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AN ELEMENTARY APPROACH TO NONEXISTENCE OF SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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Dedicated to the cherished memory of Andrzej Hulanicki

Abstract. This note presents an elementary approach to the nonexistence of solutions of linear parabolic initial-boundary value problems considered in the Feller test.

1. Introduction. Our goal is to study the solvability of the initial-boundary value problem for a linear partial differential equation of parabolic type in one space dimension of the form

$$u_t = a(x)u_{xx} + b(x)u_x, \tag{1}$$

$$u(0,t) = 0, \quad u(1,t) = 1, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

where $x \in (0,1)$, $t > 0$, a and b are given measurable nonnegative functions on $(0,1)$, and u_0 is a continuous nonnegative function satisfying the compatibility conditions $u_0(0) = 0$, $u_0(1) = 1$.

The question of the existence of a solution satisfying the boundary conditions (2) is delicate. William Feller developed tools which permitted him to give a complete answer to that question in his seminal 1952 paper [5]. The theory presented there, culminating in the famous *Feller test*, is by no means elementary, and is based on a fine probabilistic analysis of stochastic processes associated with linear parabolic equations with diffusion and drift of the form (1) considered on $(\ell, r) \subset \mathbb{R}$, $-\infty \le \ell < r \le +\infty$, with initial conditions as in (3) and suitable boundary conditions even more general than (2).

The question of solvability of (1) depends on the relations between the given measurable nonnegative diffusion and drift coefficients a, b. Without entering into details one might say that b/a should not be "too large" near the boundaries in order to have solutions of (1) satisfying (2). Precise conditions

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on a and b are the content of the Feller test for (1) (cf., e.g., [7, Ch. 15, Sec. 6], [6, Th. 5.29], [9, Ch. II, Th. 2.6]). In fact, the Feller test says that solutions of the *evolution* problem (1)–(3) can be constructed if and only if a *stationary* solution of (1)–(2) exists. Intuitively, if either the diffusion coefficient a is very small or the drift coefficient b is very large in the vicinity of, say, $x = 0$, then the Dirichlet boundary condition (2) at $x = 0$ cannot be preserved; a kind of instantaneous loss of boundary conditions occurs. Another point of view is that solutions of (1) with $u(0,t)$ kept fixed must be constant.

Let us stress that for *linear* autonomous parabolic equations like (1) solutions of the initial-boundary value problem either exist globally in time or do not exist at all. In the case of *nonlinear* parabolic equations, one may encounter another behavior of solutions: they may exist locally in time but may cease to exist after some time (see [1], [3] for examples).

Our note is devoted to an *elementary analytic* proof of *nonexistence* of solutions to (1)–(3) in Section 2 under suitable assumptions on the coefficients a and b. This extends the result on the equation $u_t = (xu_x)_x$ in [4].

The *existence* part in the Feller test involves much more technical tools from the theory of semigroups of linear operators. The only entirely analytic proof of the Feller test we know can be found in the monograph [8], and it is neither short nor elementary. In fact, a pretty large part of the general Hille–Yosida–Phillips semigroup theory has been developed to handle the solvability questions for (1) supplemented with boundary conditions.

According to our experience, the Feller test is also useful in the analysis of (nonexistence issues for) *nonlinear* elliptic and parabolic equations: see examples of application in [2, Prop. 3], [3, Th. 1], [1]. However, this powerful tool seems not to be widely known in the parabolic PDEs' community.

2. Main result. First, we rewrite the problem (1)–(3) in the self-adjoint form

$$(Wu)_t = a(Wu_x)_x, \tag{4}$$

$$u(0,t) = 0, \qquad u(1,t) = 1, \tag{5}$$

$$u(x,0) = u_0(x), \tag{6}$$

where the function W is defined by

$$W(x) = \exp\left(-\int_x^1 \frac{b(z)}{a(z)}\,dz\right). \tag{7}$$

Moreover, we define the so-called *scale function*

$$S(x) = e + \int_x^1 \frac{dz}{W(z)}. \tag{8}$$

Here, we assume that a and b are measurable functions such that $0 \le a^{-1}$ and $0 \le b$ are both bounded from above on $(\mu, 1]$ for each $\mu > 0$. In particular, $(1+b)/a$ is locally integrable on $(0,1)$, a standing assumption in the Feller test. Thus, we may encounter difficulties only in the vicinity of $x = 0$.

Recall that the Feller test in the context of the initial-boundary value problem (4)–(6) for partial differential equations reads (cf. [6–9]): If the scale function S is unbounded, i.e. the integral $\int_\varepsilon^1 dx/W(x)$ diverges as $\varepsilon \searrow 0$ (so that, in particular, $\lim_{x\searrow 0} W(x) = 0$ since W is monotone, or, in other words, $\lim_{\varepsilon\searrow 0} \int_\varepsilon^1 (b(x)/a(x))\, dx = \infty$), then the Dirichlet boundary condition (5) at $x = 0$ cannot be satisfied for solutions of the linear parabolic equation (4). In probabilistic terms, one says that $x = 0$ corresponds either to an *entrance point* or to a *natural boundary* for the diffusion process associated with (1). Roughly speaking, in that interpretation, the point $x = 0$ is not accessible from the interior of the domain $(0,1)$ for the trajectories of that process. Thus, no boundary condition at $x = 0$ can be imposed.

Now, under slightly more restrictive conditions (12), (13)–(14) below which simplify the reasoning, we formulate a result close in spirit to the Feller test, proved below with a purely analytical argument.

THEOREM. *Assume that*

$$0 \le a^{-1}, b \in L^\infty(\mu, 1) \quad \textit{for each } \mu > 0, \tag{9}$$

$$\lim_{\varepsilon\searrow 0} S(\varepsilon) = \infty, \tag{10}$$

the integrals

$$0 \le I_k = \int_0^1 \frac{W(x)}{a(x)} S(x)(\log S(x))^k\, dx \tag{11}$$

exist for each $k = 0, 1, 2, \ldots,$ and satisfy the growth estimate

$$I_k \le C^k k! \tag{12}$$

for some constant C. Moreover, suppose that the quantities

$$\frac{W(x)^2}{a(x)} S(x)^2 (\log S(x))^2 \tag{13}$$

as well as

$$\frac{x}{W(x)S(x)} \tag{14}$$

are bounded on $(0,1)$. Then the problem (1)–(3) cannot possess any classical local in time solution u.

Comments on the assumptions. (10) is the original condition in the Feller test. Relation (12) is satisfied quite frequently, e.g., if

$$\left(\frac{W^2S^2}{a}\right)_x \geq \varrho_0\,\frac{WS}{a} \tag{15}$$

for a number $\varrho_0 > 0$. Indeed, by assumptions (9) and (15),

$$\begin{aligned}\int_\varepsilon^1 \frac{W(x)}{a(x)} S(x)(\log S(x))^k\,dx &= -\frac{1}{k+1}\int_\varepsilon^1 \frac{W(x)^2}{a(x)} S(x)^2((\log S(x))^{k+1})_x\,dx\\ &= \frac{1}{k+1}\,\frac{W(\varepsilon)^2}{a(\varepsilon)} S(\varepsilon)^2(\log S(\varepsilon))^{k+1} - \frac{1}{k+1}\,\frac{e^2}{a(1)}\\ &\quad + \frac{1}{k+1}\int_\varepsilon^1 \left(\frac{W(x)^2S(x)^2}{a(x)}\right)_x (\log S(x))^{k+1}\,dx,\end{aligned}$$

so the inequality $(k+1)I_k + \delta \geq \varrho_0 I_{k+1}$ holds for $\delta = e^2/a(1) > 0$ and some $\varrho_0 > 0$. The estimate (12) is then an immediate consequence of that inequality.

Note that the relation

$$\inf_{x\in(0,1)} -\sqrt{a(x)}(\log\log S(x))_x \left(= \inf_{x\in(0,1)} \sqrt{a(x)}\frac{1}{W(x)S(x)\log S(x)}\right) > 0$$

is equivalent to the boundedness of (13).

DEFINITION. To fix ideas, we recall that a *classical* solution is a function $u \in C^{2,1}_{x,t}((0,1)\times(0,T))$ for some $T > 0$, satisfying (1) pointwise, and (1)–(3) as limits

$$\lim_{x\searrow 0} u(x,t) = 1 - \lim_{x\nearrow 1} u(x,t) = 0, \qquad \lim_{t\searrow 0} u(x,t) = u_0(x).$$

Two particular cases of the Theorem have been touchstones in the proof below.

EXAMPLE 1. All the assumptions of the Theorem are satisfied for the equation $u_t = xu_{xx} + \beta u_x$ with $\beta > 1$. Here $W(x) = x^\beta$ and $S(x) \sim x^{1-\beta}$. In the case $\beta = 1$ we have the equation $u_t = (xu_x)_x$ with $S(x) = e - \log x$. The Theorem in [4] covers that case, with a proof slightly different than the one below. Note that if $\beta < 1$, then this equation supplemented with (5)–(6) does have a solution.

EXAMPLE 2. Similarly, if $a(x) = x^\sigma$, $b(x) = \sigma x^{\sigma-1}$, i.e. (1) becomes $u_t = (x^\sigma u_x)_x$, with $\sigma \in (1,2)$, the assumptions of the Theorem are satisfied. Here $W(x) = x^\sigma$, and again $S \sim x^{1-\sigma}$. In particular, in both examples $I_k \leq C^k k!$ with some $C > 0$, as in [4].

Proof of Theorem. Assume the existence of a solution $u(x,t)$ to the problem (1)–(3) on an interval $[0,T)$ with some $T > 0$. For $\varepsilon \geq 0$, $0 \leq t \leq T$ and

$k = 0, 1, 2, \ldots$ we define

$$V_\varepsilon^k(t) = \int_0^t\int_\varepsilon^1 \frac{W(x)}{a(x)} S(x)(\log S(x))^k u(x,s)\,dx\,ds. \tag{16}$$

One may call $V_\varepsilon^k(t)$, $\varepsilon > 0$, the truncated *regularized logarithmic moments* of u. Our goal is to show for each $k = 1, 2, \ldots$ a differential inequality resembling

$$\frac{d}{dt} V_0^k \geq \varrho k(k-1) V_0^k + \gamma$$

with some $\varrho, \gamma > 0$ independent of k.

Of course, for any classical solution we have, by the maximum principle, $0 \leq u(x,t) \leq M \equiv \max_{0 \leq x \leq 1} u_0(x)$, therefore $V_0^k \leq Mt\, I_k$ are finite (see (11)).

We begin with some preliminary estimates. Integrating (4) over $(\varepsilon, \eta) \times (0,t)$, $0 < \varepsilon < \eta < 1$, we obtain

$$\int_\varepsilon^\eta \frac{W(x)}{a(x)}(u(x,t) - u_0(x))\,dx = \int_0^t (W(\eta)u_x(\eta,s) - W(\varepsilon)u_x(\varepsilon,s))\,ds. \tag{17}$$

Passing to the limit $\eta \nearrow 1$ in (17), we get the existence of a finite limit

$$H(t) \equiv \lim_{\eta \nearrow 1} W(\eta) \int_0^t u_x(\eta,s)\,ds.$$

Therefore, the limit

$$A(t) \equiv \lim_{\varepsilon \searrow 0} W(\varepsilon) \int_0^t u_x(\varepsilon,s)\,ds$$

exists, and we have

$$A(t) = H(t) + \int_0^1 \frac{W(x)}{a(x)}(u_0(x) - u(x,t))\,dx; \tag{18}$$

note that $\int_0^1 (W(x)/a(x))\,dx < \infty$ by the finiteness of (11). Next, multiplying (4) by S, integrating over $(\varepsilon, \eta) \times (0,t)$ and passing to the limit as $\eta \nearrow 1$, we get

$$\begin{aligned} W(\varepsilon)S(\varepsilon)\int_0^t u_x(\varepsilon,s)\,ds = \int_\varepsilon^1 &\frac{W(x)}{a(x)} S(x)(u_0(x) - u(x,t))\,dx \\ &+ eH(t) + t - \int_0^t u(\varepsilon,s)\,ds. \end{aligned} \tag{19}$$

Since by the assumption (11), $\int_0^1 (W(x)/a(x))S(x)\,dx < \infty$, the right hand side of (19) has a finite limit $B(t)$ as $\varepsilon \searrow 0$, so that $A(t) = 0$, because of (10). Moreover, $H(t)$ is uniformly bounded in t. Next, by (18), we deduce from (17) with $\eta = 1$ that

$$0 \le W(\varepsilon)\int_0^t u_x(\varepsilon, s)\,ds = \int_0^\varepsilon \frac{W(x)}{a(x)}(u(x,t) - u_0(x))\,dx,$$

and thus

$$\begin{aligned}(20)\qquad 0 \le B(t) &\le \limsup_{\varepsilon\searrow 0} S(\varepsilon)\int_0^\varepsilon \frac{W(x)}{a(x)}|u(x,t) - u_0(x)|\,dx \\ &\le \limsup_{\varepsilon\searrow 0}\int_0^\varepsilon \frac{W(x)}{a(x)}S(x)|u(x,t) - u_0(x)|\,dx = 0,\end{aligned}$$

because of the finiteness of I_0.

Remark. At this stage, we may easily deduce from (19) with $\varepsilon \searrow 0$ that if u exists at time t then

$$\begin{aligned}(21)\qquad t + eH(t) &= \int_0^1 \frac{W(x)}{a(x)}S(x)(u(x,t) - u_0(x))\,dx \\ &\le M\int_0^1 \frac{W(x)}{a(x)}S(x)\,dx < \infty,\end{aligned}$$

so that the maximal time of existence T of u is finite. We will prove below that actually $T = 0$.

As a further consequence of the reasoning in (20), we obtain the following regularity result for any solution u and each $k = 1, 2, \ldots:$

$$(22)\qquad \lim_{\varepsilon\searrow 0} W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k\int_0^t u_x(\varepsilon, s)\,ds = 0$$

and

$$(23)\qquad \lim_{\varepsilon\searrow 0}(\log S(\varepsilon))^k\int_0^t u(\varepsilon, s)\,ds = 0.$$

Indeed, (22) follows from

$$\begin{aligned}0 &\le W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k\int_0^t u_x(\varepsilon, s)\,ds \\ &\le \int_0^\varepsilon \frac{W(x)}{a(x)}S(x)(\log S(x))^k|u(x,t) - u_0(x)|\,dx \to 0\end{aligned}$$

as $\varepsilon \searrow 0$, because of the finiteness of I_k in (11), just as in (20).

Next, using (19) and (21) we may estimate

$$
\begin{aligned}
(24)\qquad 0 &\le (\log S(\varepsilon))^k \int_0^t u(\varepsilon,s)\,ds \\
&= (\log S(\varepsilon))^k (t+eH(t)) \\
&\quad - W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k \int_0^t u_x(\varepsilon,s)\,ds \\
&\quad + (\log S(\varepsilon))^k \int_\varepsilon^1 \frac{W(x)}{a(x)} S(x)(u_0(x)-u(x,t))\,dx \\
&= -W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k \int_0^t u_x(\varepsilon,s)\,ds \\
&\quad + (\log S(\varepsilon))^k \int_0^\varepsilon \frac{W(x)}{a(x)} S(x)(u(x,t)-u_0(x))\,dx \\
&\le -W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k \int_0^t u_x(\varepsilon,s)\,ds \\
&\quad + \int_0^\varepsilon \frac{W(x)}{a(x)} S(x)(\log S(x))^k |u(x,t)-u_0(x)|\,dx \to 0
\end{aligned}
$$

as $\varepsilon \searrow 0$, by the finiteness of I_k and (22).

Calculating the evolution of V_ε^k, we obtain, for each $k \ge 2$,

$$
\begin{aligned}
\frac{d}{dt}V_\varepsilon^k(t) - v_\varepsilon^k(0) &= \int_0^t\int_\varepsilon^1 S(x)(\log S(x))^k (W(x)u_x(x,s))_x\,dx\,ds \\
&= \int_0^t W(x)S(x)(\log S(x))^k u_x(x,s)\,\Big|_\varepsilon^1\ ds \\
&\quad - \int_0^t\int_\varepsilon^1 (S(x)(\log S(x))^k)_x W(x)u_x(x,s)\,dx\,ds \\
&\ge -\int_0^t W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k u_x(\varepsilon,s)\,ds \\
&\quad + \int_0^t\int_\varepsilon^1 (\log S(x))^k u_x(x,s)\,dx\,ds \\
&\quad + k\int_0^t\int_\varepsilon^1 (\log S(x))^{k-1} u_x(x,s)\,dx\,ds
\end{aligned}
$$

$$\begin{aligned}
&\ge -\int_0^t W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k u_x(\varepsilon,s)\,ds\\
&\quad - k\int_0^t (\log S(\varepsilon))^{k-1}u(\varepsilon,s)\,ds\\
&\quad + k(k-1)\int_0^t\int_\varepsilon^1 \frac{1}{W(x)S(x)}(\log S(x))^{k-2}u(x,s)\,dx\,ds\\
&\ge -W(\varepsilon)S(\varepsilon)(\log S(\varepsilon))^k\int_0^t u_x(\varepsilon,s)\,ds\\
&\quad - k\int_0^t (\log S(\varepsilon))^{k-1}u(\varepsilon,s)\,ds\\
&\quad + \varrho k(k-1)\int_0^t\int_\varepsilon^1 \frac{W(x)}{a(x)}S(x)(\log S(x))^k u(x,s)\,dx\,ds
\end{aligned}$$

with

$$v_\varepsilon^k(0) = \int_\varepsilon^1 \frac{W(x)}{a(x)}S(x)(\log S(x))^k u_0(x)\,dx,$$

because, by (13), the inequality

$$\frac{1}{W(x)S(x)} \ge \varrho\frac{W(x)}{a(x)}S(x)(\log S(x))^2$$

is satisfied for each $x\in(0,1)$ and some $\varrho>0$. Passing to the limit as $\varepsilon\searrow 0$ and using (22)–(23), we arrive at

$$\frac{d}{dt}V_0^k(t) \ge \varrho k(k-1)V_0^k(t) + v_0^k(0)$$

with

$$v_0^k(0) = \int_0^1 \frac{W(x)}{a(x)}S(x)(\log S(x))^k u_0(x)\,dx \ge \int_0^1 \frac{W(x)}{a(x)}S(x)u_0(x)\,dx \equiv \gamma > 0.$$

This leads immediately to

$$V_0^k(t) \ge v_0^k(0)\frac{e^{\varrho k(k-1)t}-1}{\varrho k(k-1)}$$

for $0<t<T$. On the other hand, from the obvious relation

$$V_0^k(t) \le I_k\,Mt \le I_k\,MT,$$

taking logarithms, we obtain for large k

$$\varrho k(k-1)t \le \log(I_k k^2 MT\varrho\gamma^{-1}).$$

Now, the Stirling formula

$$n! \sim \sqrt{2\pi}\, n^{n+1/2} e^{-n} \quad \text{as } n \to \infty \tag{25}$$

immediately implies that $\log I_k \le Ck + (k+1/2)\log k$, and therefore

$$t \le T \le \liminf_{k\to\infty} Ck^{-2} \log(I_k k^2) = 0$$

for a constant C, contradicting the existence of any local in time solution u satisfying (5). Note that the natural assumption (12) can be relaxed, e.g. to $\lim_{k\to\infty} k^{-2}\log I_k = 0$. ∎

Remark. It can be checked that the proof of the Theorem also applies to *weak* solutions from the Sobolev space $H^1(0,1)$, i.e. those satisfying $u - x \in L^\infty((0,T), H^1_0(0,1))$ and the integral identity

$$-\int_0^1 \frac{W(x)}{a(x)} u_0(x)\varphi(x,0)\,dx - \int_0^T\!\!\int_0^1 \frac{W(x)}{a(x)} u(x,t)\varphi_t(x,t)\,dx\,dt = -\int_0^T\!\!\int_0^1 W(x)u_x(x,t)\varphi_x(x,t)\,dx\,dt$$

for each test function $\varphi \in C^1((0,1)\times[0,T))$ with compact support in $(0,1)\times[0,T)$.

Of course, classical solutions solve the problem (1)–(3) also in the sense of semigroup theory (in the spaces of continuous functions on $(0,1)$) considered by Feller, i.e. they are *mild* solutions, which, in turn, are also weak solutions.

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