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GENERALIZED RIESZ PRODUCTS PRODUCED FROM ORTHONORMAL TRANSFORMS

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Abstract. Let $\mathcal{M}_p = \{m_k\}_{k=0}^{p-1}$ be a finite set of step functions or real valued trigonometric polynomials on $\mathbb{T} = [0,1)$ satisfying a certain orthonormality condition. We study multiscale generalized Riesz product measures μ defined as weak-* limits of elements $\mu_N \in V_N$ ($N \in \mathbb{N}$), where V_N are p^N-dimensional subspaces of $L_2(\mathbb{T})$ spanned by an orthonormal set which is produced from dilations and multiplications of elements of $\mathcal{M}_p$ and $\overline{\bigcup_{N\in\mathbb{N}} V_N} = L_2(\mathbb{T})$. The results involve mutual absolute continuity or singularity of such Riesz products extending previous results on multiscale Riesz products.

1. Introduction. Riesz products provide a strong tool for the construction of singular measures on compact Abelian groups. In 1918 F. Riesz proved that the pointwise limit of the sequence of functions

$$f_N(x) = \int_0^x \prod_{n=1}^{N} (1 + \cos(2\pi 4^n \gamma))\, d\gamma \tag{1.1}$$

is a continuous function f of bounded variation on $\mathbb{T}$ whose Fourier–Stieltjes coefficients do not vanish at infinity. Hereafter we denote by $\mathbb{T}$ the quotient $\mathbb{R}/\mathbb{Z}$. We identify elements of the space $L_2(\mathbb{T})$ with 1-periodic functions over $\mathbb{R}$. The space $L_2(\mathbb{T})$ is equipped with the usual inner product $\langle \cdot, \cdot \rangle_{L_2}$ and norm $\|\cdot\|_{L_2}$. Also we denote by $\|\cdot\|_\infty$ the norm in the space of all measurable essentially bounded functions on $\mathbb{T}$.

Over the years Riesz's original construction was broadened and many examples of measures with desired properties were produced. Let us briefly discuss the basic modifications and extensions of (1.1). A. Zygmund [16] introduced a bounded sequence $\{a_j\}_{j\in\mathbb{N}}$ of coefficients and a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of scales and he proved that the sequence

$$\mu_N(\gamma) = \prod_{j=1}^{N} (1 + a_j \cos(2\pi \lambda_j \gamma))$$

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converges to a measure μ in the weak-* topology of the Banach algebra $\mathcal{M}(\mathbb{T})$ of all bounded measures on $\mathbb{T}$. In addition a characterization of mutual absolute continuity or singularity of two such measures was established according to whether the sequence $\{a_j\}_{j\in\mathbb{N}}$ was square summable or not. This was the first of the so-called dichotomy results (see [5, 15] and references therein). Another basic modification of (1.1) was to replace the generator function $\cos(2\pi\gamma)$ with another generator function [2, 3, 10, 11].

If the sequence $\{\lambda_j\}$ of scales satisfies $\lambda_j = p^j$ for some $p \geq 2$ and if $H : \mathbb{T} \to \mathbb{R}$ is a bounded function satisfying $\int_{\mathbb{T}} H(\gamma)\, d\gamma = 0$ then we may define partial *multiscale* Riesz polynomials by

$$\mu_N(\gamma) = \prod_{j=0}^{N-1} (1 + a_j H(p^j\gamma)), \tag{1.2}$$

provided that $\{a_j\}_{j\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that $1 + a_j H(\gamma) \geq 0$ for any $\gamma \in \mathbb{T}$. In this direction the authors of [2] worked with real valued step functions $H = \sum_{k=0}^{p-1} a_k \mathbf{1}_{[k/p,(k+1)/p)}$ satisfying $a_0 + \cdots + a_{p-1} = 0$, where $\mathbf{1}_A$ is the characteristic function of a subset A of the real line. In this case weak-* convergence of μ_N is obtained from the property that the span of the set $\{\mathbf{1}_{[k/p^N,(k+1)/p^N)}(\cdot)\}_{k=0}^{p^N-1}$ is dense in $L_2(\mathbb{T})$ for sufficiently large N. For a more general selection of H, weak-* convergence of (1.2) is established under an additional lacunarity assumption; see also [13] and [4] for related dichotomy results.

Definition (1.2) establishes a bridge between Riesz products and certain multiscale constructions, like wavelets. In [9] certain measures were induced by representations of the Cuntz relations and proved to dictate the wavepacket analysis in $L_2(\mathbb{R})$, whereas in [7] Riesz products were constructed as a by-product of a multiresolution analysis on $L_2(\mathbb{R}, (dx)^s)$, where $(dx)^s$ is the Hausdorff measure. In [1] Haar-type Riesz products were constructed from properly selected dilation and translation operators on matrices.

Notice that we can use more than one generator to construct Riesz products. In this case we talk about *generalized Riesz products*. In [6, 8, 14] trigonometric generalized Riesz product measures

$$d\rho(\gamma) = \Big(\prod_{j=1}^{\infty} P_{m_j}(\lambda_j\gamma)\Big)\, d\gamma$$

were studied, where $P_m(\gamma) = \sum_{|k|<m} c_{m,k} e^{ik\gamma}$ is a positive trigonometric polynomial supported in $I_j = \{m\lambda_j : |m| < m_j\}$ and I_j is a dissociate sequence in the sense that every integer m admits at most one decomposition as a sum $\sum_j m_j$, $m_j \in I_j$ and $m_j = 0$ for all but a finite number of j's. We mention here the important role of the trigonometric system $\{e^{2\pi in\gamma}\}_{n\in\mathbb{Z}}$ (which is an orthonormal basis of $L_2(\mathbb{T})$) for establishing convergence and dichotomy results in this direction.

Motivated by the aforementioned works on multiscale Riesz product measures (especially [2]) and trigonometric generalized Riesz product measures, in this paper we construct multiscale generalized Riesz product measures

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty}\Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma)\Big)\Big)\, d\gamma \tag{1.3}$$

as a by-product of an orthonormal transform on $L_2(\mathbb{T})$. Here the functions m_k are assumed to be either step functions or real valued trigonometric polynomials on $\mathbb{T}$ satisfying a certain orthonormality condition (see (2.1) for details) and $\mathbf{c} = \{\mathbf{c}_j = \{c_{j,k}\}_{k=0}^{p-1} : j \in \mathbb{N}\}$ is a sequence in $\mathbb{R}^p$ with a certain bound.

In Section 2 we use the above functions m_k as generators of an orthonormal basis of $L_2(\mathbb{T})$ produced from dilations and multiplications of the generators m_k and we obtain a multiscale analysis of $L_2(\mathbb{T})$ which is reminiscent of (although different from) a multiresolution analysis of $L_2(\mathbb{T})$. In other words we construct a nested sequence $\{V_N\}_{N\in\mathbb{N}}, V_N \subset V_{N+1}$, of p^N-dimensional subspaces of $L_2(\mathbb{T})$ so that each V_N is the span of an orthonormal set produced from dilations and multiplications of the generators m_k and $\overline{\bigcup_{N\in\mathbb{N}} V_N} = L_2(\mathbb{T})$. In Section 3 we define our generalized Riesz product measures (1.3) as weak-* limits of elements $\mu_N \in V_N$. Based on this interpetation we use the results of Section 2 to obtain statements involving weak-* convergence, continuity and mutual absolute continuity or singularity of such measures. Finally in Section 4 we present examples which show that our construction extends previous work on multiscale Riesz products [2] to a richer family of measures.

2. On a class of orthonormal transforms on $L_2(\mathbb{T})$. Let $p \geq 2$ be a natural number and $\mathcal{M}_p = \{m_k : k = 0, \ldots, p-1\}$ be a set of measurable functions on $\mathbb{T}$ such that for almost every γ in $\mathbb{T}$ with respect to the Lebesgue measure we have

$$\frac{1}{p}\sum_{j=0}^{p-1} m_r\Big(\frac{\gamma+j}{p}\Big)\overline{m_s\Big(\frac{\gamma+j}{p}\Big)} = \delta_{r,s}, \qquad r, s = 0, \ldots, p-1, \tag{2.1}$$

where $\delta_{r,s}$ is the Kronecker delta. A straightforward consequence of (2.1) is that $\mathcal{M}_p$ is an orthonormal set in $L_2(\mathbb{T})$. This *orthonormality condition* on $\mathcal{M}_p$ usually appears in subband coding schemes and wavelet analysis. Indeed if (2.1) holds then a multiresolution analysis of $L_2(\mathbb{R})$ can be obtained (see [7]).

For each $\mathcal{M}_p$ as above and for any $i \in \mathbb{N}$ we define a new set of functions on $\mathbb{T}$ by

$$(2.2) \qquad \widetilde{m}_k(\gamma) = \prod_{j=0}^{i-1} m_{\varepsilon_j^{(k)}}(p^j\gamma), \qquad k = 0, \ldots, p^i - 1,$$

where $k = (\varepsilon_{i-1}^{(k)} \ldots \varepsilon_0^{(k)})_p$ is the unique expression of k in base p, i.e. $k = \sum_{j=0}^{i-1} \varepsilon_j^{(k)} p^j$ and $\varepsilon_j^{(k)} \in \{0, \ldots, p-1\}$ are the digits of k in base p.

Lemma 1. *For any $i \in \mathbb{N}$ the set $\mathcal{M}_{p,i} = \{\widetilde{m}_k\}_{k=0}^{p^i-1}$ is orthonormal in $L_2(\mathbb{T})$.*

Proof. Let

$$S_j : L_2(\mathbb{T}) \to L_2(\mathbb{T}), \qquad S_j f(\gamma) = m_j(\gamma) f(p\gamma).$$

Then (2.1) implies that $S_j^* S_l = \delta_{l,j} I$, where S_j^* is the adjoint operator of S_j and I is the identity operator (see the proof of Lemma 2.1 in [9]). Therefore if $k, l = 0, \ldots, p^i - 1$ and $\widetilde{m}_k, \widetilde{m}_l$ are as in (2.2) then

$$\begin{aligned}
\langle \widetilde{m}_k, \widetilde{m}_l \rangle_{L_2} &= \big\langle (S_{\varepsilon_0^{(k)}} S_{\varepsilon_1^{(k)}} \ldots S_{\varepsilon_{i-2}^{(k)}}) m_{\varepsilon_{i-1}^{(k)}}, (S_{\varepsilon_0^{(l)}} S_{\varepsilon_1^{(l)}} \ldots S_{\varepsilon_{i-2}^{(l)}}) m_{\varepsilon_{i-1}^{(l)}} \big\rangle_{L_2} \\
&= \big\langle (S^*_{\varepsilon_{i-2}^{(l)}} S^*_{\varepsilon_{i-3}^{(l)}} \ldots S^*_{\varepsilon_0^{(l)}} S_{\varepsilon_0^{(k)}} S_{\varepsilon_1^{(k)}} \ldots S_{\varepsilon_{i-2}^{(k)}}) m_{\varepsilon_{i-1}^{(k)}}, m_{\varepsilon_{i-1}^{(l)}} \big\rangle_{L_2} \\
&= \delta_{\varepsilon_0^{(k)}, \varepsilon_0^{(l)}} \ldots \delta_{\varepsilon_{i-2}^{(k)}, \varepsilon_{i-2}^{(l)}} \big\langle m_{\varepsilon_{i-1}^{(k)}}, m_{\varepsilon_{i-1}^{(l)}} \big\rangle_{L_2}.
\end{aligned}$$

Since the set $\mathcal{M}_p$ is orthonormal, the proof is complete. ∎

From now on we write $\Omega_{j,N}$ for the p-adic intervals $[j/p^N, (j+1)/p^N)$, $j = 0, \ldots, p^N - 1$. The space of all real valued step functions with constant values on $\Omega_{j,1}$ ($j = 0, \ldots, p-1$) will be denoted by $\mathbf{S}_p$. Also for odd p we define $\mathbf{T}_p = \{\sum_{|n| \le (p-1)/2} a_n e^{2\pi i n\gamma} : a_n \in \mathbb{C}\}$, the space of all trigonometric polynomials of order $\le (p-1)/2$.

Definition 1. Let $\mathcal{M}_p = \{m_k\}_{k=0}^{p-1}$ be a set of real valued functions satisfying (2.1) and let $\mathbf{S}_p$, $\mathbf{T}_p$ be as above. If $m_0 = \mathbf{1}_{\mathbb{T}}$ and if $\mathcal{M}_p \subset \mathbf{S}_p$ or $\mathcal{M}_p \subset \mathbf{T}_p$ then we say that $\mathcal{M}_p$ is an *admissible* set.

Note 1. The term *admissible set* is borrowed from [2, Def. 4].

Example 1. Let $\mathbf{U} = \{U_{k,j} : k, j = 0, \ldots, p-1\}$ be a $p \times p$ matrix over $\mathbb{R}$ such that the matrix $(1/\sqrt{p})\mathbf{U}$ is orthonormal and the first row of $\mathbf{U}$ is constant with entries $U_{0,j} = 1$ for any $j = 0, \ldots, p-1$. Then (2.1) holds and the functions

$$(2.3) \qquad m_k = \sum_{j=0}^{p-1} U_{k,j} \mathbf{1}_{\Omega_{j,1}}, \qquad k = 0, \ldots, p-1,$$

form an admissible set.

EXAMPLE 2. Let p be an odd number and $\mathbf{U} = \{U_{k,j} : k, j = 0, \ldots, p-1\}$ be a $p \times p$ unitary matrix whose first row satisfies $U_{0,j} = 1$ for $j = (p-1)/2$ and $U_{0,j} = 0$ otherwise. Then the set $\mathcal{M}_p$ defined by

$$m_k(\gamma) = \sum_{j=0}^{p-1} U_{k,j} e^{2\pi i(j-(p-1)/2)\gamma}$$

is admissible provided that the elements m_k are real valued. For $p = 3$ and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & i/\sqrt{2} \end{pmatrix},$$

we obtain the admissible set $\mathcal{M}_3 = \{1, \sqrt{2}\cos(2\pi\gamma), \sqrt{2}\sin(2\pi\gamma)\}$.

PROPOSITION 1. *Let $\mathcal{M}_p$ be an admissible set as above and $\{V_i\}_{i\in\mathbb{N}}$, $V_i \subset V_{i+1}$, be a nested sequence of p^i-dimensional subspaces of $L_2(\mathbb{T})$ spanned by the orthonormal sets $\mathcal{M}_{p,i}$ as in Lemma* 1. *Then $\overline{\bigcup_{i\in\mathbb{N}} V_i} = L_2(\mathbb{T})$.*

Proof. Let $N \in \mathbb{N}$. Clearly $\bigcup_{i=1}^N V_i = V_N$. First we consider the case $\mathcal{M}_p \subset \mathbf{S}_p$. Then $\widetilde{m}_k$, $k = 0, \ldots, p^N - 1$, are step functions with constant values on $\Omega_{j,N}$ $(j = 0, \ldots, p^N - 1)$ and from Lemma 1 we deduce that they form an orthonormal set in $\mathbf{S}_{p^N}$. In other words, V_N coincides with $\mathbf{S}_{p^N}$. Since for large N the space $\mathbf{S}_{p^N}$ is dense in $L_2(\mathbb{T})$, the proof is complete. If $\mathcal{M}_p \subset \mathbf{T}_p$ then it is easy to see that V_N coincides with the space $\mathbf{T}_{p^N}$ (of all trigonometric polynomials of order $(p^N - 1)/2$), so $\overline{\bigcup_{N\in\mathbb{N}} V_N} = L_2(\mathbb{T})$. ■

REMARK 1. Proposition 1 enables us to build multiscale transforms on $L_2(\mathbb{T})$ from admissible sets $\mathcal{M}_p$ based on a dilation operator $D_p : L_2(\mathbb{T}) \to L_2(\mathbb{T})$, $D_p f(\gamma) = f(p\gamma)$, and a set of multiplication operators $T_i : L_2(\mathbb{T}) \to L_2(\mathbb{T})$, $T_i f(\gamma) = m_i(\gamma) f(\gamma)$, $i = 0, \ldots, p-1$. Indeed the basis elements $\widetilde{m}_k$ of the p^{i+1}-dimensional space V_{i+1} result from the basis elements of the coarse space V_i by

$$\begin{aligned}\widetilde{m}_k(\gamma) &= \prod_{j=0}^{i} m_{\varepsilon_j^{(k)}}(p^j\gamma) = m_{\varepsilon_0^{(k)}}(\gamma) \prod_{j=1}^{i} m_{\varepsilon_j^{(k)}}(p^j\gamma) \\ &= m_{\varepsilon_0^{(k)}}(\gamma)\widetilde{m}_{[k/p]}(p\gamma) = (T_{\varepsilon_0^{(k)}} D_p)\widetilde{m}_{[k/p]}(\gamma), \qquad k = 0, \ldots, p^{i+1} - 1,\end{aligned}$$

where $[x]$ is the floor of a real number x. Observe that the difference from a multiresolution analysis of $L_2(\mathbb{T})$ is the use of multiplication operators T_i instead of the usual translation operator.

3. Generalized Riesz products produced from admissible sets. In this section we produce a class of generalized Riesz product measures $d\mu(\gamma) = (\prod_{j\geq 0} f_j(p^j\gamma))\, d\gamma$, where the functions f_j are linear combinations of elements in a certain admissible set $\mathcal{M}_p$ and $d\gamma$ is the usual Lebesgue measure on $\mathbb{T}$ (see Definition 2 and Theorem 1 below for details). Also we give a characterization of mutual absolute continuity or singularity of two such measures.

Let $\mathcal{B}_R = \{\mathbf{d} = \{d_0, \ldots, d_{p-1}\} : \|\mathbf{d}\|_2 \le R\}$ be the closed R-disk in $\mathbb{R}^p$.

DEFINITION 2. Let $\mathcal{M}_p = \{m_k\}_{k=0}^{p-1}$ be an admissible set as above and $\mathbf{c} = \{\mathbf{c}_j = \{c_{j,k}\}_{k=0}^{p-1}\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}_R$ for some $R < \sqrt{p}$ such that $c_{j,0} = 1$ and $\sum_{k=0}^{p-1} c_{j,k} m_k(\gamma) \ge 0$ for any $j \in \mathbb{N}$ and $\gamma \in \mathbb{T}$. For any $N \in \mathbb{N}$ we call the function

$$\mu_N(\gamma) = \prod_{j=0}^{N-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma)\Big) \tag{3.1}$$

a partial generalized Riesz polynomial with respect to the pair $(\mathcal{M}_p, \mathbf{c})$.

LEMMA 2. *Let μ_N be a partial generalized Riesz polynomial with respect to a pair $(\mathcal{M}_p, \mathbf{c})$ and $\mathcal{M}_{p,N} = \{\widetilde{m}_k\}_{k=0}^{p^N-1}$ be as in Lemma 1. Then:*

(i) $\mu_N(\gamma) = \sum_{k=0}^{p^N-1} b_k \widetilde{m}_k(\gamma)$, *where* $b_k = \prod_{j=0}^{N-1} c_{j,\varepsilon_j^{(k)}}$. *In other words* $\mu_N \in V_N$, *where V_N is defined in Proposition 1.*

(ii) *Let $N, M \in \mathbb{N}$ with $N < M$. Fix $0 \le s \le N$. Then for any $0 \le i < p^s$ we have*

$$\int_{\Omega_{i,s}} \Big(\prod_{j=N}^{M-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma)\Big)\Big)\, d\gamma = \frac{1}{p^s}.$$

Proof. (i) Direct calculation from (3.1) taking into account (2.2).

(ii) Let N, M and s be as above. Using (i) we have

$$\begin{aligned}\prod_{j=N}^{M-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^{j-s}\gamma)\Big) &= \prod_{j=N-s}^{M-1-s} \Big(\sum_{k=0}^{p-1} d_s(j,k) m_k(p^j\gamma)\Big)\\ &= \sum_{l\in I_{s,N,M}} b_s(l)\widetilde{m}_l(\gamma),\end{aligned}$$

where $d_s(j,k) = c_{j+s,k}$, $b_s(l) = \prod_{j=0}^{M-1-s} d_s(j,\varepsilon_j^{(l)}) = \prod_{j=N}^{M-1} c_{j,\varepsilon_{j-s}^{(l)}}$ and $I_{s,N,M} = \{0 \le l \le p^{M-1-s} : l = (\varepsilon_{M-1-s}^{(l)}, \ldots, \varepsilon_{N-s}^{(l)}, 0, \ldots, 0)\}$. Since $\widetilde{m}_0(\gamma) = \mathbf{1}_{\mathbb{T}}(\gamma)$ and $\mathcal{M}_{p,M-1-s}$ is orthonormal we have

$$\langle \widetilde{m}_0, \widetilde{m}_l\rangle_{L_2} = \int_{\mathbb{T}} \widetilde{m}_l(\gamma)\, d\gamma = 0$$

for all $l \neq 0$. Therefore

$$\int_{\Omega_{i,s}} \Big(\prod_{j=N}^{M-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma)\Big)\Big)\, d\gamma = \frac{1}{p^s} \int_{\mathbb{T}} \Big(\prod_{j=N}^{M-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^{j-s}\gamma)\Big)\Big)\, d\gamma$$
$$= \frac{1}{p^s} \int_{\mathbb{T}} \Big(\sum_{l\in I_{s,N,M}} b_s(l)\widetilde{m}_l(\gamma)\Big)\, d\gamma = \frac{b_s(0)}{p^s} = \frac{1}{p^s}$$

(recall that $c_{j,0} = 1$, see Def. 2). ∎

Theorem 1. *The sequence $\{\mu_N\}_{N\in\mathbb{N}}$ of partial generalized Riesz polynomials with respect to a pair $(\mathcal{M}_p, \mathbf{c})$ converges weak-* to a continuous probability measure*

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma)\Big)\Big)\, d\gamma$$

called the generalized Riesz product measure associated with the pair $(\mathcal{M}_p, \mathbf{c})$.

Proof. Let f be a continuous function on $\mathbb{T}$. We shall prove that $\{\int_{\mathbb{T}} f(\gamma)\mu_N(\gamma)\, d\gamma\}$ is a Cauchy sequence in $\mathbb{C}$. For any $\varepsilon > 0$ we can find a 1-Lipschitz continuous function h such that $\|f-h\|_\infty < \varepsilon/2$. If we take $N = 0$ in Lemma 2(ii) we see that $\|\mu_M\|_{L_1} = \int_{\mathbb{T}} \mu_M(\gamma)\, d\gamma = 1$ for any $M \in \mathbb{N}$ and so if μ exists then $\mu(\mathbb{T}) = 1$. Using these observations and the above selection of h, for any $N, M \in \mathbb{N}$ with $N < M$ we have

$$\Big|\int_{\mathbb{T}} f(\gamma)(\mu_M(\gamma) - \mu_N(\gamma))\, d\gamma\Big| < \varepsilon + \Big|\int_{\mathbb{T}} h(\gamma)(\mu_M(\gamma) - \mu_N(\gamma))\, d\gamma\Big|.$$

Let P_N be the orthogonal projection from $L_2(\mathbb{T})$ to V_N as in Proposition 1. Then for the above h and for any $\varepsilon' > 0$ we can find a sufficiently large N such that $\|P_N h - h\|_\infty < \varepsilon'/2$ pointwise on $\mathbb{T}$ (recall that $P_N h$ is either a step function if $\mathcal{M}_p \subset \mathbf{S}_p$ or a trigonometric polynomial if $\mathcal{M}_p \subset \mathbf{T}_p$). Therefore

$$\int_{\mathbb{T}} h(\gamma)(\mu_M(\gamma) - \mu_N(\gamma))\, d\gamma = \int_{\mathbb{T}} (h(\gamma) - P_N h(\gamma))(\mu_M(\gamma) - \mu_N(\gamma))\, d\gamma$$
$$+ \int_{\mathbb{T}} P_N h(\gamma)(\mu_M(\gamma) - \mu_N(\gamma))\, d\gamma.$$

The first term on the right hand side is bounded by ε', while the second term vanishes, because by Lemma 2(i) we have $\mu_M - \mu_N = \sum_{k=p^N}^{p^M-1} b_k \widetilde{m}_k$ and so $\mu_M - \mu_N \in V_N^{\perp}$. Therefore μ converges weak-*.

For the continuity of μ take a sufficiently large N and consider an element $\gamma_0 \in \mathbb{T}$. Obviously γ_0 belongs to a p-adic interval $\Omega_{i,N}$ for some

$i = 0, \dots, p^N - 1$. Using Lemma 2(ii) (for $s = N$) we compute

$$\mu(\{\gamma_0\}) \le \mu(\Omega_{i,N}) = \lim_M \int_{\Omega_{i,N}} \mu_M(\gamma)\, d\gamma \tag{3.2}$$
$$\le \|\mu_N\|_\infty \lim_M \int_{\Omega_{i,N}} \Big(\prod_{j=N}^{M-1} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j \gamma) \Big)\Big)\, d\gamma = \frac{\|\mu_N\|_\infty}{p^N}.$$

From (2.1) we deduce that the matrix

$$M(\gamma) = \left\{ \frac{1}{\sqrt{p}} m_k\left(\frac{\gamma + j}{p} \right) \right\}_{k,j=0}^{p-1}$$

is unitary for a.e. γ in $\mathbb{T}$ and so $M^*(\gamma)M(\gamma) = I_p$, where $M^*(\gamma)$ is the Hermitian transpose of $M(\gamma)$. Hence $\sum_{k=0}^{p-1} \left| m_k\left(\frac{\gamma+j}{p}\right)\right|^2 = p$ for any $j = 0, \dots, p-1$ and for a.e. γ in $\mathbb{T}$, or equivalently $\sum_{k=0}^{p-1} |m_k(\gamma')|^2 = p$ for a.e. γ in $\mathbb{T}$.

Now we apply the Cauchy–Schwarz inequality and the above estimate to get $\|\mu_N\|_\infty \le (R\sqrt{p})^N$, where R is as in Definition 2. Indeed,

$$|\mu_N(\gamma)| = \prod_{j=0}^{N-1} \Big| \sum_{k=0}^{p-1} c_{j,k} m_k(p^j \gamma) \Big| \le \prod_{j=0}^{N-1} (R\sqrt{p}) = (R\sqrt{p})^N.$$

Substituting this estimate in (3.2) we obtain $\mu(\{\gamma_0\}) \le (R/\sqrt{p})^N \to 0$ because $R < \sqrt{p}$ by assumption (see Def. 2). Therefore μ is continuous. ∎

In order to prove that two generalized Riesz product measures as above are mutually singular we need the following

LEMMA 3. *Let $d\mu(\gamma) = (\prod_{j=0}^{\infty} (\sum_{k=0}^{p-1} c_{j,k} m_k(p^j \gamma)))\, d\gamma$ be a generalized Riesz product measure as in Theorem 1. Then:*

(i) *For any $j \in \mathbb{N}$ we have $\int_{\mathbb{T}} m_k(p^j \gamma)\, d\mu = c_{j,k}$, $k = 0, \dots, p-1$.*
(ii) *If $j, j' \in \mathbb{N}$ and $j \ne j'$ we have $\int_{\mathbb{T}} m_k(p^j \gamma) m_{k'}(p^{j'} \gamma)\, d\mu = c_{j,k} c_{j',k'}$ for $k, k' = 0, \dots, p-1$.*

Proof. (i) Fix $j \in \mathbb{N}$ and take $M > j$. Then for any $k = 0, \dots, p-1$,

$$\int_{\mathbb{T}} m_k(p^j \gamma)\, d\mu = \lim_M \int_{\mathbb{T}} \widetilde{m}_{kp^j}(\gamma) \mu_M(\gamma)\, d\gamma = \lim_M \sum_{i=0}^{p^M - 1} b_i \int_{\mathbb{T}} \widetilde{m}_{kp^j}(\gamma) \widetilde{m}_i(\gamma)\, d\gamma$$
$$= b_{kp^j} \|\widetilde{m}_{kp^j}\|_2^2 = b_{kp^j} = c_{j,k}$$

(recall (2.2) and use Lemma 2(i)).

(ii) Take $M > \max\{j,j'\}$. Since $j \neq j'$ we have $m_k(p^j\gamma)m_{k'}(p^{j'}\gamma) = \widetilde{m}_{kp^j+k'p^{j'}}(\gamma)$ and so we work as above to obtain

$$\int_{\mathbb{T}} m_k(p^j\gamma)m_{k'}(p^{j'}\gamma)\,d\mu = b_{kp^j+k'p^{j'}}\|\widetilde{m}_{kp^j+k'p^{j'}}\|_2^2 = b_{kp^j+k'p^{j'}} = c_{j,k}c_{j',k'}. \blacksquare$$

Theorem 2. *Let μ,ν be continuous generalized Riesz product measures with respect to $(\mathcal{M}_p,\mathbf{c})$ and $(\mathcal{M}_p,\mathbf{d})$ respectively. If there exists at least one index $1 \le k_0 \le p-1$ such that*

$$\sum_{j\in\mathbb{N}} |c_{j,k_0} - d_{j,k_0}|^2 = \infty, \tag{3.3}$$

then μ and ν are mutually singular.

Proof. We modify the proof in [12, pp. 68–69] (see also [15]). Take $\varepsilon > 0$ and select $M \in \mathbb{N}$ such that $M > 1/\varepsilon$. By (3.3) we can find a natural number n_0 such that for any $N > n_0$ we have $z_N = \sum_{j=0}^{N-1}\sum_{k=1}^{p-1}|c_{j,k} - d_{j,k}|^2 > M$. Set $q_{j,k} = (c_{j,k} - d_{j,k})/z_N$. Then $\sum_{j=0}^{N-1}\sum_{k=1}^{p-1} q_{j,k}(c_{j,k} - d_{j,k}) = 1$ and

$$\sum_{j=0}^{N-1}\sum_{k=1}^{p-1}|q_{j,k}|^2 < \frac{1}{M} < \varepsilon. \tag{3.4}$$

Now we define

$$f_N(\gamma) = \sum_{j=0}^{N-1}\sum_{k=1}^{p-1} q_{j,k}(m_k(p^j\gamma) - c_{j,k}),$$
$$h_N(\gamma) = \sum_{j=0}^{N-1}\sum_{k=1}^{p-1} q_{j,k}(m_k(p^j\gamma) - d_{j,k}),$$

and compute

$$\begin{aligned}
&\int_{\mathbb{T}} |f_N(\gamma)|^2\,d\mu\\
&\quad= \sum_{j,j'=0}^{N-1}\sum_{k,k'=1}^{p-1} q_{j,k}q_{j',k'}\int_{\mathbb{T}}(m_k(p^j\gamma) - c_{j,k})(m_{k'}(p^{j'}\gamma) - c_{j',k'})\,d\mu\\
&\quad= \sum_{j=0}^{N-1}\sum_{k,k'=1}^{p-1} q_{j,k}q_{j,k'}\int_{\mathbb{T}}(m_k(p^j\gamma) - c_{j,k})(m_{k'}(p^j\gamma) - c_{j',k'})\,d\mu\\
&\qquad+ \sum_{j,j'=0,j\neq j'}^{N-1}\sum_{k,k'=1}^{p-1} q_{j,k}q_{j',k'}\int_{\mathbb{T}}(m_k(p^j\gamma) - c_{j,k})(m_{k'}(p^{j'}\gamma) - c_{j',k'})\,d\mu.
\end{aligned}$$

From Lemma 3 we deduce that the second term on the right hand side vanishes. Therefore using the Cauchy–Schwarz inequality and Lemma 3(i) we obtain

$$\begin{aligned}
\int_{\mathbb{T}} |f_N(\gamma)|^2\,d\mu &= \sum_{j=0}^{N-1} \int_{\mathbb{T}} \Big|\sum_{k=1}^{p-1} q_{j,k}(m_k(p^j\gamma) - c_{j,k})\Big|^2\,d\mu \\
&\le \sum_{j=0}^{N-1} \Big(\sum_{k=1}^{p-1} |q_{j,k}|^2\Big) \int_{\mathbb{T}} \sum_{k=1}^{p-1} |m_k(p^j\gamma) - c_{j,k}|^2\,d\mu \\
&= \sum_{j=0}^{N-1} \Big(\sum_{k=1}^{p-1} |q_{j,k}|^2\Big) \int_{\mathbb{T}} \sum_{k=1}^{p-1} (|m_k(p^j\gamma)|^2 - |c_{j,k}|^2)\,d\mu \\
&= \sum_{j=0}^{N-1} \Big(\sum_{k=1}^{p-1} |q_{j,k}|^2\Big) \int_{\mathbb{T}} \Big((p-1) - \sum_{k=1}^{p-1} |c_{j,k}|^2\Big)\,d\mu < (p-1)\varepsilon,
\end{aligned}$$

where the last bound follows from (3.4) and the fact that

$$\int_{\mathbb{T}} \sum_{l=1}^{p-1} |m_k(p^j\gamma)|^2\,d\mu = \sum_{k=1}^{p-1} \|\widetilde{m}_{kp^j}\|_{L_2}^2 = p-1.$$

The same estimate holds for h_N, i.e. $\int_{\mathbb{T}} |h_N(\gamma)|^2\,d\nu(\gamma) < (p-1)\varepsilon$. Let $E_f = \{\gamma \in \mathbb{T} : \lim_{N\to\infty} f_N(\gamma) = 0\}$ and $E_g = \{\gamma \in \mathbb{T} : \lim_{N\to\infty} h_N(\gamma) = 0\}$. Since

$$h_N(\gamma) - f_N(\gamma) = \sum_{j=0}^{N-1} \sum_{k=1}^{p-1} q_{j,k}(c_{j,k} - d_{j,k}) = 1 \tag{3.5}$$

for any N, we deduce that the sets E_f and E_g are disjoint. Furthermore from

$$\int_{E_f} |f_N(\gamma)|^2\,d\mu + \int_{\mathbb{T}-E_f} |f_N(\gamma)|^2\,d\mu = \int_{\mathbb{T}} |f_N(\gamma)|^2\,d\mu \to 0$$

and (3.5) we deduce that μ is concentrated on E_f. Using the same arguments we deduce that ν is concentrated on E_g. Since these two sets are disjoint, μ and ν are mutually singular. ∎

THEOREM 3. *Let μ be a continuous generalized Riesz product measure with respect to a pair $(\mathcal{M}_p, \mathbf{c})$. If $\sum_{j=0}^{\infty} |c_{j,k}|^2 < \infty$ for any $k = 1,\dots,p-1$ then μ is absolutely continuous with respect to the Lebesgue measure; otherwise it is singular with respect to the Lebesgue measure.*

Proof. We follow the proof in [15]. The singularity part is derived from Theorem 2 because the Lebesgue measure ν corresponds to the case $d_{j,k} = 0$ for any $k = 1,\dots,p-1$ and $j \in \mathbb{N}$. Now if $\sum_{j=0}^{\infty} |c_{j,k}|^2 < \infty$ for every

$k = 1, \dots, p-1$, from Lemma 2(i) we obtain

$$\begin{aligned}\|\mu_N\|_{L_2}^2 &= \sum_{k=0}^{p^N-1} |b_k|^2 = \Big(\sum_{\varepsilon_0=0}^{p-1} \cdots \sum_{\varepsilon_{N-1}=0}^{p-1}\Big) \prod_{j=0}^{N-1} |c_{j,\varepsilon_j}|^2 \\ &= \prod_{j=0}^{N-1} \sum_{k=0}^{p-1} |c_{j,k}|^2 \le \prod_{j=0}^{\infty} \Big(1 + \sum_{k=1}^{p-1} |c_{j,k}|^2\Big) < \infty.\end{aligned}$$

Therefore $\{\mu_N\}_{N\in\mathbb{N}}$ is convergent in $L_2(\mathbb{T})$ and hence in $L_1(\mathbb{T})$, and the result follows. ■

4. Applications

4.1. Bernoulli measures. Let μ be a generalized Riesz product measure with respect to a pair $(\mathcal{M}_p, \mathbf{c})$, where the generators $m_k \in \mathcal{M}_p$ are step functions produced from a $p \times p$ unitary matrix $\mathbf{U}$ as in (2.3). Using matrix notation we can write

$$\mathcal{M}_p = \mathbf{U}\mathbf{X}_p, \tag{4.1}$$

where the sets $\mathcal{M}_p$ and $\mathbf{X}_p = \{\mathbf{1}_{[0,1/p)}, \dots, \mathbf{1}_{[(p-1)/p,1)}\}$ are considered as $1 \times p$ column vectors.

Let us consider a Bernoulli scheme, i.e. a sequence $\{X_j\}_{j\in\mathbb{N}}$ of independent random variables such that each random variable X_j may take p outcomes (states) with probabilities $\mathbf{P}_j = \{p_{j,0}, \dots, p_{j,p-1}\}$ satisfying $\sum_{i=0}^{p-1} p_{j,i} = 1$ for every $j \in \mathbb{N}$. Using (4.1) we can easily verify that this process can be expressed by a generalized Riesz product measure

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty} \Big(\sum_{k=0}^{p-1} (\mathbf{P}_j\mathbf{U}^*)_k m_k(p^j\gamma)\Big)\Big)\, d\gamma. \tag{4.2}$$

Notice that $\sum_{k=0}^{p-1} |(\mathbf{P}_j\mathbf{U}^*)_k|^2 = p\sum_{k=0}^{p-1} |p_{j,k}|^2$ and so if $\sum_{k=0}^{p-1} |p_{j,k}|^2 < 1$ for any j, then μ is continuous in accordance with the related assumption of Theorem 1.

We mention that the generalized Riesz product (4.2) is a natural generalization of the multiscale Riesz product construction presented in [2] and covers well known classes of Riesz products such as the Rademacher Riesz products [3]. We mention two illustrative examples.

(i) *Non-homogeneous Cantor measures.* Let $p = 3$ and $\mathbf{P}_j = \{p_{j,0}, 0, p_{j,1}\}$ with $p_{j,0} + p_{j,1} = 1$ and $p_{j,0}^2 + p_{j,1}^2 < 1$. Consider an admissible set $\mathcal{M}_3$ whose elements are produced from (2.3) and are defined for any $\gamma \in \mathbb{T}$ by

$m_0(\gamma) = \mathbf{1}_{\mathbb{T}}(\gamma)$ and

$$m_1(\gamma) = \begin{cases} 1/\sqrt{2}, & \gamma \in [0,1/3) \cup [2/3,1), \\ -\sqrt{2}, & \gamma \in [1/3,2/3), \end{cases}$$

$$m_2(\gamma) = \begin{cases} \sqrt{3}/\sqrt{2}, & \gamma \in [0,1/3), \\ -\sqrt{3}/\sqrt{2}, & \gamma \in [2/3,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then the corresponding generalized Riesz product (4.2) is

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty} \Big(1 + \frac{1}{\sqrt{2}} m_1(3^j\gamma) + \sqrt{\frac{3}{2}} (p_{j,0} - p_{j,1}) m_2(3^j\gamma) \Big) \Big)\, d\gamma$$

and produces an inhomogeneous Cantor measure not covered by [2]. We note that in [2] the authors were not able to construct inhomogeneous Cantor measures. This is due to the fact that they dealt with multiscale Riesz products (1.2) produced from dilations of only one function. Here we use more than one generator and we obtain a richer family of measures. In this spirit our generalized Riesz product measures provide a natural generalization of [2].

(ii) *Generalized Rademacher Riesz products.* For any $p > 2$ and $\gamma \in \mathbb{T}$ define generators $m_k \in \mathcal{M}_p$ as in (2.3) by $m_0(\gamma) = \mathbf{1}_{\mathbb{T}}(\gamma)$ and

$$m_l(\gamma) = \begin{cases} \dfrac{\sqrt{p}}{\sqrt{(p-l)(p-l+1)}}, & \gamma \in [0,(p-l)/p), \\ -\dfrac{\sqrt{p(p-1)}}{\sqrt{p-l+1}}, & \gamma \in [(p-1)/p,(p-l+1)/p), \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le l < p,$$

and consider the corresponding generalized Riesz product

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty} \Big(\sum_{k=0}^{p-1} c_{j,k} m_k(p^j\gamma) \Big) \Big)\, d\gamma.$$

If $c_{j,k} = 0$ whenever $k = 2, \ldots, p-1$ then we obtain the (normalized) p-adic Rademacher Riesz product defined in [3].

4.2. Trigonometric measures. For odd p we may work as in Example 2 to produce an admissible set $\mathcal{M}_p$ such that $m_0(\gamma) = \mathbf{1}_{\mathbb{T}}(\gamma)$, $m_k(\gamma) = \sqrt{2}\cos(2\pi k\gamma)$ whenever $k = 1, \ldots, (p-1)/2$ and $m_k(\gamma) = \sqrt{2}\sin(2\pi(k-(p-1)/2)\gamma)$ whenever $k = (p-1)/2+1, \ldots, p-1$. Then the corresponding trigonometric Riesz product measure can be written as

$$d\mu(\gamma) = \Big(\prod_{j=0}^{\infty}\Big(1+\sqrt{2}\sum_{k=1}^{(p-1)/2}(c_{j,k}\cos(2\pi kp^j\gamma)+c_{j,k+(p-1)/2}\sin(2\pi kp^j\gamma))\Big)\Big)\,d\gamma.$$

For example if $p = 3$, $c_{j,2} = 0$ and $c_{j,1} = 1/\sqrt{2}$ then we obtain Jorgensen's example [7, pp. 160].

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REFERENCES

[1] N. D. Atreas and C. Karanikas, *Multiscale Haar orthonormal matrices with the corresponding Riesz products and a characterization of Cantor-type languages*, J. Fourier Anal. Appl. 13 (2007), 197–210.

[2] J. J. Benedetto, E. Bernstein and I. Konstantinidis, *Multiscale Riesz Products and their Support Properties*, Acta Appl. Math. 88 (2005), 201–227.

[3] A. Bisbas and C. Karanikas, *On the Hausdorrf dimension of Rademacher Riesz products*, Monatsh. Math. 110 (1990), 15–21.

[4] R. T. Bott, *Singularity of generalized Riesz products*, Math. Proc. Cambridge Philos. Soc. 93 (1983), 335–338.

[5] G. Brown and W. Moran, *On orthogonality of Riesz products*, Proc. Cambridge Philos. Soc. 76 (1974), 173–181.

[6] A. H. Dooley and S. J. Eigen, *A family of generalized Riesz products*, Canad. J. Math. 48 (1996), 302–315.

[7] D. Dutkay and P. Jorgensen, *Wavelets on fractals*, Rev. Mat. Iberoamer. 22 (2006), 131–180.

[8] B. Host, J.-F. Méla and F. Parreau, *Nonsingular transformations and spectral analysis of measures*, Bull Soc. Math. France 119 (1991), 33–90.

[9] P. E. T. Jorgensen, *Certain representations of the Cuntz relations, and a question on wavelets, decompositions*, in: Operator Theory, Operator Algebras, and Applications, D. G. Han et al. (eds.), Contemp. Math. 414, Amer. Math. Soc., Providence, RI, 2006, 165–188.

[10] C. Karanikas, *Examples of Riesz products-type measures on metrizable groups*, Boll. Un. Mat. Ital. A (7) 4 (1990), 331–341.

[11] C. Karanikas, *The Hausdorff dimension of very weak self-similar fractals described by the Haar wavelet system*, in: Integrability and Chaos in Discrete Systems (Brussels, 1997), Chaos Solitons Fractals 11 (2000), 275–280.

[12] S. J. Kilmer and S. Saeki, *On Riesz product measures; mutual absolute continuity and singularity*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 2, 63–93.

[13] L. Modica and S. Mortola, *Singular measures and Riesz products*, J. London Math. Soc. (2) 25 (1982), 115–121.

[14] F. Parreau, *Ergodicité et pureté des produits de Riesz*, Ann. Inst. Fourier (Grenoble) 40 (1990), 391–405.

[15] J. Peyrière, *Sur les produits de Riesz*, C. R. Acad. Sci. Paris Sér. A-B 276 (1973), A1417–A1419.

[16] A. Zygmund, *Trigonometric Series. Vols. I-II*, 2nd ed., Cambridge Univ. Press, New York, 1959.

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