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WHEN A FIRST ORDER T HAS LIMIT MODELS

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Abstract. We sort out to a large extent when a (first order complete theory) T has a superlimit model in a cardinal λ. Also we deal with related notions of being limit.

Annotated content

0. Introduction. We give background and basic definitions. We then present existence results for stable T which have models that are saturated or close to being saturated.

1. On countable superstable non-$\aleph_0$-stable. Consistently $2^{\aleph_1} \geq \aleph_2$ and some such (complete first order) T has a superlimit (non-saturated) model of cardinality $\aleph_1$. This shows that we cannot prove a non-existence result fully complementary to the results in 0.9.

2. A strictly stable consistent example. Consistently $\aleph_1 < 2^{\aleph_0}$ and some countable stable not superstable T has a (non-saturated) model of cardinality $\aleph_1$ which satisfies some relatives of being superlimit.

3. On the non-existence of limit models. The proofs here are in ZFC. If T is unstable, it has no superlimit models of cardinality λ when $\lambda \geq \aleph_1 + |T|$. For unsuperstable T we have similar results but with "few" exceptional cardinals λ on which we do not know: $\lambda < \lambda^{\aleph_0}$ which are $< \beth_\omega$. Moreover, if T is superstable and $\lambda \geq |T| + 2^{|T|}$ then T has a superlimit model of cardinality λ iff $|D(T)| \leq \lambda$ iff T has a saturated model. Lastly, we get weaker results on weaker relatives of superlimit.

0. Introduction

0A. Background and content. Recall that ([15, Ch. III]) if T is (first order complete and) superstable then for $\lambda \geq 2^{|T|}, T$ has a saturated model M of cardinality λ and moreover

(∗) if $\langle M_\alpha : \alpha < \delta \rangle$ is $\prec$-increasing, δ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\bigcup\{M_\alpha : \alpha < \delta\}$ is isomorphic to M.

When investigating categoricity of an a.e.c. (abstract elementary class) $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$, the following property turns out to be central: M is a $\leq_\mathfrak{k}$-universal model of cardinality λ with the property (∗) above (called superlimit), possibly with additional parameter $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$);

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we also consider some relatives of the "superlimit" notion, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [13, 3.1]; see also the revised version [3, 3.3] and [19], or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete T, we know $\{\lambda : T$ has a saturated model of T of cardinality $\lambda\}$, namely, it is $\{\lambda : \lambda^{<\lambda} \geq |D(T)|$ or T is stable in $\lambda\}$; for the definitions of $D(T)$ and other notions see 0B below. What if we replace saturated by superlimit (or some relative)? Let $\mathrm{EC}_\lambda(T)$ be the class of models M of T of cardinality λ.

If there is a saturated $M \in \mathrm{EC}_\lambda(T)$ we have considerable knowledge on the existence of a limit model for the cardinal λ, by [15], as mentioned in [3, 3.6] (see 0.9(1),(2)). E.g. for superstable T in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [3, 3.6] whether it exhausts the possibilities of $(\lambda, *)$-superlimit and (λ, κ)-superlimit models for elementary classes. Clearly the cases of the existence of such models of a (first order complete) theory T where there are no saturated (or special) models are rare, because even the weakest version of Definition [13, 3.1] = [3, 3.3] or here Definition 0.7 for λ implies that T has a universal model of cardinality λ, which is rare (see Kojman–Shelah [2] which includes earlier history and recently Džamonja [1]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T' \supseteq T$ of cardinality λ such that $\mathrm{PC}(T', T)$ is categorical in λ (see 0.4(3)). By [12] we have consistency results for such T_1 so naturally we first deal with the consistency results from [12]. In §1 we deal with the case of the countable superstable T_0 from [12] which is not $\aleph_0$-stable. By [12] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T'_0 \supseteq T_0$ of cardinality $\aleph_1$, $\mathrm{PC}(T'_0, T_0)$ is categorical in $\aleph_1$. We use this to get the consistency of "T_0 has a superlimit model of cardinality $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$".

In §2 for some stable non-superstable countable T_1 we have a parallel but weaker result. We reconsider the old consistency results of "some $\mathrm{PC}(T'_1, T_1), |T'_1| = \aleph_1 > |T_1|$, is categorical in $\aleph_1$" from [12]. From this we deduce that in this universe, T_1 has a strongly $(\aleph_1, \aleph_0)$-limit model.

It is a reasonable thought that we can similarly have a consistency result for the theory of linear orders, but this is still unclear.

In §3 we show that if T has a superlimit model in $\lambda \geq |T| + \aleph_1$ then T is stable and T is superstable except possibly under some severe restrictions on the cardinal λ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by 1.1 + 3.1 we have:

Conclusion 0.1. *Assume* $\lambda \geq |T| + \beth_\omega$. *Then* T *has a superlimit model of cardinality* λ *iff* T *is superstable and* $\lambda \geq |D(T)|$.

In subsequent work we shall show that for some unstable T (e.g. the theory of linear orders), if $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$, then T has a medium (λ, κ)-limit model, whereas if T has the independence property, even weak (λ, κ)-limit models do not exist; see [5] and more in [6], [20], [4], [9].

0B. Basic definitions

Notation 0.2. Let T denote a complete first order theory which has infinite models but T_1, T' etc. are not necessarily complete.

If M, N denote models, then $|M|$ is the universe of M and $\|M\|$ its cardinality and $M \prec N$ means M is an elementary submodel of N.

Let $\tau_T = \tau(T)$, $\tau_M = \tau(M)$ be the vocabularies of T, M respectively.

Let $M \models$ "$\varphi[\bar{a}]^{(\mathrm{stat})}$" mean that the model M satisfies $\varphi[\bar{a}]$ if the statement stat is true (or is 1 rather than 0).

Definition 0.3. For $\bar{a} \in {}^{\omega>}|M|$ and $B \subseteq M$ let

$$\mathrm{tp}(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{b} \in {}^{\ell g(\bar{y})}B \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}.$$

Let

$$D(T) = \{\mathrm{tp}(\bar{a}, \emptyset, M) : M \text{ a model of } T \text{ and } \bar{a} \text{ a finite sequence from } M\}.$$

If $A \subseteq M$ then

$$\mathbf{S}^m(A, M) = \{\mathrm{tp}(\bar{a}, A, N) : M \prec N \text{ and } \bar{a} \in {}^m N\};$$

if $m = 1$ we may omit it.

A model M is λ-*saturated* when: if $A \subseteq M, |A| < \lambda$ and $p \in \mathbf{S}(A, M)$ then p is realized by some $a \in M$, i.e. $p \subseteq \mathrm{tp}(a, A, M)$; if $\lambda = \|M\|$ we may omit it.

A model M is *special* when letting $\lambda = \|M\|$, there is an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ of cardinals with limit λ and a $\prec$-increasing sequence $\langle M_i : i < \mathrm{cf}(\lambda) \rangle$ of models with union M such that M_{i+1} is λ_i-saturated of cardinality λ_{i+1} for $i < \mathrm{cf}(\lambda)$.

Definition 0.4. For any T let

$$\mathrm{EC}(T) = \{M : M \text{ is a } \tau_T\text{-model of } T\},$$
$$\mathrm{EC}_\lambda(T) = \{M \in \mathrm{EC}(T) : M \text{ is of cardinality } \lambda\}.$$

For $T \subseteq T'$ let

$$\mathrm{PC}(T', T) = \{M{\restriction}\tau_T : M \text{ is model of } T'\},$$
$$\mathrm{PC}_\lambda(T', T) = \{M \in \mathrm{PC}(T', T) : M \text{ is of cardinality } \lambda\}.$$

We say M is *λ-universal* for T_1 when it is a model of T_1 and every $N \in \mathrm{EC}_\lambda(T)$ can be elementarily embedded into M; if $T_1 = \mathrm{Th}(M)$ we may omit it.

We say $M \in \mathrm{EC}(T)$ is *universal* when it is λ-universal for $\lambda = \|M\|$.

We are here mainly interested in

Definition 0.5. Given T and $M \in \mathrm{EC}_\lambda(T)$ we say that M is a *superlimit* or *λ-superlimit model* when: M is universal and if $\delta < \lambda^+$ is a limit ordinal, $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\prec$-increasing continuous, and M_α is isomorphic to M for every $\alpha < \delta$, then M_δ is isomorphic to M.

Remark 0.6. Concerning the following definition we shall use strongly limit in 2.14(1), medium limit in 2.14(2).

Definition 0.7. Let λ be a cardinal $\geq |T|$. For parts (3)–(7) below, but not (8), to simplify the presentation we assume the axiom of global choice and that $\mathbf{F}$ is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$.

(1) For non-empty $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda \text{ and } \mu \text{ is regular}\}$ and $M \in \mathrm{EC}_\lambda(T)$ we say that M is *(λ, Θ)-superlimit* when: M is universal and

if $\langle M_i : i \leq \mu\rangle$ is $\prec$-increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$,
then $\bigcup\{M_i : i < \mu\} \cong M$.

(2) If Θ is a singleton, say $\Theta = \{\theta\}$, we may say that M is *(λ, θ)-superlimit.*

(3) Let $S \subseteq \lambda^+$ be stationary. A model $M \in \mathrm{EC}_\lambda(T)$ is called *S-strongly limit* or *(λ, S)-strongly limit* when for some function $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- (a) for $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$,
- (b) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta\rangle$ is a $\prec$-increasing continuous sequence ([1]) in $\mathrm{EC}_\lambda(T)$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, then $M \cong \bigcup\{M_i : i < \delta\}$.

(4) Let $S \subseteq \lambda^+$ be stationary. $M \in \mathrm{EC}_\lambda(T)$ is called *S-limit* or *(λ, S)-limit* if for some function $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- (a) for every $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$,

([1]) No loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M')$ for $M' \cong M$ is required.

(b) if $\langle M_i : i < \lambda^+\rangle$ is a $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ such that $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ for $i < \lambda^+$ then for some closed unbounded ([2]) subset C of λ^+,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

(5) We define ([3]) "S-weakly limit", "S-medium limit" like "S-limit", "S-strongly limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\prec$-increasing continuous sequences of members of $\mathrm{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1\rangle) \prec M_{i+2}$".

(6) If $S = \lambda^+$ then we may omit S (in (3)–(5)).

(7) For non-empty $\Theta \subseteq \{\mu : \mu \leq \lambda$ and μ is regular$\}$, M is (λ, Θ)-*strongly limit* ([4]) if M is $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit. Similarly for the other notions. If we do not write λ we mean $\lambda = \|M\|$.

(8) We say that $M \in K_\lambda$ is *invariantly strong limit* when in (3), $\mathbf{F}$ is just a subset of $\{(M, N)/{\cong} : M \prec N$ are from $\mathrm{EC}_\lambda(T)\}$ and in (3)(b) we replace "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M, N)/{\cong}) \in \mathbf{F})$". But abusing notation we still write $N = \mathbf{F}(M)$ instead of $((M, N)/{\cong}) \in \mathbf{F}$. Similarly with the other notions, so we use the isomorphism type of $\bar{M}\hat{\ }\langle N\rangle$ for "weakly limit" and "medium limit".

(9) In the definitions above we may say "$\mathbf{F}$ witnesses M is ..."

Observation 0.8. (1) *Assume* $\mathbf{F}_1, \mathbf{F}_2$ *are as above and* $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (*or* $\mathbf{F}_1(\bar{N}) \prec \mathbf{F}_2(\bar{N})$) *whenever defined. If* $\mathbf{F}_1$ *is a witness then so is* $\mathbf{F}_2$.

(2) *All versions of limit models imply being a universal model in* $\mathrm{EC}_\lambda(T)$.

(3) (The obvious implications diagram) *For non-empty* $\Theta \subseteq \{\theta : \theta$ *is regular* $\leq \lambda\}$ *and stationary* $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$:

superlimit $= (\lambda, \{\mu : \mu \leq \lambda \text{regular}\})$-*superlimit*

↓

(λ, Θ)-*superlimit*

↓

S_1-*strongly limit*

↓ ↓

S_1-*medium limit* S_1-*limit*

↓ ↓

S_1-*weakly limit*

([2]) Alternatively, we can use as a parameter a filter on λ^+ extending the co-bounded filter.

([3]) Note that M is (λ, S)-strongly limit iff M is $(\{\lambda, \mathrm{cf}(\delta) : \delta \in S\})$-strongly limit.

([4]) In [3] we replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$", "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1\rangle) \prec M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \prec M_{i+1}$", "$M_i \prec \mathbf{F}(\langle M_j : j \leq i\rangle) \prec M_{i+1}$" respectively. But $(\mathrm{EC}(T), \prec)$ has amalgamation.

Lemma 0.9. *Let T be a first order complete theory.*

(1) *If λ is regular and M a saturated model of T of cardinality λ, then M is (λ,λ)-superlimit.*

(2) *If T is stable, and M is a saturated model of T of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda$ and μ is regular$\})$, then M is (λ,Θ)-superlimit* (for $\kappa(T)$, see [15, III, §3]).

(3) *If T is stable in λ and $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ then T has an invariantly strongly (λ,κ)-limit model.*

Remark 0.10. Concerning 0.9(2), note that by [15] if λ is singular or just $\lambda < \lambda^{<\lambda}$ and T has a saturated model of cardinality λ then T is stable (even stable in λ) and $\mathrm{cf}(\lambda) \geq \kappa(T)$).

Proof. (1) Let M_i be a λ-saturated model of T of cardinality λ for $i < \lambda$ with $\langle M_i : i < \lambda\rangle$ $\prec$-increasing and set $M_\lambda = \bigcup_{i<\lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$, so every $p \in \mathbf{S}(A, M_\lambda)$ is realized in M_i, hence in M_λ; so clearly M_λ is λ-saturated. Remembering the uniqueness of a λ-saturated model of T of cardinality λ we finish.

(2) Use [15, III, 3.11]: if M_i is a λ-saturated model of T with $\langle M_i : i < \delta\rangle$ increasing and $\mathrm{cf}(\delta) \geq \kappa(T)$ then $\bigcup_{i<\delta} M_i$ is λ-saturated.

(3) Let $\mathbf{K}_{\lambda,\kappa} = \{\bar{M} : \bar{M} = \langle M_i : i \leq \kappa\rangle$ is $\prec$-increasing continuous, $M_i \in \mathrm{EC}_\lambda(T)$ and $(M_{i+2}, c)_{c\in M_{i+1}}$ is saturated for every $i < \kappa\}$. Clearly $\bar{M}, \bar{N} \in \mathbf{K}_{\lambda,\kappa} \Rightarrow M_\kappa \cong N_\kappa$. Also for every $M \in \mathrm{EC}_\lambda(T)$ there is N such that $M \prec N$ and $(N, c)_{c\in M}$ is saturated, as also $\mathrm{Th}((M, c)_{c\in M})$ is stable in λ; so there is an invariant $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ such that $M \prec \mathbf{F}(M)$ and $(\mathbf{F}(M), c)_{c\in M}$ is saturated; such $\mathbf{F}$ witnesses the desired conclusion. $\blacksquare_{0.9}$

Definition 0.11. For a regular uncountable cardinal λ let

$\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E, \bar{a})$ witnesses $S \in \check{I}[\lambda]$, see below$\}$.

We say that $(E, \bar{u})$ is a *witness for* $S \in \check{I}[\lambda]$ iff:

- E is a club of the regular cardinal λ,
- $\bar{u} = \langle u_\alpha : \alpha < \lambda\rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$,
- for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $\mathrm{cf}(\delta)$ (and δ is a limit ordinal).

By [16, §1] we have

Claim 0.12. *If $\kappa^+ < \lambda$ and κ,λ are regular then some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ belongs to $\check{I}[\lambda]$.*

By [11] we have

Claim 0.13. *If $\lambda = \mu^+$, $\theta = \mathrm{cf}(\theta) \leq \mathrm{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \leq \mu$ then $S^\lambda_\theta \in \check{I}[\lambda]$.*

1. On superstable non-$\aleph_0$-stable T. We first note that superstable T tend to have superlimit models.

CLAIM 1.1. *Assume T is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then T has a superlimit model of cardinality λ iff T has a saturated model of cardinality λ iff T has a universal model of cardinality λ iff $\lambda \geq |D(T)|$.*

Proof. By [15, III, §5] we know that T is stable in λ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of T of cardinality λ, hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then T is stable in λ, hence has a saturated model of cardinality λ by [15, III] (hence universal) and the class of λ-saturated models of T is closed under increasing elementary chains by [15, III], so we are done. $\blacksquare_{1.1}$

The following are the prototypical theories which we shall consider.

DEFINITION 1.2.

$$T_0 = \mathrm{Th}({}^{\omega}2, E^0_n)_{n<\omega} \quad \text{where} \quad \eta E^0_n \nu \ \Leftrightarrow\ \eta{\restriction}n = \nu{\restriction}n,$$
$$T_1 = \mathrm{Th}({}^{\omega}(\omega_1), E^1_n)_{n<\omega} \quad \text{where} \quad \eta E^1_n \nu \ \Leftrightarrow\ \eta{\restriction}n = \nu{\restriction}n,$$
$$T_2 = \mathrm{Th}(\mathbb{R}, <).$$

Recall

OBSERVATION 1.3.

(0) *T_ℓ is a countable complete first order theory for $\ell = 0, 1, 2$.*
(1) *T_0 is superstable non-$\aleph_0$-stable.*
(2) *T_1 is strictly stable, that is, stable non-superstable.*
(3) *T_2 is unstable.*
(4) *T_ℓ has elimination of quantifiers for $\ell = 0, 1, 2$.*

CLAIM 1.4. *It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \mathrm{EC}_{\aleph_1}(T_0)$ is a superlimit model.*

Proof. By [12], for notational simplicity we start with $\mathbf{V} = \mathbf{L}$.

So T_0 is defined in 1.2 and it is the T from Theorem [12, 1.1]. Let S be the set of $\eta \in ({}^{\omega}2)^{\mathbf{L}}$. We define T' (called T_1 there) as the following theory:

$\circledast_1$ (i) for each n the sentence saying E_n is an equivalence relation with 2^n equivalence classes, each E_n equivalence class divided into two by E_{n+1}, E_{n+1} refines E_n, E_0 is trivial,

(ii) the sentences saying that

(α) for every x_0, the function $z \mapsto F(x_0, z)$ is one-to-one and
(β) $x_0 E_n F(x_0, z)$ for each $n < \omega$,

(iii) $E_n(c_\eta, c_\nu)^{\mathrm{if}(\eta{\restriction}n=\nu{\restriction}n)}$ for $\eta, \nu \in S$.

In [12] it is proved that in some forcing ([5]) extension $\mathbf{L}^{\mathbb{P}}$ of $\mathbf{L}$, $\mathbb{P}$ an $\aleph_2$-c.c. proper forcing of cardinality $\aleph_2$, and in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$, the class $\mathrm{PC}(T', T_0) = \{M{\restriction}\tau_{T_0} : M \text{ is a } \tau\text{-model of } T'\}$ is categorical in $\aleph_1$.

However, letting M^* be any model from $\mathrm{PC}(T', T_0)$ of cardinality $\aleph_1$, it is easy to see that (in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$):

$\circledast_2$ the following conditions on M are equivalent:

- (a) M is isomorphic to M^*,
- (b) $M \in \mathrm{PC}(T', T_0)$,
- (c) (α) M is a model of T_0 of cardinality $\aleph_1$,
  - (β) M^* can be elementarily embedded into M,
  - (γ) for every $a \in M$ the set $\bigcap\{a/E_n^M : n < \omega\}$ has cardinality $\aleph_1$.

But

$\circledast_3$ every model M_1 of T of cardinality $\le \aleph_1$ has a proper elementary extension to a model satisfying (c), i.e., (α)–(γ) of $\circledast_2$ above,

$\circledast_4$ if $\langle M_\alpha : \alpha < \delta\rangle$ is an increasing chain of models satisfying (c) of $\circledast_2$ and $\delta < \omega_2$ then also $\bigcup\{M_\alpha : \alpha < \delta\}$ does.

Altogether we are done. $\blacksquare_{1.4}$

Naturally we ask

Question 1.5. What occurs to T_0 for $\lambda > \aleph_1$ but $\lambda < 2^{\aleph_0}$?

Question 1.6. Does the theory T_2 of linear order consistently have an $(\aleph_1, \aleph_0)$-superlimit (or only strongly limit) model? (but see §3).

Question 1.7. What is the answer for T when T is countable superstable non-$\aleph_0$-stable and $D(T)$ is countable for $\aleph_1 < 2^{\aleph_0}$ and $\aleph_2 < 2^{\aleph_0}$?

By the above for some such T, in some universe, for $\aleph_1$ the answer is yes, there is a superlimit model.

2. A strictly stable consistent example. We now look at models of T_1 (redefined below) in cardinality $\aleph_1$; recall

Definition 2.1. $T_1 = \mathrm{Th}({}^\omega(\omega_1), E_n)_{n<\omega}$ where $E_n = \{(\eta, \nu) : \eta, \nu \in {}^\omega(\omega_1) \text{ and } \eta{\restriction}n = \nu{\restriction}n\}$.

Remark 2.2. Note that T_1 has elimination of quantifiers. Moreover, if $\lambda = \sum\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, then T_1 has a $(\lambda, \aleph_0)$-superlimit model in λ (see 2.15).

Definition/Claim 2.3. Any model of T_1 of cardinality λ is isomorphic to $M_{A,h} := (\{(\eta, \varepsilon) : \eta \in A, \varepsilon < h(\eta)\}, E_n)_{n<\omega}$ for some $A \subseteq {}^\omega\lambda$ and h :

([5]) We can replace $\mathbf{L}$ by any $\mathbf{V}_0$ which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$.

${}^{\omega}\lambda \to (\mathrm{Car} \cap \lambda^+) \setminus \{0\}$ where $(\eta_1, \varepsilon_1) E_n (\eta_2, \varepsilon_2) \Leftrightarrow \eta_1 {\restriction} n = \eta_2 {\restriction} n$; pedantically we should write $E_n^{M_{A,h}} = E_n {\restriction} |M_{A,n}|$.

We write M_A for $M_{A,h}$ when A is as above and $h : A \to \{|A|\}$, so constantly $|A|$ when A is infinite.

For $A \subseteq {}^{\omega}\lambda$ and h as above the model $M_{A,h}$ is a model of T_1 iff A is non-empty and $(\forall \eta \in A)(\forall n < \omega)(\exists^{\aleph_0} \nu \in A)(\nu {\restriction} n = \eta {\restriction} n \wedge \nu(n) \neq \eta(n))$.

Above $M_{A,h}$ has cardinality λ iff $\sum \{h(\eta) : \eta \in A\} = \lambda$.

Definition 2.4. We say that A is a (T_1, λ)-*witness* when:

- $A \subseteq {}^{\omega}\lambda$ has cardinality λ,
- if $B_1, B_2 \subseteq {}^{\omega}\lambda$ are (T_1, A)-big (see below) of cardinality λ then $(B_1 \cup {}^{\omega >}\lambda, \triangleleft)$ is isomorphic to $(B_2 \cup {}^{\omega >}\lambda, \triangleleft)$.

A set $B \subseteq {}^{\omega}\lambda$ is called (T_1, A)-*big* when it is (λ, λ)-(T_1, A)-big; see below.

B is (μ, λ)-(T_1, A)-*big* means: $B \subseteq {}^{\omega}\lambda, |B| = |A| = \mu$ and for every $\eta \in {}^{\omega >}\lambda$ there is an isomorphism f from $({}^{\omega \geq}\lambda, \triangleleft)$ onto $(\{\eta\hat{\ }\nu : \nu \in {}^{\omega \geq}\lambda\}, \triangleleft)$ mapping A into $\{\nu : \eta\hat{\ }\nu \in B\}$.

$A \subseteq {}^{\omega}(\omega_1)$ is $\aleph_1$-*suitable* when:

- $|A| = \aleph_1$,
- for a club of $\delta < \omega_1$, $A \cap {}^{\omega}\delta$ is everywhere non-meagre in the space ${}^{\omega}\delta$, i.e., for every $\eta \in {}^{\omega >}\delta$ the set $\{\nu \in A \cap {}^{\omega}\delta : \eta \triangleleft \nu\}$ is a non-meagre subset of ${}^{\omega}\delta$ (that is what is really used in [12]).

Claim 2.5. *It is consistent with ZFC that* $2^{\aleph_0} > \aleph_1$ + *there is a* $(T_1, \aleph_1)$-*witness; moreover every* $\aleph_1$-*suitable set is a* $(T_1, \aleph_1)$-*witness.*

Proof. By [12, §2]. $\blacksquare_{2.5}$

Remark 2.6. The witness does not give rise to an $(\aleph_1, \aleph_0)$-limit model as for the union of any "fast enough" $\prec$-increasing ω-chain of members of $\mathrm{EC}_{\aleph_1}(T_1)$, the relevant sets are meagre.

Definition 2.7. Let A be a (T_1, λ)-witness. We define $K^1_{T_1,A}$ as the family of $M = (|M|, <^M, P^M_\alpha)_{\alpha \leq \omega}$ such that:

(α) $(|M|, <^M)$ is a tree with $\omega + 1$ levels,

(β) P^M_α is the αth level; let $P^M_{<\omega} = \bigcup \{P^M_n : n < \omega\}$,

(γ) M is isomorphic to M^1_B for some $B \subseteq {}^{\omega}\lambda$ of cardinality λ where M^1_B is defined by $|M^1_\beta| = ({}^{\omega >}\lambda) \cup B$, $P_n^{M^1_B} = {}^{n}\lambda$, $P_\omega^{M^1_B} = B$ and $<^{M^1_B} = \triangleleft {\restriction} |M^1_B|$, i.e., being an initial segment,

(δ) moreover B is such that some f satisfies:

- $f : {}^{\omega >}\lambda \to \omega$ and $f(\langle\rangle) = 0$ for simplicity,
- $\eta \trianglelefteq \nu \in {}^{\omega >}\lambda \Rightarrow f(\eta) \leq f(\nu)$,
- if $\eta \in B$ then $\langle f(\eta {\restriction} n) : n < \omega\rangle$ is eventually constant,

- if $\eta \in {}^{\omega>}\lambda$ then $\{\nu \in {}^{\omega}\lambda : \eta^\frown\nu \in B$ and $m < \omega \Rightarrow f(\eta^\frown(\nu\restriction m)) = f(\eta)\}$ is (T_1, A)-big,
- for $\eta \in {}^{\omega>}\lambda$ and $n \in [f(\eta), \omega)$ for λ ordinals $\alpha < \lambda$, we have $f(\eta^\frown\langle\alpha\rangle) = n$.

CLAIM 2.8 (The Global Axiom of Choice). *If A is a $(T_1, \aleph_1)$-witness then:*

(a) $K^1_{T_1,A} \neq \emptyset$,
(b) *any two members of $K^1_{T_1,A}$ are isomorphic,*
(c) *there is a function $\mathbf{F}$ from $K^1_{T_1,A}$ to itself (up to isomorphism, i.e., $(M, \mathbf{F}(M))$ is defined only up to isomorphism) satisfying $M \subseteq \mathbf{F}(M)$ such that $K^1_{T_1,A}$ is closed under increasing unions of sequences $\langle M_n : n < \omega\rangle$ such that $\mathbf{F}(M_n) \subseteq M_{n+1}$.*

Proof. (a): Trivial.
(b): By the definition of "A is a $(T_1, \aleph_1)$-witness" and of $K^1_{T_1,A}$.
(c): We choose $\mathbf{F}$ such that

- if $M \in K^1_{A,T_1}$ then $M \subseteq \mathbf{F}(M) \in K^1_{A,T_1}$ and for every $k < \omega$ and $a \in P^M_k$, the set $\{b \in P^{\mathbf{F}(M)}_{k+1} : a <_{\mathbf{F}(M)} b$ and $b \notin M\}$ has cardinality $\aleph_1$.

Assume $M = \bigcup\{M_n : n < \omega\}$ where $\langle M_n : n < \omega\rangle$ is $\subseteq$-increasing, $M_n \in K^1_{A,T_1}$, $\mathbf{F}(M_n) \subseteq M_{n+1}$. Clearly M is as required at the beginning of Definition 2.7, that is, satisfies clauses (α)–(γ) there. To prove (δ), we define $f : P^M_{<\omega} \to \omega$ by $f(a) = \mathrm{Min}\{n : a \in M_n\}$. Pedantically, $\mathbf{F}$ is defined only up to isomorphism. $\blacksquare_{2.8}$

CLAIM 2.9. *If A is a (T_1, λ)-witness then:*

(a) $K^1_{T_1,A} \neq \emptyset$,
(b) *any two members of $K^1_{T_1,A}$ are isomorphic,*
(c) *if $M_n \in K^1_{T_1,A}$ and $n < \omega \Rightarrow M_n \subseteq M_{n+1}$ then $M := \bigcup\{M_n : n < \omega\} \in K^1_{T_1,A}$.*

REMARK 2.10. If we omit clause (b), we can weaken the demand on the set A.

Proof. Assume $M = \bigcup\{M_n : n < \omega\}$, $M_n \subseteq M_{n+1}$, $M_n \in K^1_{T_1,A}$ and f_n witnesses $M_n \in K^1_{T_1,A}$. Clearly M satisfies clauses (α)–(γ) of Definition 2.7; we just have to find a witness f as in (δ) there.

For each $a \in M$ let $n(a) = \mathrm{Min}\{n : a \in M_n\}$; clearly if $M \models$ "$a < b < c$" then $n(a) \leq n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \to M$ be defined by: $g_n(a) = b$ iff $b \leq^M a$, $b \in M_n$ and b is $\leq^M$-maximal under those restrictions; clearly it is well defined. Now we define $f'_n : M_n \to \omega$ by induction on $n < \omega$ such that $m < n \Rightarrow f'_m \subseteq f'_n$, as follows.

If $n = 0$ let $f'_n = f_n$.

If $n = m + 1$ and $a \in M_n$ we let $f'_n(a)$ be $f'_m(a)$ if $a \in M_m$ and be $(f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1$ if $a \in M_n \backslash M_m$. Clearly $f := \bigcup\{f'_n : n < \omega\}$ is a function from M to ω, $a \leq^M b \Rightarrow f(a) \leq f(b)$, and for any $a \in M$ the set $\{b \in M : a \leq^M b$ and $f(b) = f(a)\}$ is equal to $\{b \in M_{n(a)} : f_{n(a)}(a) = f_{n(a)}(b)$ and $a \leq^M b\}$. $\blacksquare_{2.9}$

Definition 2.11. Let A be a (T_1, λ)-witness. We define $K^2_{T_1,A}$ as in Definition 2.7 but f is constantly zero.

Claim 2.12 (The Global Axiom of Choice). *If A is a $(T_1, \aleph_1)$-witness then:*

(a) $K^2_{T_1,A} \neq \emptyset$,

(b) *any two members of $K^2_{T_1,A}$ are isomorphic,*

(c) *there is a function $\mathbf{F}$ from $\bigcup\{{}^{\alpha+2}(K^2_{T_0,A}) : \alpha < \omega_1\}$ to $K^2_{T_1,A}$ which satisfies:*

  (α) *if $\bar{M} = \langle M_i : i \leq \alpha + 1\rangle$ is an $\prec$-increasing sequence of models of T then $M_{\alpha+1} \subseteq \mathbf{F}(\bar{M}) \in K^2_{T_1,A}$,*

  (β) *when $\omega_1 = \sup\{\alpha : \mathbf{F}(\bar{M}{\restriction}r(\alpha+2)) \subseteq M_{\alpha+2})$ and is a well defined embedding of M_α into $M_{\alpha+2}\}$, the union of any increasing ω_1-sequence $\bar{M} = \langle M_\alpha : \alpha < \omega_1\rangle$ of members of $K^2_{T_1,A}$ belongs to $K^2_{T_1,A}$.*

Remark 2.13. Instead of the global axiom of choice, we can restrict the models to have universe a subset of λ^+ (or just a set of ordinals).

Proof. (a): Easy.

(b): By the definition.

(c): Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \omega_1\rangle$ be an increasing sequence of subsets of ω_1 with union ω_1 such that $\varepsilon < \omega_1 \Rightarrow |\mathscr{U}_\varepsilon \backslash \bigcup_{\zeta<\varepsilon} \mathscr{U}_\zeta| = \aleph_1$. Let $M^* \in K^2_{T_1,A}$ be such that ${}^{\omega>}(\omega_1) \subseteq |M^*| \subseteq {}^{\omega\geq}(\omega_1)$ and $M^*_\varepsilon := M^*{\restriction}{}^{\omega\geq}(\mathscr{U}_\varepsilon)$ belongs to $K^2_{T_1,A}$ for every $\varepsilon < \omega_1$.

We choose a pair $(\mathbf{F}, \mathbf{f})$ of functions with domain $\{\bar{M} : \bar{M}$ an increasing sequence of members of $K^2_{T_1,A}$ of length $< \omega_1\}$ such that:

- $\mathbf{F}(\bar{M})$ is an extension of $\bigcup\{M_i : i < \ell g(\bar{M})\}$ from $K^2_{T_1,A}$,
- $\mathbf{f}(\bar{M})$ is an embedding from $M^*_{\ell g(\bar{M})}$ into $\mathbf{F}(\bar{M})$,
- if $\bar{M}^\ell = \langle M_\alpha : \alpha < \alpha_\ell\rangle$ for $\ell = 1, 2$ and $\alpha_1 < \alpha_2$, $\bar{M}^1 = \bar{M}^2{\restriction}\alpha_1$ and $\mathbf{F}(\bar{M}^1) \subseteq M_{\alpha_1}$ then $\mathbf{f}(\bar{M}^1) \subseteq \mathbf{f}(\bar{M}^2)$,
- if $a \in \mathbf{F}(\bar{M})$ and $n < \omega$ then for some $b \in M^*_{\ell g(\bar{M})}$ we have $\mathbf{F}(M) \models aE_n(\mathbf{f}(\bar{M})(b))$.

Now check. $\blacksquare_{2.12}$

Conclusion 2.14. *Assume there is a* $(T_1, \aleph_1)$*-witness* (see Definition 2.4) *for the first-order complete theory* T_1 *from* 2.1. *Then:*

(1) T_1 *has an* $(\aleph_1, \aleph_0)$*-strongly limit model.*
(2) T_1 *has an* $(\aleph_1, \aleph_1)$*-medium limit model.*
(3) T_1 *has an* $(\aleph_1, \aleph_0)$*-superlimit model.*

Proof. (1) By 2.8 the reduction of problems on $(\mathrm{EC}(T_1), \prec)$ to $K^1_{T_1,A}$ (which is easy) is exactly as in [12].

(2) By 2.12.

(3) Like part (1) using Claim 2.9. $\blacksquare_{2.14}$

Claim 2.15. *If* $\lambda = \sum\{\lambda_n : n < \omega\}$ *and* $\lambda_n = \lambda_n^{\aleph_0}$, *then* T_1 *has a* $(\lambda, \aleph_0)$*-superlimit model in* λ.

Proof. Let M_n be the model M_{A_n,h_n} where $A_n = {}^{\omega}(\lambda_n)$ and $h_n : A_n \to \lambda_n^+$ is constantly λ_n. Clearly,

$(*)_1$ M_n is a saturated model of T_1 of cardinality λ_n,
$(*)_2$ $M_n \prec M_{n+1}$,
$(*)_3$ $M_\omega = \bigcup\{M_n : n < \omega\}$ is a special model of T_1 of cardinality λ.

The main point is:

$(*)_4$ M_ω is $(\lambda, \aleph_0)$-superlimit model of T_1.

[Why? Toward this assume:

- N_n is isomorphic to M_ω, say $f_n : M_\omega \to N_n$ is an isomorphism,
- $N_n \prec N_{n+1}$ for $n < \omega$.

Let $N_\omega = \bigcup\{N_n : n < \omega\}$ and we should prove $N_\omega \cong M_\omega$, so just N_ω is a special model of T_1 of cardinality λ suffice.

Let $N'_n = N_\omega \restriction (\bigcup\{f_n(M_k) : k \le n\})$. Clearly $N'_n \prec N'_{n+1} \prec N_\omega$ and $\bigcup\{N'_n : n < \omega\} = N_{\omega_*}$ and $\|N'_n\| = \lambda_n$. So it suffices to prove that N'_n is saturated and direct inspection shows this. $\blacksquare_{2.15}$

3. On non-existence of limit models. Naturally we assume that non-existence of superlimit models for unstable T is easier to prove. For other versions we need to look more. We first show that for $\lambda \ge |T| + \aleph_1$, if T is unstable then it does not have a superlimit model of cardinality λ, and if T is unsuperstable, we show this for "most" cardinals λ. On "Φ proper for K_{or} or K^{ω}_{tr}", see [15, VII] or [7] or hopefully some day in [8, III]. We assume some knowledge of stability.

Claim 3.1. (1) *If* T *is unstable,* $\lambda \ge |T| + \aleph_1$, *then* T *has no superlimit model of cardinality* λ.

(2) *If* T *is stable non-superstable and* $\lambda \ge |T| + \beth_\omega$ *or* $\lambda = \lambda^{\aleph_0} \ge |T|$ *then* T *has no superlimit model of cardinality* λ.

Remark 3.2. We assume some knowledge of EM models for linear orders I and members of $K^\omega_{\rm tr}$ as index models (see, e.g., [15, VII]).

(2) We use the following definition in the proof, as well as a result from [17] or [18].

Definition 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal μ such that for some, equivalently for every set A of cardinality λ there is $\mathscr{P}_A \subseteq [A]^{\leq\kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality λ such that any $B \in [\lambda]^{\leq\kappa}$ is the union of $< \kappa$ members of $\mathscr{P}_A$.

Proof of Claim 3.1. (1) Towards a contradiction assume M^* is a superlimit model of T of cardinality λ. As T is unstable we can find m and $\varphi(\bar{x}, \bar{y})$ such that

- $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)}$ linearly orders some infinite $\mathbf{I} \subseteq {}^m M, M \models T$ so $\ell g(\bar{x}) = \ell g(\bar{y}) = m$.

We can find a Φ which is proper for linear orders ([15, VII]) and $F_\ell(\ell < m)$ such that $F_\ell \in \tau_\Phi \backslash \tau_T$ is a unary function symbol for $\ell < m, \tau_T \subseteq \tau(\Phi)$ and for every linear order I, $\mathrm{EM}(I, \Phi)$ has Skolem functions and its τ_T-reduct $\mathrm{EM}_{\tau(T)}(I, \Phi)$ is a model of T of cardinality $|T| + |I|$ and $\tau(\Phi)$ is of cardinality $|T| + \aleph_0$ and $\langle a_s : s \in I\rangle$ is the skeleton of $\mathrm{EM}(I, \Phi)$, that is, it is an indiscernible sequence in $\mathrm{EM}(I, \Phi)$ and $\mathrm{EM}(I, \Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $\bar{a}_s = \langle F_\ell(a_s) : \ell < m\rangle$ in $\mathrm{EM}(I, \Phi)$ we have $\mathrm{EM}_{\tau(T)}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\mathrm{if}(s<t)}$ for $s, t \in I$.

Next we can find Φ_n (for $n < \omega$) such that:

(a) Φ_n is proper for linear orders and $\Phi_0 = \Phi$,

(b) $\mathrm{EM}_{\tau(\Phi)}(I, \Phi_n) \prec \mathrm{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ for every linear order I and $n < \omega$; moreover

(b)$^+$ $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and $\mathrm{EM}(I, \Phi_n) \prec \mathrm{EM}_{\tau(\Phi_n)}(I, \Phi_{n+1})$ for every $n < \omega$ and linear order I,

(c) if $|I| \leq n$ then $\mathrm{EM}_{\tau(\Phi)}(I, \Phi_n) = \mathrm{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ and $\mathrm{EM}_{\tau(T)}(I, \Phi_n) \cong M^*$,

(d) $|\tau(\Phi_n)| = \lambda$.

This is easy. Let Φ_ω be the limit of $\langle \Phi_n : n < \omega\rangle$, i.e. $\tau(\Phi_\omega) = \bigcup\{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then $\mathrm{EM}_{\tau(\Phi_k)}(I, \Phi_\omega) = \bigcup\{\mathrm{EM}_{\tau(\Phi_k)}(I, \Phi_n) : n \in [k, \omega)\}$. So as M^* is a superlimit model, for any linear order I of cardinality λ, $\mathrm{EM}_{\tau(T)}(I, \Phi_\omega)$ is the direct limit of $\langle \mathrm{EM}_{\tau(T)}(J, \Phi_\omega) : J \subseteq I \text{ finite}\rangle$, each isomorphic to M^*, so as we have assumed that M^* is a superlimit model it follows that $\mathrm{EM}_{\tau(T)}(I, \Phi_\omega)$ is isomorphic to M^*. But by [14, III] or [7] which may eventually be [8, III] there are 2^λ many pairwise non-isomorphic models of this form varying I on the linear orders of cardinality λ, contradiction.

(2) First assume $\lambda = \lambda^{\aleph_0}$. Let $\tau \subseteq \tau_T$ be countable such that $T' = T \cap \mathbb{L}(\tau)$ is not superstable. Clearly if M^* is a $(\lambda, \aleph_0)$-limit model then

$M^*{\restriction}\tau'$ is not $\aleph_1$-saturated. [Why? As in [10, Ch. VI, §6], but we shall give full details: there are $N_* \models T$, $p = \{\varphi_n(\lambda, \bar{a}_n) : n < \omega\}$ a type in N_*, $\bar{a}_n \triangleleft \bar{a}_{n+1}$, $\bar{a}_{\langle\rangle}$ empty and $\varphi_{n+1}(x, \bar{a}_{n+1})$ forks over $\bar{a}_n$. Let $\mathbf{F}(M)$ be such that if $n < \omega$ and $\bar{b}_n \subseteq M$ realizes $\mathrm{tp}(\bar{a}_n, \emptyset, N_*)$ then for some $\bar{b}_{n+1}$ from $\mathbf{F}, M$ realizing $\mathrm{tp}(\bar{a}_{n+1}, \emptyset, N_*)$, the type $\mathrm{tp}(\bar{b}_{n+1}, M, \mathbf{F}(M))$ does not fork over b_n.] But if $\kappa = \mathrm{cf}(\kappa) \in [\aleph_1, \lambda]$ and M^* is a (λ, κ)-limit then $M^*{\restriction}\tau'$ is $\aleph_1$-saturated, contradiction.

The case $\lambda \geq |T| + \beth_\omega$ is more complicated (the assumption $\lambda \geq \beth_\omega$ is to enable us to use [17] or see [18] for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As T is stable non-superstable by [15] for some $\bar{\Delta}$:

$\circledast_1$ for any μ there are M and $\langle a_{\eta,\alpha} : \eta \in {}^\omega\mu$ and $\alpha < \mu\rangle$ such that

- (a) M is a model of T,
- (b) $\mathbf{I}_\eta = \{a_{\eta,\alpha} : \alpha < \mu\} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu \Rightarrow a_{\eta,\alpha} \neq a_{\eta,\beta}$),
- (c) $\bar{\Delta} = \langle \Delta_n : n < \omega\rangle$ and $\Delta_n \subseteq \mathbb{L}_{\tau(T)}$ infinite,
- (d) for $\eta, \nu \in {}^\omega\mu$ we have $\mathrm{Av}_{\Delta_n}(M, \mathbf{I}_\eta) = \mathrm{Av}_{\Delta_n}(M, \mathbf{I}_\nu)$ iff $\eta{\restriction}n = \nu{\restriction}n$.

Hence by [15, VIII] (or see [7] assuming M^* is a universal model of T of cardinality λ):

$\circledast_{2.1}$ there is Φ such that:

- (a) Φ is proper for $K^\omega_{\mathrm{tr}}, \tau_T \subseteq \tau(\Phi), |\tau(\Phi)| = \lambda \geq |T| + \aleph_0$,
- (b) for $I \subseteq {}^{\omega\geq}\lambda$, $\mathrm{EM}_{\tau(\Phi)}(I, \Phi)$ is a model of T and $I \subseteq J \Rightarrow \mathrm{EM}(I, \Phi) \prec \mathrm{EM}(J, \Phi)$,
- (c) for some two-place function symbol F if for $I \in K^\omega_{\mathrm{tr}}$ and $\eta \in P^I_\omega$, I a subtree of ${}^{\omega\geq}\lambda$, for transparency we let $\mathbf{I}_{I,\eta} = \{F(a_\eta, a_\nu) : \nu \in I\}$, then $\langle \mathbf{I}_{I,\eta} : \eta \in P^I_\omega\rangle$ are as in $\circledast_1$(b), (d).

Also

$\circledast_{2.2}$ if Φ_1 satisfies (a)–(c) of $\circledast_{2.1}$ and M is a universal model of T then there is Φ^*_2 satisfying (a)–(c) of $\circledast_{2.1}$ and $\Phi_1 \leq \Phi^*_2$ (see $\circledast_{2.3}$(a)) and for every finitely generated $J \in K^\omega_{\mathrm{tr}}$ (see $\circledast_{2.3}$(b)) there is $M' \cong M$ such that $\mathrm{EM}_{\tau(T)}(J, \Phi_1) \prec M' \prec \mathrm{EM}_{\tau(T)}(J, \Phi^*_2)$,

$\circledast_{2.3}$ (a) we say $\Phi_1 \leq \Phi_2$ when $\tau(\Phi_1) \subseteq \tau(\Phi_2)$ and $J \in K^\omega_{\mathrm{tr}} \Rightarrow \mathrm{EM}(J, \Phi_1) \prec \mathrm{EM}_{\tau(\Phi_1)}(J, \Phi_2)$,

(b) we say that $J \subseteq I$ is *finitely generated* if it has the form $\{\eta_\ell : \ell < n\} \cup \{\rho$: for some n, ℓ we have $\rho \in P^I_n$ and $\rho <^I \eta_\ell\}$ for some $\eta_0, \ldots, \eta_{n-1} \in P^I_\omega$,

$\circledast_{2.4}$ if $M_* \in \mathrm{EC}_\lambda(T)$ is superlimit (or just weakly S-limit, with $S \subseteq \lambda^+$ stationary) then there is Φ as in $\circledast_{2.1}$ above such that $\mathrm{EM}_{\tau(T)}(J, \Phi) \cong M_*$ for every finitely generated $J \in K^\omega_{\mathrm{tr}}$,

$\circledast_{2.5}$ we fix Φ as in $\circledast_{2.4}$ for $M_* \in \mathrm{EC}_\lambda(T)$ superlimit.

Hence (mainly by clause (b) of $\circledast_{2.1}$ and $\circledast_{2.4}$ as in the proof of part (1))

$\circledast_3$ if $I \in K^\omega_{\mathrm{tr}}$ has cardinality $\leq \lambda$ then $\mathrm{EM}_{\tau(\Phi)}(I, \Phi)$ is isomorphic to M^*.

Now by [17], we can find regular uncountable $\kappa < \beth_\omega$ such that $\lambda = \lambda^{[\kappa]}$ (see Definition 3.3).

Let $S = \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ and $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ be such that η_δ is an increasing sequence of length ω with limit δ.

For a model M of T let $\mathrm{OB}_{\bar\eta}(M) = \{\bar{\mathbf{a}} : \bar{\mathbf{a}} = \langle a_{\eta_\delta,\alpha} : \delta \in W$ and $\alpha < \kappa\rangle, W \subseteq S$ and in M they are as in $\circledast_1$(b), (d)$\}$. For $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ let $W[\bar{\mathbf{a}}]$ be W as above and let

$$\begin{aligned}\Xi(\bar{\mathbf{a}}, M) = \{\eta \in {}^\omega\kappa : {} & \text{there is an indiscernible set} \\ & \mathbf{I} = \{a_\alpha : \alpha < \kappa\} \text{ in } M \text{ such that for every } n, \\ & \text{for some } \delta \in W[\bar{\mathbf{a}}],\ \eta{\restriction}n = \eta_\delta{\restriction}n \text{ and} \\ & \mathrm{Av}_{\Delta_n}(M, \mathbf{I}) = \mathrm{Av}_{\Delta_n}(M, \{a_{\eta_\delta,\alpha} : \alpha < \kappa\})\}.\end{aligned}$$

Clearly:

$\circledast_4$ (a) if $M \prec N$ then $\mathrm{OB}_{\bar\eta}(M) \subseteq \mathrm{OB}_{\bar\eta}(N)$,
(b) if $M \prec N$ and $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ then $\Xi(\bar{\mathbf{a}}, M) \subseteq \Xi(\bar{\mathbf{a}}, N)$.

Now by the choice of κ it should be clear that:

$\circledast_5$ if $M \models T$ is of cardinality λ then we can find an elementary extension N of M of cardinality λ such that for every $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ, for some stationary $W' \subseteq W[\bar{\mathbf{a}}]$ the set $\Xi[\bar{\mathbf{a}}, N]$ includes $\{\eta \in {}^\omega\kappa : (\forall n)(\exists \delta \in W')(\eta{\restriction}n = \eta_\delta{\restriction}n)\}$ (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[\bar{\mathbf{a}}] = \bigcup\{W_\varepsilon : \varepsilon < \varepsilon^*\}$),

$\circledast_6$ we find $M \in \mathrm{EC}_\lambda(T)$ isomorphic to M^* such that for every $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ, we can find a stationary subset W' of $W[\bar{\mathbf{a}}]$ such that the set $\Xi[\bar{\mathbf{a}}, M]$ includes $\{\eta \in {}^\omega\mu : (\forall n)(\exists \delta \in W')(\eta{\restriction}n = \eta_\delta{\restriction}n)\}$.

[Why? We choose (M_i, N_i) for $i < \kappa^+$ such that:

- $M_i \in \mathrm{EC}_\lambda(T)$ is $\prec$-increasing continuous,
- M_{i+1} is isomorphic to M^*,
- $M_i \prec N_i \prec M_{i+1}$,
- (M_i, N_i) are like (M, N) in $\circledast_5$.

Now $M = \bigcup\{M_i : i < \kappa^+\}$ is as required. The model M is isomorphic to M^* as M^* is superlimit.]

Now the model from $\circledast_6$ is not isomorphic to $M' = \mathrm{EM}_{\tau(T)}({}^{\omega>}\lambda \cup \{\eta_\delta : \delta \in S\}, \Phi)$ where Φ is from $\circledast_{2.1}$. But $M' \cong M^*$ by $\circledast_3$.

Altogether we are done. $\blacksquare_{3.1}$

The following claim says in particular that if some not unreasonable pcf conjectures hold, the conclusion holds for every $\lambda \geq 2^{\aleph_0}$.

CLAIM 3.4. *Assume T is stable non-superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$.*

(1) *T has no (λ, κ)-superlimit model provided that $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\kappa^{\aleph_0} \leq \lambda$ and $\lambda = \mathbf{U}_D(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\kappa$ and for every $f : \kappa \to \lambda$ for some $u \in \mathscr{P}$ we have $\{\alpha < \kappa : f(\alpha) \in u\} \in D^+\}$, where D is a normal filter on κ to which $\{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ belongs.*

(2) *Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \leq \aleph_0\}$, $J_1 = \{u \subseteq \kappa : u \cap S^\kappa_{\aleph_0}$ non-stationary$\}$ we have $\lambda = \mathbf{U}_{J_1,J_0}(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{\aleph_0}$, and if $u \in J_1$ and $f : (\kappa \backslash u) \to \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathscr{P}$, $\mathrm{Rang}(f \restriction w) \subseteq v\}$.*

Proof. Like 3.1. ■

CLAIM 3.5. (1) *Assume T is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$, T has a weakly (λ, κ)-limit model and even a weakly (λ, S)-limit model for some stationary $S \subseteq S^\lambda_\kappa$.*

(2) *Assume T is unsuperstable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \mathrm{cf}(\kappa_2)$. Then T has no model which is a weak (λ, S)-limit where $S \subseteq \lambda$ and $S \cap S^\lambda_{\kappa_\ell}$ is stationary for $\ell = 1, 2$.*

Proof. (1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$ (see Definition 3.3) and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let m and $\varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1. Then

- $(*)$ if $M \in \mathrm{EC}_\lambda(T)$ then there is N such that:
  - (a) $N \in \mathrm{EC}_\lambda(T)$,
  - (b) $M \prec N$,
  - (c) if $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa \rangle \in {}^\kappa({}^m M)$ for $\alpha < \kappa$ then for some $\mathscr{U} \in [\kappa]^\chi$, for every uniform ultrafilter D on κ to which $\mathscr{U}$ belongs there is $\bar{a}_D \in {}^n N$ such that $\mathrm{tp}(\bar{a}_D, N, N) = \mathrm{Av}(D, \bar{\mathbf{a}}, M) = \{\psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T)$, $\bar{c} \in {}^{\ell g(\bar{z})} M$ and $\{\{\alpha < \kappa : N \models \psi[\bar{a}_{i_\alpha}, \bar{c}]\} \in D\}$.

Similarly

- $\boxplus_1$ for every function $\mathbf{F}$ with domain $\{\bar{M} : \bar{M}$ an $\prec$-increasing sequence of models of T of length $< \lambda^+$ each with universe $\in \lambda^+\}$ such that $M_i \prec \mathbf{F}(\bar{M})$ for $i < \ell g(\bar{M})$ and $\mathbf{F}(\bar{M})$ has universe $\in \lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+ \rangle$ obeying $\mathbf{F}$ such that: for every $\varepsilon < \lambda^+$ and $\bar{\mathbf{a}} \in {}^\kappa({}^m(M_\varepsilon))$ for $\alpha < \kappa$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which $\mathscr{U}$ belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\bar{\mathbf{a}}_{D,\zeta} \in {}^m(M_{\zeta+1})$ realizing $\mathrm{Av}(D, \bar{\mathbf{a}}, M_\zeta)$ in $M_{\zeta+1}$.

Hence

$\boxplus_2$ for $\langle M_\alpha : \alpha < \lambda^+\rangle$ as in $\boxplus_1$, for every limit $\delta < \lambda^+$ of cofinality $\neq \kappa$ and every $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa\rangle \in {}^{\kappa}({}^{m}(M_\delta))$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which $\mathscr{U}$ belongs, there is a sequence $\langle \bar{b}_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle \in {}^{\mathrm{cf}(\delta)}({}^{m}(M_\delta))$ such that for every $\psi(\bar{x},\bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}(M_\delta)$, and for every $\varepsilon < \mathrm{cf}(\delta)$ large enough, $M_\delta \models \psi[\bar{b}_\varepsilon, \bar{c}]$ iff $\psi(\bar{x},\bar{c}) \in \mathrm{Av}(D, \bar{\mathbf{a}}, M_\delta)$.

The rest should be clear.

(2) Combine the above and the proof of 3.1(2). $\blacksquare_{3.5}$

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