WHEN A FIRST ORDER T HAS LIMIT MODELS

BY

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Abstract. We sort out to a large extent when a (first order complete theory) $T$ has a superlimit model in a cardinal $\lambda$. Also we deal with related notions of being limit.

Annotated content

0. Introduction. We give background and basic definitions. We then present existence results for stable $T$ which have models that are saturated or close to being saturated.

1. On countable superstable non-$\aleph_0$-stable. Consistently $2^{\aleph_1} \geq \aleph_2$ and some such (complete first order) $T$ has a superlimit (non-saturated) model of cardinality $\aleph_1$. This shows that we cannot prove a non-existence result fully complementary to the results in 0.9.

2. A strictly stable consistent example. Consistently $\aleph_1 < 2^{\aleph_0}$ and some countable stable not superstable $T$ has a (non-saturated) model of cardinality $\aleph_1$ which satisfies some relatives of being superlimit.

3. On the non-existence of limit models. The proofs here are in ZFC. If $T$ is unstable, it has no superlimit models of cardinality $\lambda$ when $\lambda \geq \aleph_1 + |T|$. For unsuperstable $T$ we have similar results but with “few” exceptional cardinals $\lambda$ on which we do not know: $\lambda < \lambda^{\aleph_0}$ which are $< \beth_\omega$. Moreover, if $T$ is superstable and $\lambda \geq |T| + 2^{|T|}$ then $T$ has a superlimit model of cardinality $\lambda$ iff $|D(T)| \leq \lambda$ iff $T$ has a saturated model. Lastly, we get weaker results on weaker relatives of superlimit.

0. Introduction

0A. Background and content. Recall that ([15, Ch. III]) if $T$ is (first order complete and) superstable then for $\lambda \geq 2^{|T|}$, $T$ has a saturated model $M$ of cardinality $\lambda$ and moreover

\[ (*) \text{ if } \langle M_\alpha : \alpha < \delta \rangle \text{ is } \prec\text{-increasing, } \delta \text{ a limit ordinal } < \lambda^+ \text{ and } \alpha < \delta \Rightarrow M_\alpha \cong M \text{ then } \bigcup \{ M_\alpha : \alpha < \delta \} \text{ is isomorphic to } M. \]

When investigating categoricity of an a.e.c. (abstract elementary class) $\mathfrak{E} = (K_\mathfrak{E}, \leq_\mathfrak{E})$, the following property turns out to be central: $M$ is a $\leq_\mathfrak{E}$-universal model of cardinality $\lambda$ with the property $(\ast)$ above (called superlimit), possibly with additional parameter $\kappa = \text{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$);

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we also consider some relatives of the “superlimit” notion, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [13, 3.1]; see also the revised version [3, 3.3] and [19], or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete \( T \), we know \{ \( \lambda : T \) has a saturated model of cardinality \( \lambda \) \}, namely, it is \{ \( \lambda : \lambda^{<\lambda} \geq |D(T)| \) or \( T \) is stable in \( \lambda \) \}; for the definitions of \( D(T) \) and other notions see 0B below. What if we replace saturated by superlimit (or some relative)? Let \( EC_{\lambda}(T) \) be the class of models \( M \) of \( T \) of cardinality \( \lambda \).

If there is a saturated \( M \in EC_{\lambda}(T) \) we have considerable knowledge on the existence of a limit model for the cardinal \( \lambda \), by [15], as mentioned in [3, 3.6] (see [0,9(1),(2)]. E.g. for superstable \( T \) in \( \lambda \geq 2^{|T|} \) there is a superlimit model (the saturated one). It seems a natural question on [3, 3.6] whether it exhausts the possibilities of \((\lambda, *)\)-superlimit and \((\lambda, \kappa)\)-superlimit models for elementary classes. Clearly the cases of the existence of such models of a (first order complete) theory \( T \) where there are no saturated (or special) models are rare, because even the weakest version of Definition [13, 3.1] = [3, 3.3] or here Definition 0.7 for \( \lambda \) implies that \( T \) has a universal model of cardinality \( \lambda \), which is rare (see Kojman–Shelah [2] which includes earlier history and recently Džamonja [1]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for \( \lambda < \lambda^{<\lambda} \)), i.e., those for \( \lambda = \aleph_1 \).

E.g. a sufficient condition for some versions is the existence of \( T' \supseteq T \) of cardinality \( \lambda \) such that \( PC(T', T) \) is categorical in \( \lambda \) (see [0.4(3)]). By [12] we have consistency results for such \( T_1 \) so naturally we first deal with the consistency results from [12]. In §1 we deal with the case of the countable superstable \( T_0 \) from [12] which is not \( \aleph_0 \)-stable. By [12] consistently \( \aleph_1 < 2^{\aleph_0} \) and for some \( T'_0 \supseteq T_0 \) of cardinality \( \aleph_1 \), \( PC(T'_0, T_0) \) is categorical in \( \aleph_1 \). We use this to get the consistency of “\( T_0 \) has a superlimit model of cardinality \( \aleph_1 \) and \( \aleph_1 < 2^{\aleph_0} \)”.

In §2 for some stable non-superstable countable \( T_1 \) we have a parallel but weaker result. We reconsider the old consistency results of “some \( PC(T'_1, T_1), |T'_1| = \aleph_1 > |T_1| \), is categorical in \( \aleph_1 \)” from [12]. From this we deduce that in this universe, \( T_1 \) has a strongly \((\aleph_1, \aleph_0)\)-limit model.

It is a reasonable thought that we can similarly have a consistency result for the theory of linear orders, but this is still unclear.

In §3 we show that if \( T \) has a superlimit model in \( \lambda \geq |T| + \aleph_1 \) then \( T \) is stable and \( T \) is superstable except possibly under some severe restrictions on
the cardinal \(\lambda\) (i.e., \(\lambda < \beth_{\omega}\) and \(\lambda < \lambda^{\aleph_0}\)). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by \([1.1 + 3.1]\) we have:

**Conclusion 0.1.** Assume \(\lambda \geq |T| + \beth_{\omega}\). Then \(T\) has a superlimit model of cardinality \(\lambda\) iff \(T\) is superstable and \(\lambda \geq |D(T)|\).

In subsequent work we shall show that for some unstable \(T\) (e.g. the theory of linear orders), if \(\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)\), then \(T\) has a medium \((\lambda, \kappa)\)-limit model, whereas if \(T\) has the independence property, even weak \((\lambda, \kappa)\)-limit models do not exist; see \([5]\) and more in \([6], [20], [4], [9]\).

### 0B. Basic definitions

**Notation 0.2.** Let \(T\) denote a complete first order theory which has infinite models but \(T_1, T'\) etc. are not necessarily complete.

If \(M, N\) denote models, then \(|M|\) is the universe of \(M\) and \(\|M\|\) its cardinality and \(M \prec N\) means \(M\) is an elementary submodel of \(N\).

Let \(\tau_T = \tau(T)\), \(\tau_M = \tau(M)\) be the vocabularies of \(T, M\) respectively.

Let \(M \models \varphi[\bar{a}]^{\text{stat}}\) mean that the model \(M\) satisfies \(\varphi[\bar{a}]\) if the statement stat is true (or is 1 rather than 0).

**Definition 0.3.** For \(\bar{a} \in \omega^>|M|\) and \(B \subseteq M\) let
\[
\text{tp}(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{b} \in \ell g(\bar{y})B \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}.
\]

Let
\[D(T) = \{\text{tp}(\bar{a}, \emptyset, M) : M \text{ a model of } T \text{ and } \bar{a} \text{ a finite sequence from } M\}\.
\]

If \(A \subseteq M\) then
\[S^m(A, M) = \{\text{tp}(\bar{a}, A, N) : M \prec N \text{ and } \bar{a} \in ^mN\};\]
if \(m = 1\) we may omit it.

A model \(M\) is \(\lambda\)-saturated when: if \(A \subseteq M, |A| < \lambda\) and \(p \in S(A, M)\) then \(p\) is realized by some \(a \in M\), i.e. \(p \subseteq \text{tp}(a, A, M)\); if \(\lambda = \|M\|\) we may omit it.

A model \(M\) is special when letting \(\lambda = \|M\|\), there is an increasing sequence \(\langle \lambda_i : i < \text{cf}(\lambda) \rangle\) of cardinals with limit \(\lambda\) and a \(\prec\)-increasing sequence \(\langle M_i : i < \text{cf}(\lambda) \rangle\) of models with union \(M\) such that \(M_{i+1}\) is \(\lambda_i\)-saturated of cardinality \(\lambda_{i+1}\) for \(i < \text{cf}(\lambda)\).

**Definition 0.4.** For any \(T\) let
\[\text{EC}(T) = \{M : M \text{ is a } \tau_T\text{-model of } T\},\]
\[\text{EC}_\lambda(T) = \{M \in \text{EC}(T) : M \text{ is of cardinality } \lambda}\].
For $T \subseteq T'$ let
\[
PC(T', T) = \{M \mid \tau_T : M \text{ model of } T'\},
\]
\[
PC_\lambda(T', T) = \{M \in PC(T', T) : M \text{ is of cardinality } \lambda\}.
\]

We say $M$ is $\lambda$-universal for $T_1$ when it is a model of $T_1$ and every $N \in EC_\lambda(T)$ can be elementarily embedded into $M$; if $T_1 = \text{Th}(M)$ we may omit it.

We say $M \in EC(T)$ is universal when it is $\lambda$-universal for $\lambda = \|M\|$.

We are here mainly interested in

**Definition 0.5.** Given $T$ and $M \in EC_\lambda(T)$ we say that $M$ is a super-limit or $\lambda$-superlimit model when: $M$ is universal and if $\delta < \lambda^+$ is a limit ordinal, $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\prec$-increasing continuous, and $M_\alpha$ is isomorphic to $M$ for every $\alpha < \delta$, then $M_\delta$ is isomorphic to $M$.

**Remark 0.6.** Concerning the following definition we shall use strongly limit in [2.14](1), medium limit in [2.14](2).

**Definition 0.7.** Let $\lambda$ be a cardinal $\geq |T|$. For parts (3)–(7) below, but not (8), to simplify the presentation we assume the axiom of global choice and that $F$ is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+]$.

1. For non-empty $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda \text{ and } \mu \text{ is regular}\}$ and $M \in EC_\lambda(T)$ we say that $M$ is $(\lambda, \Theta)$-superlimit when: $M$ is universal and
   \[
   \text{if } \langle M_i : i \leq \mu \rangle \text{ is } \prec\text{-increasing, } M_i \cong M \text{ for } i < \mu \text{ and } \mu \in \Theta,
   \]
   then $\bigcup\{M_i : i < \mu\} \cong M$.

2. If $\Theta$ is a singleton, say $\Theta = \{\theta\}$, we may say that $M$ is $(\lambda, \theta)$-superlimit.

3. Let $S \subseteq \lambda^+$ be stationary. A model $M \in EC_\lambda(T)$ is called $S$-strongly limit or $(\lambda, S)$-strongly limit when for some function $F : EC_\lambda(T) \to EC_\lambda(T)$ we have:
   \[
   \begin{align*}
   &\text{(a) for } N \in EC_\lambda(T) \text{ we have } N \prec F(N), \\
   &\text{(b) if } \delta \in S \text{ is a limit ordinal and } \langle M_i : i < \delta \rangle \text{ is a } \prec\text{-increasing continuous sequence } \langle 1 \rangle \text{ in } EC_\lambda(T) \text{ and } i < \delta \Rightarrow F(M_{i+1}) \prec M_{i+2}, \text{ then } M \cong \bigcup\{M_i : i < \delta\}. \\
   \end{align*}
   \]

4. Let $S \subseteq \lambda^+$ be stationary. $M \in EC_\lambda(T)$ is called $S$-limit or $(\lambda, S)$-limit if for some function $F : EC_\lambda(T) \to EC_\lambda(T)$ we have:
   \[
   \begin{align*}
   &\text{(a) for every } N \in EC_\lambda(T) \text{ we have } N \prec F(N),
   \end{align*}
   \]

$\overset{1}{}$ No loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $F$, i.e., only $F(M')$ for $M' \cong M$ is required.
(b) if \( \langle M_i : i < \lambda^+ \rangle \) is a \( \prec \)-increasing continuous sequence of members of \( \text{EC}_\lambda(T) \) such that \( F(M_{i+1}) \prec M_{i+2} \) for \( i < \lambda^+ \) then for some closed unbounded \(^2\) subset \( C \) of \( \lambda^+ \),

\[ [\delta \in S \cap C \Rightarrow M_5 \cong M]. \]

(5) We define \[^3\]"S-weakly limit", "S-medium limit" like "S-limit", "S-strongly limit" respectively by demanding that the domain of \( F \) is the family of \( \prec \)-increasing continuous sequences of members of \( \text{EC}_\lambda(T) \) of length \( < \lambda^+ \) and replacing "\( F(M_{i+1}) \prec M_{i+2} \)" by "\( M_{i+1} \prec F(\langle M_j : j \leq i + 1 \rangle) \prec M_{i+2} \)."

(6) If \( S = \lambda^+ \) then we may omit \( S \) (in (3)–(5)).

(7) For non-empty \( \Theta \subseteq \{ \mu : \mu \leq \lambda \text{ and } \mu \text{ is regular} \} \), \( M \) is \((\lambda, \Theta)\)-strongly limit \[^4\] if \( M \) is \( \{ \delta < \lambda^+ : \text{cf}(\delta) \in \Theta \} \)-strongly limit. Similarly for the other notions. If we do not write \( \lambda \) we mean \( \lambda = \| M \| \).

(8) We say that \( M \in K_\lambda \) is invariably strong limit when in (3), \( F \) is just a subset of \( \{(M, N)/\cong : M \prec N \text{ are from } \text{EC}_\lambda(T) \} \) and in (3)(b) we replace "\( F(M_{i+1}) \prec M_{i+2} \)" by "\( (\exists N)(M_{i+1} \prec N \prec M_{i+2} \land ((M, N)/\cong) \in F) \)."

But abusing notation we still write \( N = F(M) \) instead of \( ((M, N)/\cong) \in F \). Similarly with the other notions, so we use the isomorphism type of \( M \backslash \langle N \rangle \) for "weakly limit" and "medium limit".

(9) In the definitions above we may say "\( F \) witnesses \( M \) is ..."

**Observation 0.8.** (1) Assume \( F_1, F_2 \) are as above and \( F_1(N) \prec F_2(N) \) (or \( F_1(\tilde{N}) \prec F_2(\tilde{N}) \)) whenever defined. If \( F_1 \) is a witness then so is \( F_2 \).

(2) All versions of limit models imply being a universal model in \( \text{EC}_\lambda(T) \).

(3) (The obvious implications diagram) For non-empty \( \Theta \subseteq \{ \theta : \theta \text{ is regular } \leq \lambda \} \) and stationary \( S_1 \subseteq \{ \delta < \lambda^+ : \text{cf}(\delta) \in \Theta \} \):

\[
\begin{align*}
\text{superlimit} & = (\lambda, \{ \mu : \mu \leq \lambda \text{regular} \})\text{-superlimit} \\
& \downarrow \\
(\lambda, \Theta)\text{-superlimit} & \downarrow \\
& \downarrow \\
S_1\text{-strongly limit} & \downarrow \\
& \updownarrow \\
S_1\text{-medium limit} & S_1\text{-limit} \\
& \downarrow \\
S_1\text{-weakly limit}
\end{align*}
\]

\(^2\) Alternatively, we can use as a parameter a filter on \( \lambda^+ \) extending the co-bounded filter.

\(^3\) Note that \( M \) is \((\lambda, S)\)-strongly limit iff \( M \) is \( \{ \lambda, \text{cf}(\delta) : \delta \in S \} \)-strongly limit.

\(^4\) In \[^3\] we replace "limit" by "limit\(^-\)" if "\( F(M_{i+1}) \prec M_{i+2} \)" , "\( M_{i+1} \prec F(\langle M_j : j \leq i + 1 \rangle) \prec M_{i+2} \)" are replaced by "\( F(M_i) \prec M_{i+1} \)" , "\( M_i \prec F(\langle M_j : j \leq i \rangle) \prec M_{i+1} \)" respectively. But \( (\text{EC}(T), \prec) \) has amalgamation.
LEMMA 0.9. Let $T$ be a first order complete theory.

(1) If $\lambda$ is regular and $M$ a saturated model of $T$ of cardinality $\lambda$, then $M$ is $(\lambda, \lambda)$-superlimit.

(2) If $T$ is stable, and $M$ is a saturated model of $T$ of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$, then $M$ is $(\lambda, \Theta)$-superlimit (for $\kappa(T)$, see [15 III, §3]).

(3) If $T$ is stable in $\lambda$ and $\kappa = \text{cf}(\kappa) \leq \lambda$ then $T$ has an invariantly strongly $(\lambda, \kappa)$-limit model.

REMARK 0.10. Concerning 0.9(2), note that by [15] if $\lambda$ is singular or just $\lambda < \lambda^{<\lambda}$ and $T$ has a saturated model of cardinality $\lambda$ then $T$ is stable (even stable in $\lambda$) and $\text{cf}(\lambda) \geq \kappa(T)$).

Proof. (1) Let $M_i$ be a $\lambda$-saturated model of $T$ of cardinality $\lambda$ for $i < \lambda$ with $\langle M_i : i < \lambda \rangle$ $\prec$-increasing and set $M_\lambda = \bigcup_{i<\lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$, so every $p \in S(A, M_\lambda)$ is realized in $M_i$, hence in $M_\lambda$; so clearly $M_\lambda$ is $\lambda$-saturated. Remembering the uniqueness of a $\lambda$-saturated model of $T$ of cardinality $\lambda$ we finish.

(2) Use [15 III, 3.11]: if $M_i$ is a $\lambda$-saturated model of $T$ with $\langle M_i : i < \delta \rangle$ increasing and $\text{cf}(\delta) \geq \kappa(T)$ then $\bigcup_{i<\delta} M_i$ is $\lambda$-saturated.

(3) Let $K_{\lambda, \kappa} = \{\tilde{M} : \tilde{M} = \langle M_i : i \leq \kappa \rangle \text{ is } \prec$-increasing continuous, $M_i \in EC_\lambda(T)$ and $(M_{i+1}, c)_{c \in M_{i+1}}$ is saturated for every $i < \kappa\}$. Clearly $\tilde{M}, \tilde{N} \in K_{\lambda, \kappa} \Rightarrow M_\kappa \cong N_\kappa$. Also for every $M \in EC_\lambda(T)$ there is $N$ such that $M \prec N$ and $(N, c)_{c \in M}$ is saturated, as also $\text{Th}((M, c)_{c \in M})$ is stable in $\lambda$; so there is an invariant $F : EC_\lambda(T) \rightarrow EC_\lambda(T)$ such that $M \prec F(M)$ and $(F(M), c)_{c \in M}$ is saturated; such $F$ witnesses the desired conclusion. 

DEFINITION 0.11. For a regular uncountable cardinal $\lambda$ let

$\bar{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{u}) \text{ witnesses } S \in \bar{I}[\lambda], \text{see below}\}.$

We say that $(E, \bar{u})$ is a witness for $S \in \bar{I}[\lambda]$ iff:

- $E$ is a club of the regular cardinal $\lambda$,
- $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$,
- for every $\delta \in E \cap S$, $u_\delta$ is an unbounded subset of $\delta$ of order-type $\text{cf}(\delta)$ (and $\delta$ is a limit ordinal).

By [16 §1] we have

CLAIM 0.12. If $\kappa^+ < \lambda$ and $\kappa, \lambda$ are regular then some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ belongs to $\bar{I}[\lambda]$.

By [11] we have

CLAIM 0.13. If $\lambda = \mu^+$, $\theta = \text{cf}(\theta) \leq \text{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \leq \mu$ then $S_\theta^\alpha \subseteq \bar{I}[\lambda]$.
1. On superstable non-$\aleph_0$-stable $T$. We first note that superstable $T$ tend to have superlimit models.

Claim 1.1. Assume $T$ is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ has a saturated model of cardinality $\lambda$ iff $T$ has a universal model of cardinality $\lambda$ iff $\lambda \geq |D(T)|$.

Proof. By [15, III, §5] we know that $T$ is stable in $\lambda$ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of $T$ of cardinality $\lambda$, hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then $T$ is stable in $\lambda$, hence has a saturated model of cardinality $\lambda$ by [15, III] (hence universal) and the class of $\lambda$-saturated models of $T$ is closed under increasing elementary chains by [15, III], so we are done. □

The following are the prototypical theories which we shall consider.

Definition 1.2.

$T_0 = \text{Th}(\omega^2, E^0_n)_{n<\omega}$ where $\eta E^0_n \nu \iff \eta n = \nu n$,

$T_1 = \text{Th}(\omega, (E^1_n)_{n<\omega})$ where $\eta E^1_n \nu \iff \eta n = \nu n$,

$T_2 = \text{Th}(\mathbb{R}, <)$.

Recall

Observation 1.3.

(0) $T_\ell$ is a countable complete first order theory for $\ell = 0, 1, 2$.
(1) $T_0$ is superstable non-$\aleph_0$-stable.
(2) $T_1$ is strictly stable, that is, stable non-superstable.
(3) $T_2$ is unstable.
(4) $T_\ell$ has elimination of quantifiers for $\ell = 0, 1, 2$.

Claim 1.4. It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \text{EC}_{\aleph_1}(T_0)$ is a superlimit model.

Proof. By [12], for notational simplicity we start with $V = L$.

So $T_0$ is defined in 1.2 and it is the $T$ from Theorem [12, 1.1]. Let $S$ be the set of $\eta \in (\omega^2)^L$. We define $T'$ (called $T_1$ there) as the following theory:

$\oplus_1$ (i) for each $n$ the sentence saying $E_n$ is an equivalence relation with $2^n$ equivalence classes, each $E_n$ equivalence class divided into two by $E_{n+1}$, $E_{n+1}$ refines $E_n$, $E_0$ is trivial,

(ii) the sentences saying that

(a) for every $x_0$, the function $z \mapsto F(x_0, z)$ is one-to-one and

(b) $x_0 E_n F(x_0, z)$ for each $n < \omega$,

(iii) $E_n(c_\eta, c_\nu)^i_{(\eta n = \nu n)}$ for $\eta, \nu \in S$. 

In [12] it is proved that in some forcing\(^{(9)}\) extension \(L^P\) of \(L\), \(P\) an \(\aleph_2\)-c.c. proper forcing of cardinality \(\aleph_2\), and in \(V = L^P\), the class \(PC(T',T_0) = \{M | \tau_{T_0} : M \text{ is a } \tau\text{-model of } T'\}\) is categorical in \(\aleph_1\).

However, letting \(M^*\) be any model from \(PC(T',T_0)\) of cardinality \(\aleph_1\), it is easy to see that (in \(V = L^P\)):

\(\exists\) the following conditions on \(M\) are equivalent:

(a) \(M\) is isomorphic to \(M^*\),
(b) \(M \in PC(T',T_0)\),
(c) (\(\alpha\) \(M\) is a model of \(T_0\) of cardinality \(\aleph_1\),
(\(\beta\) \(M^*\) can be elementarily embedded into \(M\),
(\(\gamma\) for every \(a \in M\) the set \(\bigcap\{a/E_n^M : n < \omega\}\) has cardinality \(\aleph_1\).

But

\(\exists\) every model \(M_1\) of \(T\) of cardinality \(\leq \aleph_1\) has a proper elementary extension to a model satisfying (c), i.e., (\(\alpha\)–(\(\gamma\)) of \(\exists\) above,

\(\exists\) if \(\langle M_\alpha : \alpha < \delta \rangle\) is an increasing chain of models satisfying (c) of \(\exists\) and \(\delta < \omega_2\) then also \(\bigcup\{M_\alpha : \alpha < \delta\}\) does.

Altogether we are done. \(\blacksquare\)

Naturally we ask

**Question 1.5.** What occurs to \(T_0\) for \(\lambda > \aleph_1\) but \(\lambda < 2^{\aleph_0}\)?

**Question 1.6.** Does the theory \(T_2\) of linear order consistently have an \((\aleph_1,\aleph_0)\)-superlimit (or only strongly limit) model? (but see \S 3).

**Question 1.7.** What is the answer for \(T\) when \(T\) is countable superstable non-\(\aleph_0\)-stable and \(D(T)\) is countable for \(\aleph_1 < 2^{\aleph_0}\) and \(\aleph_2 < 2^{\aleph_0}\)?

By the above for some such \(T\), in some universe, for \(\aleph_1\) the answer is yes, there is a superlimit model.

**2. A strictly stable consistent example.** We now look at models of \(T_1\) (redefined below) in cardinality \(\aleph_1\); recall

**Definition 2.1.** \(T_1 = \text{Th}(\omega(\omega_1), E_n)_{n<\omega}\) where \(E_n = \{(\eta,\nu) : \eta,\nu \in \omega(\omega_1)\text{ and } \eta|n = \nu|n\}\).

**Remark 2.2.** Note that \(T_1\) has elimination of quantifiers. Moreover, if \(\lambda = \sum\{\lambda_n : n < \omega\}\) and \(\lambda_n = \lambda^{\aleph_0}_n\), then \(T_1\) has a \((\lambda,\aleph_0)\)-superlimit model in \(\lambda\) (see 2.15).

**Definition/Claim 2.3.** Any model of \(T_1\) of cardinality \(\lambda\) is isomorphic to \(M_{A,h} := \{(\eta,\varepsilon) : \eta \in A, \varepsilon < h(\eta)\}, E_n)_{n<\omega}\) for some \(A \subseteq \omega \lambda\) and \(h : \)

\(^{(9)}\) We can replace \(L\) by any \(V_0\) which satisfies \(2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2\).
\( \omega \lambda \rightarrow (\text{Car} \cap \lambda^+) \setminus \{0\} \) where \((\eta_1, \varepsilon_1) E_n(\eta_2, \varepsilon_2) \Leftrightarrow \eta_1 \mid n = \eta_2 \mid n \); pedantically we should write \( E_n^{M_{A,h}} = E_n \mid |M_{A,n}| \).

We write \( M_A \) for \( M_{A,h} \) when \( A \) is as above and \( h : A \rightarrow \{|A|\} \), so constantly \( |A| \) when \( A \) is infinite.

For \( A \subseteq \omega \lambda \) and \( h \) as above the model \( M_{A,h} \) is a model of \( T_1 \) iff \( A \) is non-empty and \(( \forall \eta \in A ) \forall n < \omega ) (\exists n^0 \nu \in A ) (\nu \mid n = \eta \mid n \land \nu(n) \neq \eta(n)) \).

Above \( M_{A,h} \) has cardinality \( \lambda \) iff \( \sum \{h(\eta) : \eta \in A \} = \lambda \).

**Definition 2.4.** We say that \( A \) is a \((T_1, \lambda)\)-witness when:

- \( A \subseteq \omega \lambda \) has cardinality \( \lambda \),
- if \( B_1, B_2 \subseteq \omega \lambda \) are \((T_1, A)\)-big (see below) of cardinality \( \lambda \) then \((B_1 \cup \omega > \lambda, <) \) is isomorphic to \((B_2 \cup \omega > \lambda, <) \).

A set \( B \subseteq \omega \lambda \) is called \((T_1, A)\)-big when it is \((\lambda, \lambda)-(T_1, A)\)-big; see below.

\( B \) is \((\mu, \lambda)-(T_1, A)\)-big means: \( B \subseteq \omega \lambda, |B| = |A| = \mu \) and for every \( \eta \in \omega > \lambda \) there is an isomorphism \( f \) from \((\omega \geq \lambda, <) \) onto \((\{\eta^* \nu : \nu \in \omega \geq \lambda\}, <) \) mapping \( A \) into \( \{\nu : \eta^* \nu \in B\} \).

\( A \subseteq \omega (\omega_1) \) is \( \aleph_1 \)-suitable when:

- \( |A| = \aleph_1 \),
- for a club of \( \delta < \omega_1, A \cap \omega^* \delta \) is everywhere non-meagre in the space \( \omega^* \delta \), i.e., for every \( \eta \in \omega^* \delta \), the set \( \{\nu \in A \cap \omega^* \delta : \eta < \nu\} \) is a non-meagre subset of \( \omega^* \delta \) (that is what is really used in \([12]\))

**Claim 2.5.** It is consistent with ZFC that \( 2^{\aleph_0} > \aleph_1 + \) there is a \((T_1, \aleph_1)\)-witness; moreover every \( \aleph_1 \)-suitable set is a \((T_1, \aleph_1)\)-witness.

**Proof.** By \([12] \S 2\). \[\square\]

**Remark 2.6.** The witness does not give rise to an \((\aleph_1, \aleph_0)\)-limit model for the union of any “fast enough” \( \prec \)-increasing \( \omega \)-chain of members of \( \text{EC}_{\aleph_1}(T_1) \), the relevant sets are meagre.

**Definition 2.7.** Let \( A \) be a \((T_1, \lambda)\)-witness. We define \( K^1_{T_1, A} \) as the family of \( M = (|M|, <^M, P^M_\alpha)_{\alpha \leq \omega} \) such that:

- \((\alpha) \) \( |M|, <^M \) is a tree with \( \omega + 1 \) levels,
- \((\beta) \) \( P^M_\alpha \) is the \( \alpha \)th level; let \( P^M_{<\omega} = \bigcup \{P^M_n : n < \omega\} \),
- \((\gamma) \) \( M \) is isomorphic to \( M^1_B \) for some \( B \subseteq \omega \lambda \) of cardinality \( \lambda \) where \( M^1_B \) is defined by \( |M^1_B| = (\omega > \lambda) \cup B, P^M_B = n^L, P^M_{\omega B} = B \) and \( <^M_B = \prec |M^1_B| \), i.e., being an initial segment,
- \((\delta) \) moreover \( B \) is such that some \( f \) satisfies:
  - \( f : \omega > \lambda \rightarrow \omega \) and \( f(\langle \rangle) = 0 \) for simplicity,
  - \( \eta \leq \nu \in \omega > \lambda \Rightarrow f(\eta) \leq f(\nu) \),
  - if \( \eta \in B \) then \( \langle f(\eta \mid n) : n < \omega \rangle \) is eventually constant,
\begin{itemize}
  \item if $\eta \in \omega^\lambda$ then $\{\nu \in \omega^\lambda : \eta \prec \nu \in B \text{ and } m < \omega \Rightarrow f(\eta \prec (\nu|m)) = f(\eta)\}$ is $(T_1, A)$-big,
  \item for $\eta \in \omega^\lambda$ and $n \in [f(\eta), \omega)$ for $\lambda$ ordinals $\alpha < \lambda$, we have $f(\eta \prec (\alpha)) = n$.
\end{itemize}

**Claim 2.8 (The Global Axiom of Choice).** If $A$ is a $(T_1, \mathcal{N}_1)$-witness then:

(a) $K^1_{T_1,A} \neq \emptyset$,

(b) any two members of $K^1_{T_1,A}$ are isomorphic,

(c) there is a function $F$ from $K^1_{T_1,A}$ to itself (up to isomorphism, i.e., $(M, F(M))$ defined only up to isomorphism) satisfying $M \subseteq F(M)$ such that $K^1_{T_1,A}$ is closed under increasing unions of sequences $\langle M_n : n < \omega \rangle$ such that $F(M_n) \subseteq M_{n+1}$.

**Proof.** (a): Trivial.

(b): By the definition of “$A$ is a $(T_1, \mathcal{N}_1)$-witness” and of $K^1_{T_1,A}$.

(c): We choose $F$ such that

- if $M \in K^1_{A,T_1}$ then $M \subseteq F(M) \in K^1_{A,T_1}$ and for every $k < \omega$ and $a \in P^M_k$, the set $\{b \in P_{k+1}^{F(M)} : a <_{F(M)} b \text{ and } b \notin M\}$ has cardinality $\mathcal{N}_1$.

Assume $M = \bigcup\{M_n : n < \omega\}$ where $\langle M_n : n < \omega \rangle$ is $\subseteq$-increasing, $M_n \in K^1_{A,T_1}$, $F(M_n) \subseteq M_{n+1}$. Clearly $M$ is as required at the beginning of Definition 2.7, that is, satisfies clauses (a)-(γ) there. To prove (δ), we define $f : P^M_{\omega_\omega} \to \omega$ by $f(a) = \text{Min}\{n : a \in M_n\}$. Pedantically, $F$ is defined only up to isomorphism.

**Claim 2.9.** If $A$ is a $(T_1, \lambda)$-witness then:

(a) $K^1_{T_1,A} \neq \emptyset$,

(b) any two members of $K^1_{T_1,A}$ are isomorphic,

(c) if $M_n \in K^1_{T_1,A}$ and $n < \omega \Rightarrow M_n \subseteq M_{n+1}$ then $M := \bigcup\{M_n : n < \omega\} \in K^1_{T_1,A}$.

**Remark 2.10.** If we omit clause (b), we can weaken the demand on the set $A$.

**Proof.** Assume $M = \bigcup\{M_n : n < \omega\}$, $M_n \subseteq M_{n+1}$, $M_n \in K^1_{T_1,A}$ and $f_n$ witnesses $M_n \in K^1_{T_1,A}$. Clearly $M$ satisfies clauses (α)-(γ) of Definition 2.7; we just have to find a witness $f$ as in (δ) there.

For each $a \in M$ let $n(a) = \text{Min}\{n : a \in M_n\}$; clearly if $M \models "a < b < c"$ then $n(a) \leq n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \to M$ be defined by: $g_n(a) = b$ iff $b \leq^M a$, $b \in M_n$ and $b$ is $\leq^M$-maximal under those restrictions; clearly it is well defined. Now we define $f'_n : M_n \to \omega$ by induction on $n < \omega$ such that $m < n \Rightarrow f'_m \subseteq f'_n$, as follows.
If \( n = 0 \) let \( f'_n = f_n \).

If \( n = m + 1 \) and \( a \in M_n \) we let \( f'_n(a) \) be \( f'_m(a) \) if \( a \in M_m \) and be 
\((f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1 \) if \( a \in M_n \setminus M_m \). Clearly \( f := \bigcup \{ f'_n : n < \omega \} \) is a function from \( M \) to \( \omega, a \leq^M b \implies f(a) \leq f(b) \), and for any \( a \in M \) the set \( \{ b \in M : a \leq^M b \text{ and } f(b) = f(a) \} \) is equal to \( \{ b \in M_{n(a)} : f_n(a)(a) = f_n(a)(b) \text{ and } a \leq^M b \} \).

**Definition 2.11.** Let \( A \) be a \((T_1, \lambda)\)-witness. We define \( K^2_{T_1,A} \) as in Definition 2.7 but \( f \) is constantly zero.

**Claim 2.12 (The Global Axiom of Choice).** If \( A \) is a \((T_1, \aleph_1)\)-witness then:

(a) \( K^2_{T_1,A} \neq \emptyset \),

(b) any two members of \( K^2_{T_1,A} \) are isomorphic,

(c) there is a function \( F \) from \( \bigcup \{ \alpha^+2(K^2_{T_0,A}) : \alpha < \omega_1 \} \) to \( K^2_{T_1,A} \) which satisfies:

• (a) if \( \bar{M} = \langle M_i : i \leq \alpha + 1 \rangle \) is an \( \omega \)-increasing sequence of models of \( T \) then \( M_{\alpha+1} \subseteq F(M) \subseteq K^2_{T_1,A} \).

• (b) when \( \omega_1 = \sup\{ \alpha : F(\bar{M}\uplus r(\alpha+2)) \subseteq M_{\alpha+2} \} \) and is a well defined embedding of \( M_\alpha \) into \( M_{\alpha+2} \), the union of any increasing \( \omega_1 \)-sequence \( \bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle \) of members of \( K^2_{T_1,A} \) belongs to \( K^2_{T_1,A} \).

**Remark 2.13.** Instead of the global axiom of choice, we can restrict the models to have universe a subset of \( \lambda^+ \) (or just a set of ordinals).

**Proof.** (a): Easy.

(b): By the definition.

(c): Let \( \langle \mathcal{U}_\varepsilon : \varepsilon < \omega_1 \rangle \) be an increasing sequence of subsets of \( \omega_1 \) with union \( \omega_1 \) such that \( \varepsilon < \omega_1 \Rightarrow |\mathcal{U}_\varepsilon \setminus \bigcup_{\zeta < \varepsilon} \mathcal{U}_\zeta| = \aleph_1 \). Let \( M^* \in K^2_{T_1,A} \) be such that \( \omega^>(\omega_1) \subseteq |M^*| \subseteq \omega^>(\omega_1) \) and \( M^*_\varepsilon := M^*|_{\omega^>(\mathcal{U}_\varepsilon)} \) belongs to \( K^2_{T_1,A} \) for every \( \varepsilon < \omega_1 \).

We choose a pair \( (F, f) \) of functions with domain \( \{ \bar{M} : \bar{M} \text{ an increasing sequence of members of } K^2_{T_1,A} \text{ of length } \omega_1 \} \) such that:

- \( F(\bar{M}) \) is an extension of \( \bigcup \{ M_i : i < \ell g(\bar{M}) \} \) from \( K^2_{T_1,A} \),
- \( f(\bar{M}) \) is an embedding from \( M^*_{\ell g(M)} \) into \( F(\bar{M}) \),
- if \( \bar{M}^\ell = \langle M_\alpha : \alpha < \alpha_\ell \rangle \) for \( \ell = 1, 2 \) and \( \alpha_1 < \alpha_2, \bar{M}^1 = \bar{M}^2|_{\alpha_1} \) and \( F(\bar{M}^1) \subseteq M_{\alpha_1} \) then \( f(\bar{M}^1) \subseteq f(\bar{M}^2) \),
- if \( a \in F(\bar{M}) \) and \( n < \omega \) then for some \( b \in M^*_{\ell g(M)} \) we have \( F(M) \models aE_n(f(\bar{M})(b)) \).

Now check. ■ 2.12
CONCLUSION 2.14. Assume there is a \((T_1, \aleph_1)\)-witness (see Definition 2.4) for the first-order complete theory \(T_1\) from 2.4. Then:

1. \(T_1\) has an \((\aleph_1, \aleph_0)\)-strongly limit model.
2. \(T_1\) has an \((\aleph_1, \aleph_1)\)-medium limit model.
3. \(T_1\) has an \((\aleph_1, \aleph_0)\)-superlimit model.

Proof. (1) By 2.8 the reduction of problems on \((EC(T_1), \prec)\) to \(K_{T_1,A}\) (which is easy) is exactly as in [12].

(2) By 2.12.

(3) Like part (1) using Claim 2.9.

CLAIM 2.15. If \(\lambda = \sum \{\lambda_n : n < \omega\}\) and \(\lambda_n = \lambda_n^{\aleph_0}\), then \(T_1\) has a \((\lambda, \aleph_0)\)-superlimit model in \(\lambda\).

Proof. Let \(M_n\) be the model \(M_{\omega,A_n,h_n}\) where \(A_n = \omega(\lambda_n)\) and \(h_n : A_n \to \lambda_n^{+}\) is constantly \(\lambda_n\). Clearly,

\((*)_1 M_n\) is a saturated model of \(T_1\) of cardinality \(\lambda_n\),
\((*)_2 M_n \prec M_{n+1},\)
\((*)_3 M_\omega = \bigcup \{M_n : n < \omega\}\) is a special model of \(T_1\) of cardinality \(\lambda\).

The main point is:

\((*)_4 M_\omega\) is \((\lambda, \aleph_0)\)-superlimit model of \(T_1\).

[Why? Toward this assume:
- \(N_n\) is isomorphic to \(M_\omega\), say \(f_n : M_\omega \to N_n\) is an isomorphism,
- \(N_n \prec N_{n+1}\) for \(n < \omega\).

Let \(N_\omega = \bigcup \{N_n : n < \omega\}\) and we should prove \(N_\omega \cong M_\omega\), so just \(N_\omega\) is a special model of \(T_1\) of cardinality \(\lambda\) suffice.

Let \(N'_n = N_\omega \upharpoonright (\bigcup \{f_n(M_k) : k \leq n\})\). Clearly \(N'_n \prec N'_{n+1} \prec N_\omega\) and \(\bigcup \{N'_n : n < \omega\} = N_\omega\) and \(\|N'_n\| = \lambda_n\). So it suffices to prove that \(N'_n\) is saturated and direct inspection shows this.]

3. On non-existence of limit models. Naturally we assume that non-existence of superlimit models for unstable \(T\) is easier to prove. For other versions we need to look more. We first show that for \(\lambda \geq |T| + \aleph_1\), if \(T\) is unstable then it does not have a superlimit model of cardinality \(\lambda\), and if \(T\) is unsuperstable, we show this for “most” cardinals \(\lambda\). On “\(\Phi\) proper for \(K_\text{or} or K_{\text{tr}}\)”, see [15 VII] or [7] or hopefully some day in [8 III]. We assume some knowledge of stability.

CLAIM 3.1. (1) If \(T\) is unstable, \(\lambda \geq |T| + \aleph_1\), then \(T\) has no superlimit model of cardinality \(\lambda\).

(2) If \(T\) is stable non-superstable and \(\lambda \geq |T| + \beth_\omega\) or \(\lambda = \lambda^{\aleph_0} \geq |T|\) then \(T\) has no superlimit model of cardinality \(\lambda\).
Remark 3.2. We assume some knowledge of EM models for linear orders $I$ and members of $K_{ir}^\omega$ as index models (see, e.g., [15, VII]).

(2) We use the following definition in the proof, as well as a result from [17] or [18].

Definition 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal $\mu$ such that for some, equivalently for every set $A$ of cardinality $\lambda$ there is $\mathcal{P}_A \subseteq [A]^{\leq \kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality $\lambda$ such that any $B \in [\lambda]^{\leq \kappa}$ is the union of $< \kappa$ members of $\mathcal{P}_A$.

Proof of Claim 3.1. (1) Towards a contradiction assume $M^*$ is a superlimit model of $T$ of cardinality $\lambda$. As $T$ is unstable we can find $m$ and $\varphi(\bar{x}, \bar{y})$ such that

- $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)}$ linearly orders some infinite $I \subseteq m \cdot M, M \models T$ so $\ell g(\bar{x}) = \ell g(\bar{y}) = m$.

We can find a $\Phi$ which is proper for linear orders ([15, VII]) and $F_\ell(\ell < m)$ such that $F_\ell \in \tau_\Phi \setminus \tau_T$ is a unary function symbol for $\ell < m, \tau_T \subseteq \tau(\Phi)$ and for every linear order $I$, $\text{EM}(I, \Phi)$ has Skolem functions and its $\tau_T$-reduct $\text{EM}_{\tau(T)}(I, \Phi)$ is a model of $T$ of cardinality $|T| + |I|$ and $\tau(\Phi)$ is of cardinality $|T| + \aleph_0$ and $\langle a_s : s \in I \rangle$ is the skeleton of $\text{EM}(I, \Phi)$, that is, it is an indiscernible sequence in $\text{EM}(I, \Phi)$ and $\text{EM}(I, \Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $\bar{a}_s = \langle F_\ell(a_s) : \ell < m \rangle$ in $\text{EM}(I, \Phi)$ we have $\text{EM}_{\tau(T)}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\text{iff}(s < t)}$ for $s, t \in I$.

Next we can find $\Phi_n$ (for $n < \omega$) such that:

- $\Phi_n$ is proper for linear orders and $\Phi_0 = \Phi$,
- $\text{EM}_{\tau(\Phi)}(I, \Phi_n) \prec \text{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ for every linear order $I$ and $n < \omega$; moreover
- $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and $\text{EM}(I, \Phi_n) \prec \text{EM}_{\tau(\Phi_n)}(I, \Phi_{n+1})$ for every $n < \omega$ and linear order $I$.
- if $|I| \leq n$ then $\text{EM}_{\tau(\Phi)}(I, \Phi_n) \models \text{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ and $\text{EM}_{\tau(T)}(I, \Phi_n) \cong \text{EM}_{\tau(T)}(I, \Phi_{n+1})$
- if $|\tau(\Phi_n)| = \lambda$.

This is easy. Let $\Phi_\omega$ be the limit of $\langle \Phi_n : n < \omega \rangle$, i.e. $\tau(\Phi_\omega) = \bigcup \{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then $\text{EM}_{\tau(\Phi_k)}(I, \Phi_\omega) = \bigcup \{\text{EM}_{\tau(\Phi_k)}(I, \Phi_n) : n \in [k, \omega)\}$. So as $\text{EM}(I, \Phi_\omega)$ is a superlimit model, for any linear order $I$ of cardinality $\lambda$, $\text{EM}_{\tau(T)}(I, \Phi_\omega)$ is the direct limit of $\langle \text{EM}_{\tau(T)}(J, \Phi_\omega) : J \subseteq I \text{ finite} \rangle$, each isomorphic to $\text{EM}(I, \Phi_\omega)$, so as we have assumed that $\text{EM}(I, \Phi_\omega)$ is a superlimit model it follows that $\text{EM}_{\tau(T)}(I, \Phi_\omega)$ is isomorphic to $\text{EM}(I, \Phi_\omega)$, contradiction. But by [14, III] or [7] which may eventually be [8, III] there are $2^\lambda$ many pairwise non-isomorphic models of this form varying $I$ on the linear orders of cardinality $\lambda$, contradiction.

(2) First assume $\lambda = \lambda^{\aleph_0}$. Let $\tau \subseteq \tau_T$ be countable such that $T' = T \cap \mathbb{L}(\tau)$ is not superstable. Clearly if $\text{EM}(I, \Phi_\omega)$ is a $(\lambda, \aleph_0)$-limit model then
$M^* | \tau'$ is not $\aleph_1$-saturated. [Why? As in \cite{10} Ch. VI, \S 6], but we shall give full details: there are $N_* \models T$, $p = \{ \varphi_n(\lambda, a_n) : n < \omega \}$ a type in $N_*$, $\bar{a}_n \subseteq \bar{a}_{n+1}$ empty and $\varphi_{n+1}(x, \bar{a}_{n+1})$ forks over $\bar{a}_n$. Let $F(M)$ be such that if $n < \omega$ and $\bar{b}_n \subseteq M$ realizes $tp(\bar{a}_n, \emptyset, N_*)$ then for some $\bar{b}_{n+1}$ from $F, M$ realizing $tp(\bar{a}_{n+1}, \emptyset, N_*)$, the type $tp(\bar{b}_{n+1}, M, F(M))$ does not fork over $\bar{b}_n$.) But if $\kappa = cf(\kappa) \in [\aleph_1, \lambda]$ and $M^*$ is a $(\lambda, \kappa)$-limit then $M^* | \tau'$ is $\aleph_1$-saturated, contradiction.

The case $\lambda \geq |T| + \beth_\omega$ is more complicated (the assumption $\lambda \geq \beth_\omega$ is to enable us to use \cite{17} or see \cite{18} for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As $T$ is stable non-superstable by \cite{15} for some $\Delta$:

$\dag_1$ for any $\mu$ there are $M$ and $\langle a_{n,\alpha} : n \in \omega, \mu \rangle$ such that

(a) $M$ is a model of $T$,

(b) $I_\eta = \{ a_{n,\alpha} : \alpha < \mu \} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu$ implies $a_{n,\alpha} \neq a_{n,\beta}$),

(c) $\Delta = \{ a_{n,\alpha} : n < \omega \}$ and $\Delta_n \subseteq \Pi_\tau(T)$ infinite,

(d) for $\eta, \nu \in \omega, \mu$ we have $Av_{\Delta_n}(M, I_\eta) = Av_{\Delta_n}(M, I_\nu)$ iff $\eta \mid n = \nu \mid n$.

Hence by \cite{15} VIII (or see \cite{7} assuming $M^*$ is a universal model of $T$ of cardinality $\lambda$):

$\dag_{2.1}$ there is $\Phi$ such that:

(a) $\Phi$ is proper for $K^\omega_{tr}, \tau_T \subseteq \tau(\Phi), |\tau(\Phi)| = \lambda \geq |T| + \aleph_0$,

(b) for $I \subseteq \omega \geq \lambda$, $EM(\tau(\Phi), I, \Phi)$ is a model of $T$ and $I \subseteq J \Rightarrow EM(I, \Phi) \prec EM(J, \Phi)$,

(c) for some two-place function symbol $F$ if for $I \subseteq K^\omega_{tr}$ and $\eta \in P^I_\omega$, $I$ a subtree of $\omega \geq \lambda$, for transparency we let $I_{I,\eta} = \{ F(a_\eta, a_\nu) : \nu \in I \}$, then $\langle I_{I,\eta} : \eta \in P^I_\omega \rangle$ are as in $\dag_1(b), (d)$.

Also

$\dag_{2.2}$ if $\Phi_1$ satisfies (a)--(c) of $\dag_{2.1}$ and $M$ is a universal model of $T$ then there is $\Phi_2$ satisfying (a)--(c) of $\dag_{2.1}$ and $\Phi_1 \leq \Phi_2$ (see $\dag_{2.3}(a)$) and for every finitely generated $J \subseteq K^\omega_{tr}$ (see $\dag_{2.3}(b)$) there is $M' \cong M$ such that $EM(\tau_T(I, \Phi_1) \prec M' \prec EM(\tau_T(I, \Phi_2)$,

$\dag_{2.3}$ (a) we say $\Phi_1 \leq \Phi_2$ when $\tau(\Phi_1) \subseteq \tau(\Phi_2)$ and $J \subseteq K^\omega_{tr} \Rightarrow EM(J, \Phi_1) \prec EM(J, \Phi_2)$,

(b) we say that $J \subseteq I$ is finitely generated if it has the form $\{ \eta_\ell : \ell < n \} \cup \{ \rho : \text{for some } n, \ell \text{ we have } \rho \in P^I_\omega \text{ and } \rho \prec \eta_\ell \}$ for some $\eta_0, \ldots, \eta_{n-1} \in P^I_\omega$,

$\dag_{2.4}$ if $M_* \in EC_\lambda(T)$ is superlimit (or just weakly $S$-limit, with $S \subseteq \lambda^+$ stationary) then there is $\Phi$ as in $\dag_{2.1}$ above such that $EM_{\tau(T)}(J, \Phi) \cong M_*$ for every finitely generated $J \subseteq K^\omega_{tr}$,
\( \circ \)_{2.5} \) we fix \( \Phi \) as in \( \circ \)_{2.4} for \( M_* \in EC_{\lambda}(T) \) superlimit.

Hence (mainly by clause (b) of \( \circ \)_{2.1} and \( \circ \)_{2.4} as in the proof of part (1))

\( \circ \)_{3} \) if \( I \in K_{tr}^{\omega} \) has cardinality \( \leq \lambda \) then \( EM_{T}(I, \Phi) \) is isomorphic to \( M^* \).

Now by \([17] \), we can find regular uncountable \( \kappa < \beth \) such that \( \lambda = \lambda^{[\kappa]} \) (see Definition \( 3.3 \)).

Let \( S = \{ \delta < \kappa : cf(\delta) = \aleph_0 \} \) and \( \bar{\eta} = \{ \eta_\delta : \delta \in S \} \) be such that \( \eta_\delta \) is an increasing sequence of length \( \omega \) with limit \( \delta \).

For a model \( M \) of \( T \) let \( OB_{\bar{\eta}}(M) = \{ \bar{a} : \bar{a} = \langle a_{\eta_\delta, \alpha} : \delta \in W \text{ and } \alpha < \kappa \rangle, W \subseteq S \text{ and in } M \text{ they are as in } \circ \)_{1}(b), (d) \}. \) For \( \bar{a} \in OB_{\bar{\eta}}(M) \) let \( W[\bar{a}] \) be \( W \) as above and let

\[ \Xi(\bar{a}, M) = \{ \eta \in \omega^\kappa : \text{there is an indiscernible set} \]
\[ \quad \quad \quad I = \{ a_\alpha : \alpha < \kappa \} \text{ in } M \text{ such that for every } n, \]
\[ \quad \quad \quad \text{for some } \delta \in W[\bar{a}], \eta|n = \eta_\delta|n \text{ and} \]
\[ \quad \quad \quad Av_{\Delta_n}(M, I) = Av_{\Delta_n}(M, \{ a_{\eta_\delta, \alpha} : \alpha < \kappa \}) \}. \]

Clearly:

\( \circ \)_{4} \) (a) if \( M \prec N \) then \( OB_{\bar{\eta}}(M) \subseteq OB_{\bar{\eta}}(N) \),

(b) if \( M \prec N \) and \( \bar{a} \in OB_{\bar{\eta}}(M) \) then \( \Xi(\bar{a}, M) \subseteq \Xi(\bar{a}, N) \).

Now by the choice of \( \kappa \) it should be clear that:

\( \circ \)_{5} \) if \( M \models T \) is of cardinality \( \lambda \) then we can find an elementary extension \( N \) of \( M \) of cardinality \( \lambda \) such that for every \( \bar{a} \in OB_{\bar{\eta}}(M) \) with \( W[\bar{a}] \) a stationary subset of \( \kappa \), for some stationary \( W' \subseteq W[\bar{a}] \) the set \( \Xi[\bar{a}, N] \) includes \( \{ \eta \in \omega^\kappa : (\forall n)(\exists \delta \in W')(\eta|n = \eta_\delta|n) \} \) (moreover we can even find \( \varepsilon^* < \kappa \) and \( W_\varepsilon \subseteq W \) for \( \varepsilon < \varepsilon^* \) satisfying \( W[\bar{a}] = \bigcup \{ W_\varepsilon : \varepsilon < \varepsilon^* \} \)).

\( \circ \)_{6} \) we find \( M \in EC_{\lambda}(T) \) isomorphic to \( M^* \) such that for every \( \bar{a} \in OB_{\bar{\eta}}(M) \) with \( W[\bar{a}] \) a stationary subset of \( \kappa \), we can find a stationary subset \( W' \) of \( W[\bar{a}] \) such that the set \( \Xi[\bar{a}, M] \) includes \( \{ \eta \in \omega^\mu : (\forall n)(\exists \delta \in W')(\eta|n = \eta_\delta|n) \} \).

[Why? We choose \( (M_i, N_i) \) for \( i < \kappa^+ \) such that:

- \( M_i \in EC_{\lambda}(T) \) is \( \prec \)-increasing continuous,
- \( M_{i+1} \) is isomorphic to \( M^* \),
- \( M_i \prec N_i \prec M_{i+1} \),
- \( (M_i, N_i) \) are like \( (M, N) \) in \( \circ \)_{5} \).

Now \( M = \bigcup \{ M_i : i < \kappa^+ \} \) is as required. The model \( M \) is isomorphic to \( M^* \) as \( M^* \) is superlimit.]

Now the model from \( \circ \)_{6} \) is not isomorphic to \( M' = EM_{\tau}(T)(\omega \setminus \lambda \cup \{ \eta_\delta : \delta \in S \}, \Phi) \) where \( \Phi \) is from \( \circ \)_{2.1}. But \( M' \cong M^* \) by \( \circ \)_{3}.

Altogether we are done. \( \end{proof} \)
The following claim says in particular that if some not unreasonable pcf conjectures hold, the conclusion holds for every $\lambda \geq 2^{\aleph_0}$.

**Claim 3.4.** Assume $T$ is stable non-superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \text{cf}(\kappa) > \aleph_0$.

1. $T$ has no $(\lambda, \kappa)$-superlimit model provided that $\kappa = \text{cf}(\kappa) > \aleph_0$, $\kappa^{\aleph_0} \leq \lambda$ and $\lambda = \mathcal{U}_D(\lambda) := \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\kappa}\}$ and for every $f : \kappa \to \lambda$ for some $u \in \mathcal{P}$ we have $\{\alpha < \kappa : f(\alpha) \in u\} \in D^+$, where $D$ is a normal filter on $\kappa$ to which $\{\delta < \kappa : \text{cf}(\delta) = \aleph_0\}$ belongs.

2. Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \leq \aleph_0\}$, $J_1 = \{u \subseteq \kappa : u \cap S_{\lambda}^{k_0} \text{ non-stationary}\}$ we have $\lambda = \mathcal{U}_{J_1,J_0}(\lambda) := \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\aleph_0}\}$, and if $u \in J_1$ and $f : (\kappa \setminus u) \to \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathcal{P}$, $\text{Rang}(f|w) \subseteq v$.

**Proof.** Like 3.1.

**Claim 3.5.** (1) Assume $T$ is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$, $T$ has a weakly $(\lambda, \kappa)$-limit model and even a weakly $(\lambda, S)$-limit model for some stationary $S \subseteq S_{\lambda}^\kappa$.

2. Assume $T$ is unsuperstable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \text{cf}(\kappa_2)$. Then $T$ has no model which is a weak $(\lambda, S)$-limit where $S \subseteq \lambda$ and $S \cap S_{\lambda}^{\kappa_1}$ is stationary for $\ell = 1, 2$.

**Proof.** (1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$ (see Definition 3.3) and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let $m$ and $\varphi(\vec{x}, \vec{y})$ be as in the proof of [3.1] Then

(*) if $M \in EC_{\lambda}(T)$ then there is $N$ such that:

(a) $N \in EC_{\lambda}(T)$,
(b) $M \prec N$,
(c) if $\vec{a} = \langle a_i : i < \kappa \rangle \in \kappa(m,M)$ for $\alpha < \kappa$ then for some $\mathcal{U} \in [\kappa]^\lambda$, for every uniform ultrafilter $D$ on $\kappa$ to which $\mathcal{U}$ belongs there is $\vec{a}_D \in nN$ such that $\text{tp}(\vec{a}_D, N, N) = \text{Av}(D, \vec{a}, M) = \{\psi(\vec{x}, \vec{c}) : \psi(\vec{x}, \vec{z}) \in \text{L}(\tau_T), \vec{c} \in \ell^g(\vec{z})M\}$ and $\{\alpha < \kappa : N \models \psi[\vec{a}_{i_\alpha}, \vec{c}]\} \subseteq D$.

Similarly

$\beth_1$ for every function $F$ with domain $\{M : M \prec \text{F}(M)\}$ an $\prec$-increasing sequence of models of $T$ of length $< \lambda^+$ each with universe $\in \lambda^+$ such that $M_i \prec F(M)$ for $i < \ell^g(M)$ and $F(M)$ has universe $\in \lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+ \rangle$ obeying $F$ such that: for every $\varepsilon < \lambda^+$ and $\vec{a} \in \kappa(m(M_{\varepsilon}))$ for $\alpha < \kappa$, there is $\mathcal{U} \in [\kappa]^\kappa$ such that for every ultrafilter $D$ on $\kappa$ to which $\mathcal{U}$ belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\vec{a}_{D,\zeta} \in m(M_{\zeta+1})$ realizing $\text{Av}(D, \vec{a}, M_\zeta)$ in $M_{\zeta+1}$. 

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Hence

$\forall_2$ for $\langle M_\alpha : \alpha < \lambda^+ \rangle$ as in $\forall_1$, for every limit $\delta < \lambda^+$ of cofinality $\neq \kappa$ and every $a = \langle a_i : i < \kappa \rangle \in \kappa(\mathcal{m}(M_\delta))$, there is $\mathcal{V} \in [\kappa]^{\kappa}$ such that for every ultrafilter $D$ on $\kappa$ to which $\mathcal{V}$ belongs, there is a sequence $\langle b_\varepsilon : \varepsilon \in \text{cf}(\delta) \rangle \in \mathcal{C}(\mathcal{m}(M_\delta))$ such that for every $\psi(\bar{x}, \bar{z}) \in \mathcal{L}(\tau_T)$ and $\bar{c} \in \ellg(\mathcal{L})(M_\delta)$, and for every $\varepsilon < \text{cf}(\delta)$ large enough, $M_\delta \models \psi[b_\varepsilon, \bar{c}]$ iff $\psi(\bar{x}, \bar{c}) \in \text{Av}(D, \bar{a}, M_\delta)$.

The rest should be clear.

(2) Combine the above and the proof of 3.1(2).

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