number of exceptional v is at most $3\delta x$, as claimed. ■

For a given α, we call the natural number n an *α-primitive* number if $s(n)/n \geq \alpha$ while $s(d)/d < \alpha$ for every proper divisor d of n. (Traditionally, these are called *primitive* $(1+\alpha)$*-abundant numbers*, but the shorter name will be typographically convenient.) We use the following estimate of Erdős [6, p. 6], which is proved by the method of [4]. A weaker result, still strong enough for our purposes, could be derived by a small modification of the proof in [3].

Lemma 2.3. *Fix a positive rational number α. There is a constant $c = c(\alpha) > 0$ and an $x_0 = x_0(\alpha)$ so that for $x > x_0$, the number of α-primitive numbers not exceeding x is bounded by*

$$\frac{x}{\exp(c\sqrt{\log x \log_2 x})}.$$

We also need the following technical lemma, which asserts the rarity of numbers with an inordinately large number of prime factors.

Lemma 2.4. *We have*

$$\sum_{a\colon \omega(a) > 10\ell \log_2 a} \frac{\ell^{\omega(a)}}{a} < \infty.$$

Proof. For $j \geq 1$, let S_j denote that portion of the series corresponding to those values of a with $2^{j-1} < a \leq 2^j$. For large values of j, we have, with $y := 2^j$,

$$S_j \leq \sum_{\substack{a \leq y \\ \omega(a) \geq 9\ell \log_2 y}} \frac{\ell^{\omega(a)}}{a} \leq \sum_{k \geq 9\ell \log_2 y} \frac{1}{k!}\left(\sum_{p \leq y} \ell\left(\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right)\right)^k$$
$$\leq \sum_{k \geq 9\ell \log_2 y} \frac{1}{k!}(\ell(\log_2 y + O(1)))^k.$$

Considering the ratio of successive terms, we see that the remaining sum is dominated by the smallest value of k. So putting $k_0 = \lceil 9\ell \log_2 y \rceil$, we have

$$S_j \ll \frac{1}{k_0!}(\ell(\log_2 y + O(1)))^{k_0} \le \left(\frac{e\ell(\log_2 y + O(1))}{k_0}\right)^{k_0}$$
$$\le \frac{1}{3^{k_0}} \le \frac{1}{(\log y)^{9\ell \log 3}} < \frac{1}{(\log y)^9} \ll \frac{1}{j^9},$$

once j is large. Hence, $\sum_{j \ge 1} S_j < \infty$, as desired. ∎

Proof of the upper bound in Theorem 2.1. We may assume that $0<\epsilon<1$. Let $\delta > 0$ be given. Fix $\eta \in (0,1)$ small enough that for all large x, all of the $v \le x$, with at most δx exceptions, satisfy

$$P(v) > x^{\eta} \quad \text{and} \quad P(v) \,\|\, v. \tag{3}$$

To see that such a choice of η is possible, one can invoke standard results on smooth numbers (e.g., Dickman's theorem) or appeal to Brun's sieve. Next, choose a fixed number $B \ge 1$ so that all but δx of the $v \le x$ satisfy

$$\sigma(dv^{\ell})/dv^{\ell} \le B; \tag{4}$$

that such a choice of B is possible was justified in the lower-bound half of the proof. Fix positive rational numbers α_1 and α_2 satisfying

$$\alpha_1 \le \frac{\epsilon}{4B}, \qquad \alpha_2 \le \frac{\alpha_1 \eta}{32\ell}.$$

Finally, fix a natural number T which is sufficiently large with respect to the α_i, η, B, d, and ℓ; the exact meaning of "sufficiently large" will be specified in the course of the proof.

Suppose that the right-hand inequality in (1) fails for $n = dv^{\ell}$. We assume that v satisfies both (3) and (4). Then

$$\frac{\sigma(s(dv^{\ell}))}{s(dv^{\ell})} \ge \frac{\sigma(dv^{\ell})}{dv^{\ell}} + \epsilon.$$

We now apply Lemma 2.2, which allows us to assume, at the cost of excluding δx values of $v \le x$, that $n_1 = N_1$. Thus,

$$\frac{\sigma(N_2)/N_2}{\sigma(n_2)/n_2} = \frac{\sigma(s(dv^{\ell}))/s(dv^{\ell})}{\sigma(dv^{\ell})/dv^{\ell}} \ge 1 + \frac{\epsilon}{\sigma(dv^{\ell})/dv^{\ell}} \ge 1 + \frac{\epsilon}{B} \ge 1 + 4\alpha_1.$$

In particular,

$$\frac{\sigma(N_2)}{N_2} \ge 1 + 4\alpha_1. \tag{5}$$

We can assume our choice of T was such that, apart from at most δx exceptional v,

$$\frac{\sigma(n_2)}{n_2} \le 1 + \alpha_1. \tag{6}$$

Indeed, the argument for the analogous claim in the proof of the lower bound shows it is sufficient that $T > \max\{d, (\delta \log(1+\alpha_1))^{-1}\}$. Henceforth, we assume (6). Now write $N_2 = N_3 N_4$, where every prime dividing N_3 divides dv^ℓ, while N_4 is coprime to dv^ℓ. Note that every prime dividing N_3 divides n_2. Hence,

$$\begin{aligned}\frac{\sigma(N_3)}{N_3} &\leq \prod_{p \mid N_3}\left(1+\frac{1}{p-1}\right) = \left(\prod_{p \mid N_3}\frac{p^2}{p^2-1}\right)\prod_{q \mid N_3}\frac{q+1}{q}\\ &\leq \left(\prod_{p>T}\frac{p^2}{p^2-1}\right)\frac{\sigma(n_2)}{n_2} \leq 1+2\alpha_1,\end{aligned}$$

using (6) and assuming an initial appropriate choice of T. So from (5),

$$\frac{\sigma(N_4)}{N_4} = \frac{\sigma(N_2)/N_2}{\sigma(N_3)/N_3} \geq \frac{1+4\alpha_1}{1+2\alpha_1} \geq 1+\alpha_1.$$

It follows that there is an α_1-primitive number a_1 dividing N_4. Note that each prime dividing a_1 exceeds T.

We next claim that there is an α_2-primitive number a_2 dividing a_1 with

$$a_2 \leq a_1^{\eta/(4\ell)}.$$

To prove this, list the distinct prime factors of a_1 in increasing order, say

$$T < q_1 < \cdots < q_t.$$

Put

$$a_0 := q_1 \cdots q_{\lfloor \eta t/(4\ell)\rfloor}.$$

Then

$$a_0 \leq (q_1 \cdots q_t)^{\lfloor \eta t/(4\ell)\rfloor/t} \leq a_1^{\eta/(4\ell)}.$$

It is enough to show that $\sigma(a_0)/a_0 \geq 1+\alpha_2$, since then a_0 will have an α_2-primitive divisor a_2 satisfying the desired bound. First, we show that $\lfloor \eta t/(4\ell)\rfloor \geq \eta t/(8\ell)$. Otherwise, $\eta t/(4\ell) < 1$, so that $t < 4\ell/\eta$, and

$$1+\alpha_1 \leq \frac{\sigma(a_1)}{a_1} \leq \prod_{q \mid a_1}\left(1+\frac{1}{q-1}\right) \leq \left(1+\frac{1}{T}\right)^{4\ell/\eta} \leq \exp\left(\frac{4\ell}{\eta T}\right),$$

which fails if we assume a suitable initial choice of T. Since

$$\frac{\sigma(a_0)}{a_0} = \prod_{1 \leq i \leq \lfloor \eta t/(4\ell)\rfloor}\frac{q_i+1}{q_i} \geq \left(\prod_{p>T}\frac{p^2-1}{p^2}\right)\prod_{1 \leq i \leq \lfloor \eta t/(4\ell)\rfloor}\frac{q_i}{q_i-1}$$

and

$$\prod_{1 \le i \le \lfloor \eta t/(4\ell) \rfloor} \frac{q_i}{q_i - 1} \ge \left(\prod_{1 \le i \le t} \frac{q_i}{q_i - 1} \right)^{\lfloor \eta t/(4\ell) \rfloor / t}$$
$$\ge \left(\frac{\sigma(a_1)}{a_1} \right)^{\eta/(8\ell)} \ge (1 + \alpha_1)^{\eta/(8\ell)} \ge 1 + \frac{\alpha_1 \eta}{16\ell},$$

we have

$$\frac{\sigma(a_0)}{a_0} \ge \left(\prod_{p > T} \frac{p^2 - 1}{p^2} \right) \left(1 + \frac{\alpha_1 \eta}{16\ell} \right) \ge 1 + \frac{\alpha_1 \eta}{32\ell} \ge 1 + \eta_2,$$

again assuming a suitable choice of T to justify the middle inequality.

Observe that a_2 satisfies

$$a_2 \le a_1^{\eta/(4\ell)} \le s(dv^\ell)^{\eta/(4\ell)} < x^{\eta/3}$$

for large x. Write $v = Pm$, where $P = P(v)$, so that (3) gives

$$m \le x/P \le x^{1-\eta}.$$

Then a_2 divides

$$\begin{aligned} s(dv^\ell) &= \sigma(dm^\ell)\sigma(P^\ell) - dm^\ell P^\ell \\ &= P^\ell s(dm^\ell) + (P^{\ell-1} + P^{\ell-2} + \cdots + 1)\sigma(dm^\ell). \end{aligned} \tag{7}$$

Moreover, since a_2 divides N_4, we see that a_2 is coprime to dv^ℓ, and so also coprime to dm^ℓ. So for each prime q dividing a_2, the condition that q divides the expression on the right of (7) places P into at most ℓ residue classes modulo q. Since a_2 divides a_0, we see that a_2 is squarefree; hence a_2 dividing the expression in (7) places P into at most $\ell^{\omega(a_2)}$ residue classes modulo a_2.

We now sum over pairs a_2 and m, for each pair counting the number of possible values of $P \le x/m$. Clearly m and P determine $v = mP$; so from the above remarks and the Brun–Titchmarsh inequality, we deduce that the number of remaining v values is

$$\begin{aligned} &\ll \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ T < a_2 \le x^{\eta/3}}} \sum_{m \le x^{1-\eta}} \frac{\ell^{\omega(a_2)} x}{m \varphi(a_2) \log(x/(a_2 m))} \\ &\ll \frac{1}{\eta} \frac{x}{\log x} \sum_{a_2} \frac{\ell^{\omega(a_2)}}{\varphi(a_2)} \sum_m \frac{1}{m}. \end{aligned} \tag{8}$$

The sum on m is clearly $\ll \log x$. To handle the sum on a_2, notice that since a_2 is α_2-primitive, we have

$$\frac{a_2}{\varphi(a_2)} \ll \frac{\sigma(a_2)}{a_2} \le \frac{3}{2}(1 + \alpha_2) \ll 1,$$

and so $\varphi(a_2) \gg a_2$. Thus, the remaining sum over a_2 is

$$\ll \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ a_2 \geq T}} \frac{\ell^{10\ell \log_2 a_2}}{a_2} + \sum_{\substack{a \geq T \\ \omega(a) \geq 10\ell \log_2 a}} \frac{\ell^{\omega(a)}}{a}$$
$$\ll \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ a_2 \geq T}} \frac{(\log a_2)^D}{a_2} + \sum_{\substack{a \geq T \\ \omega(a) \geq 10\ell \log_2 a}} \frac{\ell^{\omega(a)}}{a},$$

where

$$D := 10\ell \log \ell.$$

But if T was chosen sufficiently large, then both of the last two sums are bounded by $\eta\delta x$; for the first sum, this follows from Lemma 2.3 (and partial summation), and for the second, from Lemma 2.4. Inserting our estimates into (8) leads to an upper bound of $\ll \delta x$. Since the number of exceptional v appearing earlier in the argument is also $\ll \delta x$, we see that the total number of $v \leq x$ for which the right-hand inequality in (1) fails is $\ll \delta x$. Since $\delta > 0$ was arbitrary, the proof is complete. ∎

3. Proof of Theorem 1.1 for odd ℓ. Let $\epsilon > 0$. Suppose that $n \leq x$ is ℓ-full. If $s(n)/n > 1 + \epsilon$, then the lower bound in Theorem 2.1 gives $s(s(n)) > s(n) > n$, apart from $o(x^{1/\ell})$ exceptional n (as $x \to \infty$). Similarly, if $s(n)/n < 1 - \epsilon$, then the upper bound in Theorem 2.1 gives $s(s(n)) < s(n) < n$, ignoring $o(x^{1/\ell})$ values of n. In neither case can n be amicable. So all amicable ℓ-full numbers, except for a set of density zero relative to the set of all ℓ-full numbers, satisfy

$$1 - \epsilon < s(n)/n < 1 + \epsilon. \tag{9}$$

Let $\delta > 0$ be given. We claim that for a suitable $\epsilon > 0$, the number of ℓ-full $n \leq x$ satisfying (9) is smaller than $2\delta x^{1/\ell}$, once x is large. The result of the previous paragraph then completes the proof.

We now prove the claim. Write $n = dv^\ell$, with d an $(\ell+1)$-full number and v a squarefree integer coprime to d. We may fix y with the property that for large x, there are fewer than $\delta x^{1/\ell}$ ℓ-full values of $n \leq x$ for which $d > y$. For each fixed $d \leq y$, the inequality (9), with $n = dv^\ell$, says precisely that

$$(2 - \epsilon)\frac{d}{\sigma(d)} < \frac{\sigma(v^\ell)}{v^\ell} < (2 + \epsilon)\frac{d}{\sigma(d)}. \tag{10}$$

By making ϵ sufficiently small, we can ensure that for large x, the number of $v \leq (x/d)^{1/\ell}$ satisfying (10) is smaller than $\delta y^{-1} x^{1/\ell}$, uniformly for $d \leq y$. This follows immediately from the existence of a continuous distribution function for $\sigma(w^\ell)/w^\ell$ (as guaranteed by the Erdős–Wintner theorem of [10]). Since there are trivially $\leq y$ values of $d \leq y$, we see that the number of ℓ-full values of $n \leq x$ that arise this way is less than $\delta x^{1/\ell}$. This completes the proof of the claim and of the theorem. ∎

4. Preliminaries for the case of even ℓ. Throughout this section and the next, we assume that ℓ is a fixed, even natural number. We begin our preparations for the even case by quoting a theorem of Halász type (cf. [13]), which appears in a stronger, more quantitative form in [14].

Theorem A. *Let $\mathscr{D}$ be a closed, convex proper subset of the closed unit disc in $\mathbb{C}$, and assume that $0 \in \mathscr{D}$. Suppose that h is a complex-valued multiplicative function satisfying $|h(n)| \leq 1$ for all $n \in \mathbb{N}$ and $h(p) \in \mathscr{D}$ for all primes p. If the series*

$$\sum_{p} \frac{1 - \Re(h(p))}{p} \tag{11}$$

diverges, then h has mean value zero, i.e.,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} h(n) = 0.$$

The next theorem, extracted from [24, Corollary 2.3], is a modern estimate for character sums of a type first considered by Davenport [1].

Theorem B. *Let $f_1(T), \dots, f_k(T)$ be monic pairwise coprime polynomials in $\mathbb{F}_q[T]$ whose largest squarefree divisors have degrees $d_1, \dots, d_k$. Let $\chi_1, \dots, \chi_k$ be nontrivial multiplicative characters of the finite field $\mathbb{F}_q$. Assume that for some $1 \leq i \leq k$, the polynomial $f_i(T)$ is not of the form $g(T)^{\mathrm{ord}(\chi_i)}$ in $\mathbb{F}_q[T]$, where $\mathrm{ord}(\chi)$ is the order of χ. Then*

$$\Big|\sum_{x \in \mathbb{F}_q} \chi_1(f_1(x)) \cdots \chi_k(f_k(x))\Big| \leq \Big(\Big(\sum_{i=1}^{k} d_i\Big) - 1\Big)\sqrt{q}.$$

The next few lemmas set up for the sieving step alluded to in the introduction.

Lemma 4.1. *If p is a prime with $p \equiv 2 \pmod{(\ell+1)}$, then the polynomial $T^{\ell} + T^{\ell-1} + \dots + 1$ is squarefree over $\mathbb{F}_p$ and has no roots in $\mathbb{F}_p$.*

Proof. If $G(T) := T^{\ell} + T^{\ell-1} + \dots + 1$ has a repeated factor in $\mathbb{F}_p[T]$, then $G(T)(T-1) = T^{\ell+1} - 1$ has a multiple root in $\overline{\mathbb{F}}_p$. Thus, p divides $\ell + 1$. But $p > \ell + 1$, unless $p = 2$, in which case also $p \nmid \ell + 1$. So G is squarefree.

If G has a root modulo p, then there is an integer a for which $p \mid G(a) \mid a^{\ell+1} - 1$. So the order of a modulo p, say d, divides $\ell + 1$. If $d = 1$, then $a \equiv 1 \pmod{p}$, and so $G(a) \equiv G(1) \equiv \ell + 1 \pmod{p}$. Thus, $p \mid \ell + 1$, which we have already noted is impossible. So $d > 1$. Since $d \mid \#\mathbb{F}_p^{\times} = p - 1$, we have $p \equiv 1 \pmod{d}$. But $p \equiv 2 \pmod{d}$. This contradiction completes the proof. ∎

LEMMA 4.2. *Let m be a squarefree natural number, every prime factor p of which satisfies*

$$p \equiv 2 \pmod{\ell+1}, \qquad p > (\ell+1)^2.$$

Let d be a natural number coprime to m. Let a and b be integers prime to m. Let $\mathscr{V}$ be the set of squarefree numbers v coprime to d satisfying the simultaneous congruences

$$\sigma(v^\ell) \equiv a \pmod{m} \quad \text{and} \quad v \equiv b \pmod{m}. \tag{12}$$

Relative to the set of all squarefree natural numbers coprime to d, the set $\mathscr{V}$ has density

$$\prod_{p \mid m} \frac{p}{(p+1)(p-1)^2}.$$

Proof. We can assume $m > 1$, so that $\varphi(m) \geq (\ell+1)^2 > 1$. If χ and ψ are Dirichlet characters modulo m, define

$$S_{\chi,\psi}(x) := \sum_{\substack{v \leq x \\ v \text{ squarefree}, \gcd(v,d)=1}} \chi(\sigma(v^\ell))\psi(v). \tag{13}$$

We start by showing that if at least one of χ and ψ is not trivial, then $S_{\chi,\psi}(x) = o(x)$, as $x \to \infty$. To this end, apply Theorem A to the multiplicative function h given by

$$v \mapsto (\mathbf{1}_{v \text{ squarefree}} \cdot \mathbf{1}_{\gcd(d,v)=1})\chi(\sigma(v^\ell))\psi(v),$$

taking $\mathscr{D}$ to be the convex hull of the $\varphi(m)$th roots of unity. Note that for each prime q, either $h(q) = 1$, or

$$1 - \Re(h(q)) \geq \min\left\{1, 1 - \cos\frac{2\pi}{\varphi(m)}\right\} > 0.$$

Since $S_{\chi,\psi}$ is the summatory function of h, the claim follows from Theorem A and Dirichlet's theorem on primes in progressions if there is at least one coprime residue class $A \bmod m$ for which

$$\chi(A^\ell + A^{\ell-1} + \cdots + 1)\psi(A) \neq 1. \tag{14}$$

In what follows, the reader should keep in mind that $A^\ell + A^{\ell-1} + \cdots + 1$ is coprime to m for every A, by Lemma 4.1 and our condition on the prime divisors of m.

Let us now prove the existence of a residue class $A \bmod m$ satisfying (14). We start by writing $\chi = \prod_{p \mid m} \chi_p$ and $\psi = \prod_{p \mid m} \psi_p$, where χ_p and ψ_p are Dirichlet characters modulo p. Since χ is nontrivial, one can find some p

dividing m, say p_0, for which not both χ_{p_0} and ψ_{p_0} are trivial. By Lemma 4.1 and Theorem B, we have

$$\Big|\sum_{A \bmod p_0} \chi(A^\ell + A^{\ell-1} + \cdots + 1)\psi(A)\Big| \le \ell\sqrt{p_0}. \tag{15}$$

Since the left-hand sum in (15) has $p_0 - 1$ nonzero terms, each of absolute value 1, and $p_0 - 1 > \ell\sqrt{p_0}$, it follows that $\chi(A^\ell + A^{\ell-1} + \cdots + 1)\psi(A)$ assumes at least two different values as A ranges over $\mathbb{F}_{p_0}^\times$. Now for each prime p dividing m with $p \neq p_0$, choose a nonzero residue class $A_p \bmod p$ arbitrarily, and choose the nonzero residue class $A_{p_0} \bmod p_0$ so that

$$\chi_{p_0}(A_{p_0}^\ell + A_{p_0}^{\ell-1} + \cdots + 1)\psi(A_{p_0}) \neq \Big(\prod_{\substack{p\mid m\\ p\neq p_0}} \chi_p(A_p^\ell + A_p^{\ell-1} + \cdots + 1)\psi_p(A_p)\Big)^{-1}.$$

Choosing A so that $A \equiv A_p \pmod{p}$ for each p dividing m, we have (14).

Multiplying $S_{\chi,\psi}(x)$ through by $\overline{\chi(a)\psi(b)}$ and summing over all χ and ψ, the usual orthogonality relations give us

$$\varphi(m)^2 \sum_{\substack{v\le x\\ v \text{ squarefree}, \gcd(v,d)=1\\ (12) \text{ holds}}} 1 = \sum_{\chi,\psi} \overline{\chi(a)\psi(b)} S_{\chi,\psi}(x) = S_{\varepsilon,\varepsilon}(x) + o(x), \tag{16}$$

as $x \to \infty$; here ε denotes the trivial character modulo m. We have

$$\begin{aligned} S_{\epsilon,\epsilon}(x) &= \sum_{\substack{v\le x\\ v \text{ squarefree}, \gcd(d,v)=1}} \epsilon(\sigma(v^l))\epsilon(v) \\ &= \sum_{\substack{v\le x\\ v \text{ squarefree}, (v,dm)=1}} 1 \sim \Big(\frac{6x}{\pi^2}\prod_{p\mid d}\frac{p}{p+1}\Big)\prod_{p\mid m}\frac{p}{p+1}, \end{aligned}$$

as $x \to \infty$. (Note that $\sigma(v^l)$ is prime to m, by the remarks following (14). For the final estimate, see, e.g., the bottom of [17, p. 634].) Since the squarefree integers prime to d have density $\frac{6}{\pi^2}\prod_{p\mid d}\frac{p}{p+1}$, the lemma follows upon dividing both sides of (16) by $\varphi(m)^2 = \prod_{p\mid m}(p-1)^2$. ∎

LEMMA 4.3. *Let $p > (\ell+1)^2$ be a prime satisfying $p \equiv 2 \pmod{\ell+1}$, and let d be a natural number relatively prime to p. Let a and b be integers coprime to p. Let $\mathscr{W}$ consist of the squarefree natural numbers v coprime to d satisfying the simultaneous congruences*

$$\sigma(v^2) \equiv a \pmod{p^2} \quad \text{and} \quad v \equiv b \pmod{p^2}. \tag{17}$$

The density of $\mathscr{W}$, relative to the set of all squarefree natural numbers co-

prime to d, is precisely

$$\frac{1}{p(p-1)^2(p+1)}.$$

Proof. Let χ and ψ be multiplicative characters modulo p^2, and define $S_{\chi,\psi}(x)$ as in (13). Imitating the proof of Lemma 4.2, it is enough to show that (as $x \to \infty$) we have $S_{\chi,\psi}(x) = o(x)$, provided that not both χ and ψ are trivial. Using Theorem A, we deduce that this holds if there is an A coprime to p satisfying (14). Suppose for the sake of contradiction that there is no such A. Then

$$\chi^p(A^\ell + A^{\ell-1} + \cdots + 1)\psi^p(A) = 1$$

whenever A is coprime to p. But χ^p and ψ^p are characters modulo p. (This follows from the observation that the residue class of a^p modulo p^2 depends only on the residue class of a modulo p.) We obtain a contradiction to the character-sum estimate (15) (with χ, ψ replaced by χ^p, ψ^p, and p_0 replaced by p) unless χ^p and ψ^p are both trivial. Thus, χ and ψ both have order 1 or p.

If χ has order 1 (i.e., is trivial), then ψ is not, and there is some A coprime to p for which (14) holds. So χ must have order p. Since ψ has order dividing p and the characters modulo p^2 form a cyclic group, it follows that there is some e with $0 \le e < p$ for which $\psi = \chi^e$. Hence

$$\chi((A^\ell + A^{\ell-1} + \cdots + 1)A^e) = 1 \tag{18}$$

for every A coprime to p. Write

$$A = A_1 + A_2 p, \qquad \text{where } 0 \le A_1 < p,\ \gcd(A_1, p) = 1.$$

Putting $G(T) := (T^\ell + T^{\ell-1} + \cdots + 1)T^e$, we have

$$\begin{aligned}(A^\ell + A^{\ell-1} + \cdots + 1)A^e &= G(A_1 + A_2 p) \\ &\equiv G(A_1) + G'(A_1)A_2 p \pmod{p^2}.\end{aligned} \tag{19}$$

Now we exploit the fact that χ, as a character of order p, takes the value 1 precisely on the pth power residues modulo p^2. But for each nonzero residue class c modulo p, there is precisely one pth power residue modulo p^2 which reduces to $c \bmod p$ (namely, the class of c^p). Since (18) holds for all A coprime to p, we see that for each A_1 with $0 \le A_1 < p$ and $\gcd(A_1, p) = 1$, the residue class modulo p^2 of $G(A_1) + G'(A_1)A_2 p$ is independent of A_2. Hence

$$G'(A_1) = eA_1^{e-1}(A_1^\ell + A_1^{\ell-1} + \cdots + 1) + A_1^e(\ell A_1^{\ell-1} + \cdots + 2A_1 + 1) \equiv 0 \pmod{p},$$

and so

$$e(A_1^\ell + A_1^{\ell-1} + \cdots + 1) + A_1(\ell A_1^{\ell-1} + \cdots + 2A_1 + 1) \equiv 0 \pmod{p}.$$

This last congruence fails for some A_1: Indeed, the left-hand side is formally nonzero as polynomial in A_1 over $\mathbb{F}_p$, since either its constant term or its coefficient of A_1 is nonvanishing. So it has at most ℓ roots modulo p; but $\ell < p-1$. Thus, there is some A coprime to p for which (18) fails. ∎

5. Proof of Theorem 1.1 for even ℓ. As in the proof for odd ℓ, we partition the n under consideration according to their $(\ell+1)$-full part d. We may assume that d is fixed. As before, we reserve the letter v for integers which are squarefree and prime to d. It suffices to show that dv^ℓ is amicable for only $o(x)$ values of $v \le x$, as $x \to \infty$.

Suppose that $n = dv^\ell$ is amicable, and let $m = s(n)$. Since n and m form an amicable pair, we have

$$\sigma(m) = n + m = \sigma(n) = \sigma(d)\sigma(v^\ell). \tag{20}$$

Since ℓ is even, the number $\sigma(v^\ell)$ is odd. Thus, (20) shows that $\nu_2(\sigma(m)) = \nu_2(\sigma(d))$. Consequently, at most $R := \nu_2(\sigma(d))$ odd primes can appear to the first power in the factorization of m. But we now show that (as $x \to \infty$) all but $o(x)$ of the $v \le x$ are such that $s(dv^\ell)$ has more than R odd primes appearing to the first power.

To start the argument off, let z be a fixed, large real number, to be chosen later, and define

$$\mathscr{P} := \{p \equiv 2 \pmod{\ell+1} : z < p \le \exp(z)\}.$$

Assume to begin with that $z > 2$. Let $\omega_{\mathscr{P}}(\cdot)$ be the additive function which counts the number of distinct prime divisors of its argument that belong to $\mathscr{P}$. If $s(dv^\ell)$ has fewer than R odd primes appearing to the first power in its prime factorization, then v belongs either to

$$\mathscr{S}_1 := \{v : \omega_{\mathscr{P}}(s(dv^\ell)) \le R\}$$

or to

$$\mathscr{S}_2 := \{v : p^2 \mid s(dv^\ell) \text{ for some } p \in \mathscr{P}\}.$$

Thus, it suffices to show that the upper density of $\mathscr{S}_1 \cup \mathscr{S}_2$ can be made arbitrarily small by choosing z appropriately.

For the rest of this section, whenever we write "relative density", we mean with respect to the squarefree numbers coprime to d.

We start by estimating the density of $\mathscr{S}_1$. Assume that

$$z > \max\{d\sigma(d), (\ell+1)^2\}.$$

If m is squarefree and composed of primes in $\mathscr{P}$, then $m \mid s(dv^\ell)$ exactly when

$$\sigma(d)\sigma(v^\ell) \equiv dv^\ell \pmod{m}.$$

Note that m is coprime to $d\sigma(d)$, by our lower bound on z, and coprime to $\sigma(v^\ell)$ by Lemma 4.1. Thus, if we place v in a coprime residue class modulo

m, then there is a uniquely determined coprime residue class modulo m which $\sigma(v^\ell)$ must belong to for us to have $m \mid s(dv^\ell)$. We now apply Lemma 4.2: Since there are $\varphi(m)$ possible residue classes modulo m in which we can place v, we find that the set of v for which $m \mid s(dv^\ell)$ has relative density

$$\varphi(m)\prod_{p\mid m}\frac{p}{(p+1)(p-1)^2}=\prod_{p\mid m}\frac{p}{p^2-1}.$$

It follows (letting m range over the squarefree numbers composed of primes in $\mathscr{P}$) that $\mathscr{S}_1$ has relative density

$$\sum_{\substack{\mathscr{Q}\subset\mathscr{P}\\ \#\mathscr{Q}\le R}}\Bigg(\prod_{q\in\mathscr{Q}}\frac{q}{q^2-1}\Bigg)\Bigg(\prod_{p\in\mathscr{P}\setminus\mathscr{Q}}\bigg(1-\frac{p}{p^2-1}\bigg)\Bigg)$$
$$=\sum_{\substack{\mathscr{Q}\subset\mathscr{P}\\ \#\mathscr{Q}\le R}}\Bigg(\prod_{q\in\mathscr{Q}}\frac{q}{q^2-q-1}\Bigg)\prod_{p\in\mathscr{P}}\bigg(1-\frac{p}{p^2-1}\bigg).$$

Now

$$\prod_{p\in\mathscr{P}}\bigg(1-\frac{p}{p^2-1}\bigg)\le\prod_{p\in\mathscr{P}}\bigg(1-\frac{1}{p}\bigg)\ll_\ell\bigg(\frac{\log z}{z}\bigg)^{1/\varphi(\ell+1)},$$

where the last product is bounded using the estimate

$$\sum_{\substack{p\le y\\ p\equiv 2\ (\mathrm{mod}\ \ell+1)}}\frac{1}{p}=\frac{1}{\varphi(\ell+1)}\log_2 y+O_\ell(1)$$

(see [17, pp. 449–450]). Also, by the multinomial theorem,

$$\sum_{\substack{\mathscr{Q}\subset\mathscr{P}\\ \#\mathscr{Q}\le R}}\prod_{q\in\mathscr{Q}}\frac{q}{q^2-q-1}\le\sum_{0\le r\le R}\frac{1}{r!}\Bigg(\sum_{p\in\mathscr{P}}\frac{p}{p^2-p-1}\Bigg)^r$$
$$\ll_R\Bigg(1+\sum_{p\in\mathscr{P}}\frac{1}{p}\Bigg)^R\ll_R(\log z)^R.$$

Thus, $\mathscr{S}_1$ has relative density

$$\ll_{\ell,R}(\log z)^R\bigg(\frac{\log z}{z}\bigg)^{1/\varphi(\ell+1)}\ll(\log z)^{R+1}z^{-1/\varphi(\ell+1)}.$$

We turn now to $\mathscr{S}_2$. Let $p\in\mathscr{P}$. Arguing as in the treatment of $\mathscr{S}_1$, but replacing Lemma 4.2 with Lemma 4.3, we see that the set of v for which $p^2\mid s(dv^\ell)$ has relative density

$$\varphi(p^2)\frac{1}{p(p-1)^2(p+1)}=\frac{1}{p^2-1}.$$

Thus, $\mathscr{S}_2$ has upper relative density

$$\ll \sum_{p\in\mathscr{P}} \frac{1}{p^2-1} \ll \frac{1}{z}.$$

Combining our estimates, we see that $\mathscr{S}_1 \cup \mathscr{S}_2$ has upper relative density

$$\ll_{\ell,R} (\log z)^{R+1} z^{-1/\varphi(\ell+1)} + z^{-1},$$

and so can be made arbitrarily small. This completes the proof. ∎

6. Powerful sociable numbers. Let s_k denote the kth iterate of the sum of proper divisors function s. A natural number n is called *sociable* if the sequence $n, s(n), s_2(n), \ldots$ is purely periodic; if the period length is k, we call the set $\{n, s(n), \ldots, s_{k-1}(n)\}$ a *sociable k-cycle*, and we call n a *k-sociable number*. Sociable numbers should be viewed as higher-order generalizations of perfect and amicable numbers, which correspond respectively to the cases $k = 1$ and $k = 2$. For the history of sociable numbers, and some recent theoretical results, see [16].

We conclude this paper by proving an analogue of Theorem 1.1 for ℓ-full k-sociable numbers, when k is odd.

Theorem 6.1. *Fix an integer $\ell \geq 1$ and an odd integer $k \geq 1$. The number of k-sociable ℓ-full numbers $n \leq x$ is $o(x^{1/\ell})$, as $x \to \infty$.*

When $k = 1$, the result of Theorem 6.1 follows (with any ℓ) from the theorem of Hornfeck and Wirsing [15] that the number of perfect numbers in $[1, x]$ is $\ll_\epsilon x^\epsilon$ for any fixed $\epsilon > 0$. For the rest of the proof, we assume that k and ℓ are fixed with $k > 1$ and $\ell \geq 1$. We fix an $(\ell+1)$-full number d. We use v exclusively to denote a squarefree number prime to d. It suffices to show that as $x \to \infty$, only $o(x)$ values of $v \leq x$ are such that dv^ℓ is k-sociable.

The following lemma, which appears as [20, Lemma 5], is a variant of an observation due to Dickson (see [2]). If $\mathscr{S}$ is a set of natural numbers, we write $\gcd(\mathscr{S})$ for the greatest common divisor of the elements of $\mathscr{S}$, and we write $\sigma(\mathscr{S})$ for the set $\{\sigma(m) : m \in \mathscr{S}\}$.

Lemma 6.2. *Let $\mathscr{C}$ be a sociable cycle of odd order > 1. Then* $\gcd(\sigma(\mathscr{C}))$ *divides* $\gcd(\mathscr{C})$, *except possibly if* $2 \parallel \gcd(\sigma(\mathscr{C}))$, *in which case* $\frac{1}{2}\gcd(\sigma(\mathscr{C}))$ *divides* $\gcd(\mathscr{C})$.

The next lemma follows from elementary sieving (for details, see [21, Lemma 8.13]).

Lemma 6.3. *Let $\mathscr{Q}$ be a set of primes for which $\sum_{q\in\mathscr{Q}} 1/q$ diverges. For each $\epsilon > 0$, there is a $y > 0$ for which the following holds: For all natural numbers m, excluding a set of upper density $< \epsilon$, there is a prime $q \in \mathscr{Q} \cap [2, y]$ for which $q \parallel m$.*

Put

$$G(T) := T^{\ell} + T^{\ell-1} + \cdots + 1,$$

and put

$$\mathscr{P} := \{p : p > d,\ p \equiv 1 \pmod{\ell+1}\}.$$

LEMMA 6.4. *If M is a natural number all of whose prime divisors belong to $\mathscr{P}$, then G has a root modulo M.*

Proof. It is enough to verify this when $M = p^e$, where $p \in \mathscr{P}$ and $e \geq 1$. Observe that $(\mathbb{Z}/p^e\mathbb{Z})^{\times}$ is cyclic and that $\ell+1 \mid p-1 \mid (\mathbb{Z}/p^e\mathbb{Z})^{\times}$. Thus, there is an integer a whose order modulo p^e is precisely $\ell + 1$. For this a,

$$p^e \mid a^{\ell+1} - 1 = G(a)(a-1). \tag{21}$$

If p divides $a - 1$, then the order of a modulo p^e divides p^{e-1}; i.e., $\ell + 1$ divides p^{e-1}. But this is absurd, since p is a prime with $p > \ell + 1$. So p is coprime to $a - 1$, and we deduce from (21) that $G(a) \equiv 0 \pmod{p^e}$. ∎

The next lemma should be compared with [7, Lemma 1] (see also [21, Lemma 8.19]).

LEMMA 6.5. *Let K be a nonnegative integer. Let M be a natural number all of whose prime divisors belong to $\mathscr{P}$. Then the following is true for almost all natural numbers m: There are primes $p_0, p_1, \ldots, p_K \in \mathscr{P}$ for which*

$$p_i \,\|\, m \quad \textit{for each } i = 0, 1, \ldots, K, \tag{22}$$

and

$$G(p_0) \equiv 0 \pmod{M}, \qquad G(p_{i+1}) \equiv 0 \pmod{p_i^{2\ell}} \quad \textit{for all } 0 \leq i < K. \tag{23}$$

Proof. The lemma is a consequence of the following assertion, which we prove by induction on K:

> For each nonnegative integer K, each natural number M supported on the primes in $\mathscr{P}$, and each $\epsilon > 0$, there is a number B with the property that for all m outside of a set of upper density $< \epsilon$, one can find primes $p_0, \ldots, p_K \in \mathscr{P} \cap [1, B]$ satisfying both (22) and (23).

To start with, suppose that $K = 0$. By Lemma 6.4, there is an integer r for which $G(r) \equiv 0 \pmod{M}$. Clearly r is coprime to M. The result in this case follows from Lemma 6.3, if we take $\mathscr{Q} := \{p \in \mathscr{P} : p \equiv r \pmod{M}\}$. That the sum of the reciprocals of the primes in $\mathscr{Q}$ diverges follows from Dirichlet's theorem.

Now suppose the statement is known to hold for a certain $K \geq 0$. Suppose $\epsilon > 0$ and M are given. By the induction hypothesis, we can choose a number B_0 with the property that for all m outside of a set $\mathscr{E}_0$ (say) of upper density $< \epsilon/2$, there are primes $p_0, \ldots, p_K \in \mathscr{P} \cap [1, B_0]$ satisfying

(22) and (23). Let $P := (\prod_{p\in\mathscr{P}\cap[1,B_0]} p)^{2\ell}$. Let r' be a root of G modulo P, and apply Lemma 6.3 with

$$\mathscr{Q} := \{p \in \mathscr{P} : p \equiv r' \pmod{P}\}.$$

We find that for a suitable choice of y, all m outside of a set $\mathscr{E}_1$ (say) of upper density $< \epsilon/2$ have an exact prime divisor $p_{K+1} \equiv r' \pmod{P}$ with $p_{K+1} \in \mathscr{P} \cap [1, y]$. But then if n lies outside $\mathscr{E}_0 \cup \mathscr{E}_1$, the primes $p_0, \ldots, p_{K+1}$ satisfy (22) and (23) with K replaced by $K + 1$. Since $\mathscr{E}_0 \cup \mathscr{E}_1$ has upper density $< \epsilon$, we obtain the $(K + 1)$-case of the assertion with $B = \max\{B_0, y\}$. ∎

Proof of Theorem 6.1. Fix a real number z so large that $\sum_{p\in\mathscr{P},\, p\le z} \frac{1}{p} > 1$. Then, putting $M := \prod_{p\in\mathscr{P},\, p\le z} p$, we have

$$\frac{\sigma(M)}{M} = \prod_{\substack{p\in\mathscr{P}\\ p\le z}} \left(1 + \frac{1}{p}\right) \ge 1 + \sum_{\substack{p\in\mathscr{P}\\ p\le z}} \frac{1}{p} > 2.$$

So $s(M) > M$, i.e., M is an *abundant number*.

Suppose that dv^ℓ is k-sociable. With $K := k - 1$, assume that there are primes $p_0, \ldots, p_K \in \mathscr{P}$ all exactly dividing v and satisfying (23). Since $p_i \parallel v$ for each $i = 0, \ldots, K$, we have

$$p_i^\ell \parallel dv^\ell \tag{24}$$

for each $i = 0, \ldots, K$. Consequently, for each $i = 0, \ldots, K - 1$,

$$p_i^{2\ell} \mid G(p_{i+1}) = \sigma(p_{i+1}^\ell) \mid \sigma(dv^\ell). \tag{25}$$

Since $s(dv^\ell) = \sigma(dv^\ell) - dv^\ell$, it follows from (24) and (25) that $p_i^\ell \parallel s(dv^\ell)$ for all $i = 0, \ldots, K - 1$. Iterating the same argument, we find that $s_i(dv^\ell)$ is exactly divisible by each of $p_0^\ell, \ldots, p_{K-i}^\ell$ for each $0 \le i \le K$. In particular, p_0^ℓ exactly divides each of $dv^\ell, s(dv^\ell), s_2(dv^\ell), \ldots, s_K(dv^\ell)$; in other words, p_0^ℓ exactly divides each member of the sociable cycle $\mathscr{C}$ (say) containing dv^ℓ. Thus,

$$M \mid G(p_0) = \sigma(p_0^\ell) \mid \gcd(\sigma(\mathscr{C})).$$

Since M is odd, it follows from Lemma 6.2 that

$$M \mid \gcd(\mathscr{C}).$$

Since M is abundant, this implies that each member of $\mathscr{C}$ is abundant. But a cycle cannot consist entirely of abundant terms. This contradiction shows that v cannot have prime divisors $p_0, \ldots, p_K$ as above. By Lemma 6.5, v is restricted to a set of asymptotic density zero. ∎

Acknowledgements. The author thanks Carl Pomerance for helpful comments on the content and presentation of this paper. This work was conducted during a visit of the author to Dartmouth College. He would like to express his appreciation to the faculty and staff of the Dartmouth mathematics department for their hospitality. He also acknowledges the generous support of the National Science Foundation (under award DMS-0802970).

REFERENCES

[1] H. Davenport, *On character sums in finite fields*, Acta Math. 71 (1939), 99–121.

[2] L. E. Dickson, *Theorems and tables on the sum of divisors of a number*, Quart. J. Math. 44 (1913), 264–296.

[3] P. Erdős, *On the density of the abundant numbers*, J. London Math. Soc. 9 (1934), 278–282.

[4] —, *On primitive abundant numbers*, ibid. 10 (1935), 49–58.

[5] —, *On amicable numbers*, Publ. Math. Debrecen 4 (1955), 108–111.

[6] —, *Remarks on number theory. I. On primitive α-abundant numbers*, Acta Arith. 5 (1958), 25–33.

[7] —, *On asymptotic properties of aliquot sequences*, Math. Comp. 30 (1976), 641–645.

[8] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, *On the normal behavior of the iterates of some arithmetic functions*, in: Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math. 85, Birkhäuser Boston, Boston, MA, 1990, 165–204.

[9] P. Erdős und G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Litt. Sci. Szeged 7 (1934), 95–102.

[10] P. Erdős and A. Wintner, *Additive arithmetical functions and statistical independence*, Amer. J. Math. 61 (1939), 713–721.

[11] M. García, J. M. Pedersen, and H. te Riele, *Amicable pairs, a survey*, in: High Primes and Misdemeanours: Lectures in Honour of the 60th Birthday of Hugh Cowie Williams, Fields Inst. Comm. 41, Amer. Math. Soc., Providence, RI, 2004, 179–196.

[12] A. A. Gioia and A. M. Vaidya, *Amicable numbers with opposite parity*, Amer. Math. Monthly 74 (1967), 969–973; correction, ibid. 75 (1968), 386.

[13] G. Halász, *On the distribution of additive and the mean values of multiplicative arithmetic functions*, Studia Sci. Math. Hungar. 6 (1971), 211–233.

[14] R. R. Hall, *A sharp inequality of Halász type for the mean value of a multiplicative arithmetic function*, Mathematika 42 (1995), 144–157.

[15] B. Hornfeck und E. Wirsing, *Über die Häufigkeit vollkommener Zahlen*, Math. Ann. 133 (1957), 431–438.

[16] M. Kobayashi, P. Pollack, and C. Pomerance, *On the distribution of sociable numbers*, J. Number Theory 129 (2009), 1990–2009.

[17] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2nd ed., Chelsea, New York, 1953.

[18] F. Luca and D. E. Iannucci, *Squares whose aliquot sum is a square*, JP J. Algebra Number Theory Appl. 13 (2009), 17–26.

[19] J. M. Pedersen, *Tables of aliquot cycles*, http://amicable.homepage.dk/tables.htm.

[20] P. Pollack, *A remark on sociable numbers of odd order*, J. Number Theory 130 (2010), 1732–1736.

[21] —, *Not Always Buried Deep: A Second Course in Elementary Number Theory*, Amer. Math. Soc., Providence, RI, 2009.

[22] C. Pomerance, *On the distribution of amicable numbers*, J. Reine Angew. Math. 293/294 (1977), 217–222.

[23] C. Pomerance, *On the distribution of amicable numbers. II*, ibid. 325 (1981), 183–188.
[24] D. Wan, *Generators and irreducible polynomials over finite fields*, Math. Comp. 66 (1997), 1195–1212.

Paul Pollack
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street
Urbana, IL 61801, U.S.A.
E-mail: pppollac@illinois.edu

Received 9 July 2010;
revised 27 September 2010 (5399)