COLLOQUIUM MATHEMATICUM

VOL. 134 2014 NO. 1

COBRAIDED SMASH PRODUCT HOM-HOPF ALGEBRAS

BY

TIANSHUI MA (Xinxiang), HAIYING LI (Xinxiang) and TAO YANG (Nanjing)

Abstract. Let (A,α) and (B,β) be two Hom-Hopf algebras. We construct a new class of Hom-Hopf algebras: R-smash products $(A \natural_R B, \alpha\otimes\beta)$. Moreover, necessary and sufficient conditions for $(A \natural_R B, \alpha\otimes\beta)$ to be a cobraided Hom-Hopf algebra are given.

1. Introduction. Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have recently been intensively investigated (see [1, 3, 5, 6, 11, 12, 13]). Hom-algebras are generalizations of algebras obtained by a twisting map; they have been introduced for the first time in [5]. The associativity is replaced by Hom-associativity; Hom-coassociativity for a Hom-coalgebra can be considered in a similar way. Also definitions and properties of Hom-bialgebras and Hom-Hopf algebras have been proposed (see [1, 3, 6, 12, 13]).

Caenepeel and Goyvaerts [1] studied the Hom-structures from the point of view of monoidal categories and found that Hom-algebras coincide with algebras in a symmetric monoidal category. Yau [12] defined the notion of cobraided Hom-bialgebras and showed that each cobraided Hom-bialgebra comes with solutions of the operator quantum Hom-Yang–Baxter equations, which are twisted analogues of the operator form of the quantum Yang–Baxter equation. Solutions of the Hom-Yang–Baxter equation can be obtained from comodules of suitable cobraided Hom-bialgebras. In [11], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras.

Let H be a Hopf algebra and A an H-module algebra. Then we can construct a new Hopf algebra, their smash product $A \# H$ (see [7] or [8]). Extended forms of smash product can be found in [2, 4].

Let (H,β) be a Hom-Hopf algebra and (A,α) an (H,β)-module Hom-algebra (introduced by Yau [11]). Then it is natural to ask: How to construct the smash product Hom-Hopf algebra and when is it cobraided?

The purpose of this article is to answer the above questions.

2010 *Mathematics Subject Classification*: 16T05.
Key words and phrases: Hom-smash product, cobraided Hom-Hopf algebra,Yang–Baxter equation.

DOI: 10.4064/cm134-1-3

This article is organized as follows. In Section 2, we recall some definitions and results which will be used later. In Section 3, before constructing the smash product Hom-Hopf algebra $(A \natural H, \alpha\otimes\beta)$ (Theorem 3.3), we give a more general case, the so-called R-smash product Hom-Hopf algebra $(A \natural_R B, \alpha\otimes\beta)$ (Theorem 3.1). We remark that the smash product Hom-Hopf algebra $(A \natural H, \alpha\otimes\beta)$ in Theorem 3.3 is different from the one defined by Chen–Wang–Zhang [3], since here the construction of $(A \natural H, \alpha\otimes\beta)$ is based on the concept of the module Hom-algebra introduced by Yau [11], which differs from Chen–Wang–Zhang's [3]. Necessary and sufficient conditions for $(A \natural_R B, \alpha\otimes\beta)$ to be a cobraided Hom-Hopf algebra are derived in Section 4 (Theorems 4.8 and 4.9). In the last section, we give a concrete example.

2. Preliminaries. Throughout this paper, we follow the definitions and terminology of [1, 11, 12], with all algebraic systems supposed to be over a field K. Given a K-space M, we write id_M for the identity map on M.

We now recall some useful definitions.

Definition 2.1. A *Hom-algebra* is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a K-linear space, $\mu : A\otimes A \to A$ is a K-linear map, $1_A \in A$ and α is an automorphism of A such that

(HA1) $$\alpha(aa') = \alpha(a)\alpha(a'), \quad \alpha(1_A) = 1_A,$$

(HA2) $$\alpha(a)(a'a'') = (aa')\alpha(a''), \quad a1_A = 1_A a = \alpha(a),$$

for $a, a', a'' \in A$. Here we use the notation $\mu(a\otimes a') = aa'$.

Definition 2.2. A *Hom-coalgebra* is a quadruple $(C, \Delta, \varepsilon_C, \beta)$ (abbr. (C, β)), where C is a K-linear space, $\Delta : C \to C\otimes C$, $\varepsilon_C : C \to K$ are K-linear maps, and β is an automorphism of C, such that

(HC1) $$\beta(c)_1\otimes\beta(c)_2 = \beta(c_1)\otimes\beta(c_2), \quad \varepsilon_C\circ\beta = \varepsilon_C,$$

(HC2) $$\beta(c_1)\otimes c_{21}\otimes c_{22} = c_{11}\otimes c_{12}\otimes\beta(c_2), \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c),$$

for $c \in A$. Here we use the notation $\Delta(c) = c_1\otimes c_2$ (summation implicitly understood).

Remarks. (a) Here we use β instead of β^{-1} in [1].

(b) The first equation in (HC2) is equivalent to

(1) $$c_1\otimes c_{21}\otimes c_{22} = \beta^{-1}(c_{11})\otimes c_{12}\otimes\beta(c_2)$$

and

(2) $$c_{11}\otimes c_{12}\otimes c_2 = \beta(c_1)\otimes c_{21}\otimes\beta^{-1}(c_{22}),$$

respectively.

(c) By (1), (2) and (HC2), we have

(3) $$c_{11}\otimes c_{12}\otimes c_{21}\otimes c_{22} = \beta(c_1)\otimes\beta^{-1}(c_{211})\otimes\beta^{-1}(c_{212})\otimes c_{22}.$$

DEFINITION 2.3. A *Hom-bialgebra* is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e.

$$\Delta(hh') = \Delta(h)\Delta(h'), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \quad \varepsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \to H$ such that

$$S(h_1)h_2 = h_1 S(h_2) = \varepsilon(h)1_H \quad \text{and} \quad S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$ (abbr. (H, γ, S)) a *Hom-Hopf algebra*.

Let (H, γ) and (H', γ') be two Hom-bialgebras. A linear map $f : H \to H'$ is called a *Hom-bialgebra map* if $f \circ \gamma = \gamma' \circ f$ and at the same time f is a bialgebra map in the usual sense.

DEFINITION 2.4. Let (A, β) be a Hom-algebra. A *left* (A, β)*-Hom-module* is a triple $(M, \triangleright, \alpha)$, where M is a linear space, $\triangleright : A \otimes M \to M$ is a linear map, and α is an automorphism of M, such that

(HM1) $\quad \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m),$

(HM2) $\quad \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m), \quad 1_A \triangleright m = \alpha(m),$

for $a, a' \in A$ and $m \in M$.

REMARKS. (a) It is obvious that (A, μ, β) is a left (A, β)-Hom-module.

(b) When $\beta = \mathrm{id}_A$ and $\alpha = \mathrm{id}_M$, a left (A, β)-Hom-module is the usual left A-module.

DEFINITION 2.5. Let (H, β) be a Hom-bialgebra and (A, α) a Hom-algebra. If $(A, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $a, a' \in A$,

(HMA1) $\quad \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'),$

(HMA2) $\quad h \triangleright 1_A = \varepsilon_H(h)1_A,$

then $(A, \triangleright, \alpha)$ is called an (H, β)*-module Hom-algebra*.

REMARKS. (a) It is obvious that (H, μ_H, β) is an (H, β)-module Hom-algebra.

(b) When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, an (H, β)-module Hom-algebra is the usual H-module algebra.

(c) Similar to the case of Hopf algebras, Yau [13] showed that (HMA1) is satisfied if and only if μ_A is a morphism of H-modules for suitable H-module structures on $A \otimes A$ and A, respectively.

(d) If $\beta^2 = \mathrm{id}$ in (HMA1), then we can get (6.1) of [3]. So the two definitions of module Hom-algebra are different, which leads to the difference of smash product Hom-Hopf algebra in our Theorem 3.3 and in Definition 6.2 of [3].

DEFINITION 2.6. A *cobraided Hom-Hopf algebra* is an octuple $(H, \mu, 1_H, \Delta, \varepsilon, S, \alpha, \sigma)$ (abbr. (H, α, σ)) in which $(H, \mu, 1_H, \Delta, \varepsilon, S, \alpha)$ is a Hom-Hopf algebra and σ is a bilinear form on H (i.e., $\sigma \in \mathrm{Hom}(H \otimes H, K)$), satisfying the following axioms (for all $h, g, l \in H$):

(CHA1) $\sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon(h)$,

(CHA2) $\sigma(hg, \alpha(l)) = \sigma(\alpha(h), l_1)\sigma(\alpha(g), l_2)$,

(CHA3) $\sigma(\alpha(h), gl) = \sigma(h_1, \alpha(l))\sigma(h_2, \alpha(g))$,

(CHA4) $\sigma(h_1, g_1)h_2g_2 = g_1h_1\sigma(h_2, g_2)$,

(CHA5) $\sigma(\alpha(h), \alpha(g)) = \sigma(h, g)$.

In this case, σ is called the *Hom-cobraiding form.*

REMARKS. (a) When $\alpha = \mathrm{id}_H$, a cobraided Hom-Hopf algebra is exactly the usual cobraided (or coquasitriangular) Hopf algebra.

(b) The above definition is slightly different from the definitions in [12] or [13]. Here we replace the Hom-bialgebra by Hom-Hopf algebra and also add two conditions, (CHA1) and (CHA5). Similar to the Hopf algebra setting, the Hom-cobraiding form σ in Definition 2.6 is invertible.

(c) By Yau's results [12], each cobraided Hom-Hopf algebra comes with solutions of the operator quantum Hom-Yang–Baxter equations, which are twisted analogues of the operator form of the quantum Yang–Baxter equation.

Next, we generalize the concept of skew pairing to the Hom-setting.

DEFINITION 2.7. Let (A, α, S_A) and (B, β, S_B) be two Hom-Hopf algebras, and $\vartheta \in \mathrm{Hom}(A \otimes B, K)$ a bilinear form. A *Hom-skew pairing* is a triple (A, B, ϑ) such that

(SP1) $\vartheta(a, 1_B) = \varepsilon_A(a), \quad \vartheta(1_A, b) = \varepsilon_B(b)$,

(SP2) $\vartheta(aa', \beta(b)) = \vartheta(\alpha(a), b_1)\vartheta(\alpha(a'), b_2)$,

(SP3) $\vartheta(\alpha(a), bb') = \vartheta(a_1, \beta(b'))\vartheta(a_2, \beta(b))$,

(SP4) $\vartheta(\alpha(a), \beta(b)) = \vartheta(a, b)$,

for $a, a' \in A$ and $b, b' \in B$.

REMARKS. (a) When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_B$, we get the usual skew pairing.

(b) If (H, α, σ) is a cobraided Hom-Hopf algebra, then (H, H, σ) is a Hom-skew pairing.

(c) ϑ is (convolution) invertible with $\vartheta^{-1}(a, b) = \vartheta(S_A(a), b)$.

3. Smash product Hom-Hopf algebra. In this section, we introduce a class of Hom-Hopf algebras: R-smash products $A \natural_R B$, generalizing the R-smash product studied in [2]. As a special case, we obtain the Hom-smash product based on the structure of module Hom-algebra introduced by Yau [11], [13].

Let A and B be two linear spaces, and $R : B\otimes A \to A\otimes B$ a linear map. In the following, we write $R(b \otimes a) = \sum a_R \otimes b_R$ for all $a \in A$ and $b \in B$, and the notations $\sum a_r \otimes b_r$, $\sum a_{R'} \otimes b_{R'}$ are the copies of $\sum a_R \otimes b_R$. As usual, we omit the summation sign "$\sum$".

THEOREM 3.1. *Let $(A, \mu_A, 1_A, \alpha)$ and $(B, \mu_B, 1_B, \beta)$ be two Hom-algebras, and $R : B\otimes A \to A\otimes B$ a linear map such that for all $a \in A,\ b \in B$,*

$$\alpha(a)_R \otimes \beta(b)_R = \alpha(a_R) \otimes \beta(b_R). \tag{4}$$

Then $(A \natural_R B, \alpha\otimes\beta)$ ($A \natural_R B = A\otimes B$ as a linear space) with multiplication

$$(a \otimes b)(a' \otimes b') = a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b',$$

where $a, a' \in A,\ b, b' \in B$, and with unit $1_A \otimes 1_B$ is a Hom-algebra if and only if the following conditions hold:

$$a_R \otimes 1_{BR} = \alpha(a) \otimes 1_B, \qquad 1_{AR} \otimes b_R = 1_A \otimes \beta(b), \tag{C1}$$

$$\alpha(a)_R \otimes (bb')_R = a_{Rr} \otimes \beta^{-1}(\beta(b)_r)b'_R, \tag{C2}$$

$$\alpha((aa')_R) \otimes \beta(b)_R = \alpha(a_R)\alpha(a')_r \otimes b_{Rr}, \tag{C3}$$

where $a, a' \in A, b, b' \in B$.

We call this Hom-algebra the R-smash product Hom-algebra and denote it by $(A \natural_R B, \alpha \otimes \beta)$.

Proof. ($\Leftarrow$) For all $a, a', a'' \in A$ and, $b, b', b'' \in B$, firstly, we prove that (HA1) holds. In fact,

$$\begin{aligned}(\alpha\otimes\beta)((a\otimes b)(a'\otimes b')) &= \underline{\alpha(a\alpha^{-1}(a')_R)} \otimes \underline{\beta(\beta^{-1}(b_R)b')}\\ &\overset{\text{(HA1)}}{=} \alpha(a)\underline{\alpha(\alpha^{-1}(a')_R)} \otimes \underline{b_R}\beta(b')\\ &\overset{(4)}{=} \alpha(a)a'_R \otimes \beta^{-1}(\beta(b)_R)\beta(b')\\ &= ((\alpha\otimes\beta)(a\otimes b))((\alpha\otimes\beta)(a'\otimes b'))\end{aligned}$$

and

$$(\alpha\otimes\beta)(1_A\otimes 1_B) = \alpha(1_A)\otimes\beta(1_B) \overset{\text{(HA1)}}{=} 1_A\otimes 1_B.$$

Secondly, we prove (HA2):

$$\begin{aligned}&(\alpha(a)\otimes\beta(b))((a'\otimes b')(a''\otimes b''))\\ &\quad= \alpha(a)\underline{\alpha^{-1}(a'\alpha^{-1}(a'')_R)_r} \otimes \beta^{-1}(\underline{\beta(b)_r})(\beta^{-1}(b'_R)b'')\\ &\quad\overset{\text{(C3)}}{=} \alpha(a)\alpha^{-1}(\alpha(\alpha^{-1}(a'_r))\alpha^{-1}(a'')_{RR'}) \otimes \beta^{-1}(b_{rR'})(\beta^{-1}(b'_R)b'')\\ &\quad= \underline{\alpha(a)(\alpha^{-1}(a'_r)\alpha^{-1}(\alpha^{-1}(a'')_{RR'}))} \otimes \beta^{-1}(b_{rR'})(\beta^{-1}(b'_R)b'')\end{aligned}$$

$$\begin{aligned}
&\overset{\text{(HA2)}}{=} (a\alpha^{-1}(a'_r))\alpha^{-1}(a'')_{RR'} \otimes \underline{\beta^{-1}(b_{rR'})(\beta^{-1}(b'_R)b'')}\\
&\overset{\text{(HA2)}}{=} (a\alpha^{-1}(a'_r))\underline{\alpha^{-1}(a'')_{RR'}} \otimes \beta^{-1}(\underline{\beta^{-1}(b_{rR'})b'_R})\beta(b'')\\
&\overset{\text{(C2)}}{=} (a\alpha^{-1}(a')_r)a''_R \otimes \beta^{-1}((\beta^{-1}(b_r)b')_R)\beta(b'')\\
&= ((a\otimes b)(a'\otimes b'))(\alpha(a'')\otimes\beta(b''))
\end{aligned}$$

and

$$\begin{aligned}
(a\otimes b)(1_A\otimes 1_B) &= a\alpha^{-1}(1_A)_R \otimes \beta^{-1}(b_R)1_B\\
&\overset{\text{(HA1)}}{=} a1_{AR}\otimes\beta^{-1}(b_R)1_B\\
&\overset{\text{(C1)}}{=} a1_A\otimes b1_B\\
&\overset{\text{(HA2)}}{=} \alpha(a)\otimes\beta(b).
\end{aligned}$$

Similarly, $(1_A\otimes 1_B)(a\otimes b)=\alpha(a)\otimes\beta(b)$ holds.

($\Rightarrow$) By (HA2), we have

$$1_A\alpha^{-1}(a)_R\otimes\beta^{-1}(1_{BR})b=\alpha(a)\otimes\beta(b), \tag{5}$$

$$a\alpha^{-1}(1_A)\otimes\beta^{-1}(b_R)1_B=\alpha(a)\otimes\beta(b) \tag{6}$$

and

$$\begin{aligned}
\alpha(a)\alpha^{-1}(a'\alpha^{-1}(a'')_R)_r\otimes\beta^{-1}(\beta(b)_r)(\beta^{-1}(b'_R)b'')\\
= (a\alpha^{-1}(a')_r)a''_R\otimes\beta^{-1}((\beta^{-1}(b_r)b')_R)\beta(b'').
\end{aligned} \tag{7}$$

Letting $b=1_B$ and $a=1$ in (5) and (6), respectively, we get (C1).

Letting $a=a'=1_A$ and $b''=1_B$ in (7) and using (C1), we obtain (C2).

Likewise, (C3) can be obtained by letting $a=1_A$ and $b'=b''=1_B$ in (7). ■

When $\alpha=\mathrm{id}_A$ and $\beta=\mathrm{id}_B$, we have

Example 3.2 ([2]). Let $(A,\mu_A,1_A)$ and $(B,\mu_B,1_B)$ be two algebras, and $R: B\otimes A\to A\otimes B$ a linear map. Then $A\,\#_R\,B$ ($A\,\#_R\,B=A\otimes B$ as linear spaces) with multiplication

$$(a\otimes b)(a'\otimes b')=aa'_R\otimes b_Rb',$$

where $a,a'\in A, b,b'\in B$, and unit $1_A\otimes 1_B$ becomes an algebra if and only if the following conditions hold:

$$\begin{aligned}
&a_R\otimes 1_{BR}=a\otimes 1_B,\quad 1_{AR}\otimes b_R=1_A\otimes b,\\
&a_R\otimes(bb')_R=a_{Rr}\otimes b_rb'_R,\\
&(aa')_R\otimes b_R=a_Ra'_r\otimes b_{Rr},
\end{aligned}$$

where $a,a'\in A, b,b'\in B$.

THEOREM 3.3. *Let (H,β) be a Hom-bialgebra and $(A,\triangleright,\alpha)$ an (H,β)-module Hom-algebra. Then $(A\natural H,\alpha\otimes\beta)$ ($A\natural H = A\otimes H$ as a linear space) with multiplication*

$$(a\otimes h)(a'\otimes h') = a(h_1\triangleright \alpha^{-1}(a'))\otimes \beta^{-1}(h_2)h',$$

where $a,a'\in A$, $h,h'\in H$, and unit $1_A\otimes 1_H$ is a Hom-algebra; we call it the smash product Hom-algebra and denote it by $(A\natural H,\alpha\otimes\beta)$.

Proof. Define $R: H\otimes A\to A\otimes H$ by

$$R(h\otimes a) = h_1\triangleright a\otimes h_2, \qquad \forall a\in A,\ h\in H.$$

Firstly, for all $a\in A$ and $h\in H$,

$$\begin{aligned}\alpha(a)_R\otimes\beta(h)_R &= \underline{\beta(h)_1}\triangleright\alpha(a)\otimes\underline{\beta(h)_2}\\ &\overset{(HC1)}{=} \underline{\beta(h_1)\triangleright\alpha(a)}\otimes\beta(h_2)\\ &\overset{(HM1)}{=} \alpha(h_1\triangleright a)\otimes\beta(h_2) = \alpha(a_R)\otimes\beta(h_R),\end{aligned}$$

so (4) holds.

Secondly, we have

$$a_R\otimes 1_{HR} = 1\triangleright a\otimes 1_H \overset{(HM2)}{=} \alpha(a)\otimes 1_H,$$
$$1_{AR}\otimes h_R = h_1\triangleright 1_A\otimes h_2 \overset{(HMA2)}{=} 1_A\otimes\varepsilon(h_1)h_2 \overset{(HC2)}{=} 1_A\otimes\beta(h).$$

Thirdly, we verify (C2) and (C3): for all $a,a'\in A$ and $h,h'\in B$,

$$\begin{aligned}\alpha(a)_R\otimes(hh')_R &= (hh')_1\triangleright\alpha(a)\otimes(hh')_2\\ &= \underline{(h_1h'_1)\triangleright\alpha(a)}\otimes h_2h'_2\\ &\overset{(HM2)}{=} \underline{\beta(h_1)}\triangleright(h'_1\triangleright a)\otimes\underline{h_2}h'_2\\ &\overset{(HC1)}{=} \beta(h)_1\triangleright(h'_1\triangleright a)\otimes\beta^{-1}(\beta(h)_2)h'_2\\ &= a_{Rr}\otimes\beta^{-1}(\beta(h)_r)h'_R\end{aligned}$$

and

$$\begin{aligned}\alpha((aa')_R)\otimes\beta(h)_R &= \alpha(\underline{\beta(h)_1}\triangleright(aa'))\otimes\underline{\beta(h)_2}\\ &\overset{(HC1)}{=} \underline{\alpha(\beta(h_1)\triangleright(aa'))}\otimes\beta(h_2)\\ &\overset{(HM1)}{=} \beta^2(h_1)\triangleright\underline{\alpha(aa')}\otimes\beta(h_2)\\ &\overset{(HA1)}{=} \underline{\beta^2(h_1)\triangleright(\alpha(a)\alpha(a'))}\otimes\beta(h_2)\\ &\overset{(HMA1)}{=} (\underline{h_{11}}\triangleright\alpha(a))(\underline{h_{12}}\triangleright\alpha(a'))\otimes\underline{\beta(h_2)}\\ &\overset{(HC2)}{=} \underline{(\beta(h_1)\triangleright\alpha(a))}(h_{21}\triangleright\alpha(a'))\otimes h_{22}\\ &\overset{(HM1)}{=} \alpha(h_1\triangleright a)(h_{21}\triangleright\alpha(a'))\otimes h_{22}\\ &= \alpha(a_R)\alpha(a')_r\otimes h_{Rr}. \qquad\blacksquare\end{aligned}$$

Remarks. (a) The smash product Hom-Hopf algebra $(A \natural H, \alpha\otimes\beta)$ is different from the one defined by Chen–Wang–Zhang in [3], since here the construction of $(A \natural B, \alpha\otimes\beta)$ is based on the concept of the module Hom-algebra introduced by Yau [11], while two of conditions (6.1), (6.2) in the module Hom-algebra of [3] are as in the case of a Hopf algebra.

(b) When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, we get the usual smash product algebra $A \# H$ (see [7, 8]).

Lemma 3.4. *Let (C, α) and (D, β) be two Hom-coalgebras. Then $(C\otimes D, \alpha\otimes\beta)$ is a Hom-coalgebra with the following comultiplication and counit:*

$$\Delta(c\otimes d) = c_1\otimes d_1\otimes c_2\otimes d_2, \qquad \varepsilon(c\otimes d) = \varepsilon_C(c)\varepsilon_D(d),$$

for $c\in C$ and $d\in D$. We call it the tensor product Hom-coalgebra.

Proof. Straightforward. ■

Theorem 3.5. *Let (A, α, S_A) and (B, β, S_B) be two Hom-Hopf algebras, and $R : B\otimes A \to A\otimes B$ a linear map. Then the R-smash product Hom-algebra $(A \natural_R B, \alpha\otimes\beta)$ equipped with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if R is a coalgebra map, i.e.*

$$a_{R1}\otimes b_{R1}\otimes a_{R2}\otimes b_{R2} = a_{1R}\otimes b_{1R}\otimes a_{2r}\otimes b_{2r},$$
$$\varepsilon_A(a_R)\varepsilon_B(b_R) = \varepsilon_A(a)\varepsilon_B(b),$$

for $a\in A$ and $b\in B$.

Furthermore, the R-smash product Hom-bialgebra $(A \natural_R B, \alpha\otimes\beta)$ is a Hom-Hopf algebra with antipode $\bar S$ defined by

$$\bar S(a\otimes b) = \alpha^{-1}(S_A(a))_R \otimes \beta^{-1}(S_B(b)_R).$$

Proof. We only prove that $\bar S$ is an antipode of $(A \natural_R B, \alpha\otimes\beta)$. The rest is straightforward by direct computation. For all $a\in A$ and $b\in B$,

$$\begin{aligned}
(\bar S * \mathrm{id}_{A\natural_R B})(a\otimes b) &= (\alpha^{-1}(S_A(a_1))_R\otimes \beta^{-1}(S_B(b_1)_R))(a_2\otimes b_2)\\
&= \alpha^{-1}(S_A(a_1))_R\underline{\alpha^{-1}(a_2)_r}\otimes \beta^{-1}(\underline{\beta^{-1}(S_B(b_1)_R)_r})b_2\\
&\overset{(4)}{=} \underline{\alpha^{-1}(S_A(a_1))_R\alpha^{-1}(a_{2r})}\otimes \beta^{-2}(S_B(b_1)_{Rr})b_2\\
&\overset{(\mathrm{HA1})}{=} \alpha^{-1}(\underline{\alpha(\alpha^{-1}(S_A(a_1))_R)a_{2r}})\otimes \beta^{-2}(\underline{S_B(b_1)_{Rr}})b_2\\
&\overset{(\mathrm{C3})}{=} \alpha^{-1}(S_A(a_1)a_2)_R\otimes \beta^{-2}(\beta(S_B(b_1))_R)b_2\\
&= \underline{1_{AR}}\varepsilon_A(a)\otimes \beta^{-2}(\underline{\beta(S_B(b_1))_R})b_2\\
&\overset{(\mathrm{C1})}{=} 1_A\varepsilon_A(a)\otimes S_B(b_1)b_2\\
&= 1_A\otimes 1_B\varepsilon_A(a)\varepsilon_B(b) = 1_A\otimes 1_B\bar\varepsilon(a\otimes b)
\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_{A\natural_R B} * \bar{S})(a\otimes b) &= (a_1\otimes b_1)(\alpha^{-1}(S_A(a_2))_R\otimes\beta^{-1}(S_B(b_2)_R))\\ &= a_1\underline{\alpha^{-1}(\alpha^{-1}(S_A(a_2))_R)_r}\otimes\beta^{-1}(b_{1r})\underline{\beta^{-1}(S_B(b_2)_R)}\\ &\overset{(4)}{=} a_1\alpha^{-2}(S_A(a_2))_{Rr}\otimes\beta^{-1}(b_{1r})\beta^{-1}(S_B(b_2))_R\\ &= a_1\underline{\alpha^{-1}(\alpha^{-1}(S_A(a_2)))_{Rr}}\\ &\quad\otimes\underline{\beta^{-1}(\beta(\beta^{-1}(b_1))_r)\beta^{-1}(S_B(b_2))_R}\\ &\overset{(C2)}{=} a_1\alpha^{-1}(S_A(a_2))_R\otimes\beta^{-1}(b_1S_B(b_2))\\ &= a_1\underline{\alpha^{-1}(S_A(a_2))_R}\otimes\underline{1_{BR}}\varepsilon_B(b)\\ &\overset{(C1)}{=} a_1S_A(a_2)\otimes 1_B\varepsilon_B(b)\\ &= 1_A\otimes 1_B\varepsilon_A(a)\varepsilon_B(b) = 1_A\otimes 1_B\bar{\varepsilon}(a\otimes b),\end{aligned}$$

while

$$\begin{aligned}\bar{S}(\alpha(a)\otimes\beta(b)) &= \alpha^{-1}(S_A(\alpha(a)))_R\otimes\beta^{-1}(S_B(\beta(b))_R)\\ &= \alpha^{-1}(\alpha(S_A(a)))_R\otimes\beta^{-1}(\beta(S_B(b))_R)\\ &= S_A(a)_R\otimes\beta^{-1}(\beta(S_B(b))_R)\\ &\overset{(4)}{=} \alpha(\alpha^{-1}(S_A(a))_R)\otimes S_B(b)_R = (\alpha\otimes\beta)(\bar{S}(a\otimes b)),\end{aligned}$$

finishing the proof. ■

When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_B$, we have

EXAMPLE 3.6 ([2]). Let A and B be two Hopf algebras. Then the twisted tensor product algebra $A \#_R B$ equipped with the usual tensor product coalgebra structure is a bialgebra if and only if R is a coalgebra map.

Furthermore, the twisted tensor product bialgebra $A \#_R B$ is a Hopf algebra with antipode $S_{A\#_R B}$ defined by

$$S_{A\#_R B}(a\otimes b) = S_A(a)_R\otimes S_B(b)_R.$$

THEOREM 3.7. *Let (H,β) be a Hom-Hopf algebra and $(A,\triangleright,\alpha)$ an (H,β)-module Hom-algebra. Then the smash product Hom-algebra $(A\natural H, \alpha\otimes\beta)$ endowed with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if*

$$(h\triangleright a)_1\otimes(h\triangleright a)_2 = (h_1\triangleright a_1)\otimes(h_2\triangleright a_2),\qquad \varepsilon_A(h\triangleright a)=\varepsilon_A(a)\varepsilon_H(h) \tag{8}$$

and

$$h_1\otimes h_2\triangleright a = h_2\otimes h_1\triangleright a. \tag{9}$$

Moreover, $(A\natural H,\alpha\otimes\beta)$ is a Hom-Hopf algebra with antipode

$$S_{A\natural H}(a\otimes h) = S_H(h)_1\triangleright\alpha^{-1}(S_A(a))\otimes\beta^{-1}(S_H(h)_2).$$

Proof. Let $R(h \otimes a) = h_1 \triangleright a \otimes h_2$ for $a \in A$ and $h \in H$ in Theorem 3.5. Then R is a coalgebra map if and only if

$$(h_1 \triangleright a)_1 \otimes h_{21} \otimes (h_1 \triangleright a)_2 \otimes h_{22} = h_{11} \triangleright a_1 \otimes h_{12} \otimes h_{21} \triangleright a_2 \otimes h_{22} \tag{10}$$

and

$$\varepsilon_A(h \triangleright a) = \varepsilon_A(a)\varepsilon_H(h).$$

Moreover by (3) and (HC1), it is easy to deduce that the first equation in (8) and (9) are equivalent to (10). ∎

REMARKS. (a) Let (H, β) be a Hom-Hopf algebra. Assume that $(A, \triangleright, \alpha)$ is a Hom-coalgebra and an (H, β)-Hom-module satisfying (8). Then we call $(A, \triangleright, \alpha)$ an (H, β)-*module Hom-coalgebra.*

When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, then an (H, β)-module Hom-coalgebra is exactly a module coalgebra in the usual sense (see [7]).

(b) Theorem 3.7 is the Hom-version of the usual smash product Hopf algebra (see [7]).

4. Cobraided Hom-Hopf algebra. In this section, necessary and sufficient conditions for a smash product Hom-Hopf algebra to be cobraided are given.

PROPOSITION 4.1. *Let $(A \natural_R B, \alpha \otimes \beta)$ be a R-smash product Hom-Hopf algebra. Define*

$$i : A \to A \natural_R B, \quad i(a) = a \otimes 1_B, \qquad j : B \to A \natural_R B, \quad j(b) = 1_A \otimes b,$$

for all $a \in A$ and $b \in B$. Then i and j are both Hom-bialgebra maps.

Proof. Straightforward. ∎

Let $(A \natural_R B, \alpha \otimes \beta)$ be an R-smash product Hom-Hopf algebra, and $\sigma : A \natural_R B \otimes A \natural_R B \to K$ a bilinear form. Define

$$\begin{aligned} \tau : A \otimes A \to K, &\quad \tau(a, a') = \sigma(i \otimes i)(a \otimes a'), \\ \upsilon : B \otimes B \to K, &\quad \upsilon(b, b') = \sigma(j \otimes j)(b \otimes b'), \\ \varphi : A \otimes B \to K, &\quad \varphi(a, b) = \sigma(i \otimes j)(a \otimes b), \\ \psi : B \otimes A \to K, &\quad \psi(b, a) = \sigma(j \otimes i)(b \otimes a), \end{aligned}$$

for $a, a' \in A$ and $b, b' \in B$.

The following two lemmas are obvious.

LEMMA 4.2. *Let $(A \natural_R B, \alpha \otimes \beta)$ be an R-smash product Hom-Hopf algebra. If σ satisfies* (CHA1)*, then for $a \in A$ and $b \in B$,*

$$\begin{aligned}\tau(1_A, a) &= \tau(a, 1_A) = \varepsilon_A(a),\\ \upsilon(b, 1_B) &= \upsilon(1_B, b) = \varepsilon_B(b),\\ \varphi(1_A, b) &= \varepsilon_B(b), \quad \varphi(a, 1_B) = \varepsilon_A(a),\\ \psi(1_B, a) &= \varepsilon_A(a), \quad \psi(b, 1_A) = \varepsilon_B(b).\end{aligned}$$

LEMMA 4.3. *Let $(A \natural_R B, \alpha \otimes \beta)$ be an R-smash product Hom-Hopf algebra. If σ satisfies* (CHA5) *for $\alpha \otimes \beta$, then, for $a, a' \in A$ and $b, b' \in B$,*

$$\begin{aligned}\tau(\alpha(a), \alpha(a')) &= \tau(a, a'),\\ \upsilon(\beta(b), \beta(b')) &= \upsilon(b, b'),\\ \varphi(\alpha(a), \beta(b)) &= \varphi(a, b),\\ \psi(\beta(b), \alpha(a)) &= \psi(b, a).\end{aligned}$$

LEMMA 4.4. *Let $(A\natural_R B, \alpha\otimes\beta, \sigma)$ be a cobraided R-smash product Hom-Hopf algebra. Then, for all $a, a' \in A$ and $b, b' \in B$,*

$$\sigma(\alpha(a) \otimes \beta(b), \alpha(a') \otimes \beta(b')) = \varphi(a_1, b'_1)\tau(a_2, a'_1)\upsilon(b_1, b'_2)\psi(b_2, a'_2). \tag{11}$$

Proof. By (CHA2) and (CHA3), for all $a, a', a'', a''' \in A$ and $b, b', b'', b''' \in B$, we have

$$\begin{aligned}&\sigma(a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b', a''\alpha^{-1}(a''')_r \otimes \beta^{-1}(b''_r)b''')\\ &\qquad = \sigma(a_1 \otimes b_1, a'''_1 \otimes b'''_1)\sigma(a_2 \otimes b_2, a''_1 \otimes b''_1)\\ &\qquad\quad \times \sigma(a'_1 \otimes b'_1, a'''_2 \otimes b'''_2)\sigma(a'_2 \otimes b'_2, a''_2 \otimes b''_2).\end{aligned}$$

Letting $a' = a''' = 1_A$ and $b = b'' = 1_B$ in the above equation, we get (11). ∎

LEMMA 4.5. *Let $(A\natural_R B, \alpha\otimes\beta, \sigma)$ be a cobraided R-smash product Hom-Hopf algebra. Then, for all $a, a' \in A$ and $b, b' \in B$,*

$$\varphi(\alpha(\alpha^{-1}(a)_R), b_1)\upsilon(b'_R, b_2) = \upsilon(\beta(b'), b_1)\varphi(\alpha(a), b_2), \tag{D1}$$

$$\tau(\alpha(\alpha^{-1}(a)_R), a'_1)\psi(b_R, a'_2) = \psi(\beta(b), a'_1)\tau(\alpha(a), a'_2), \tag{D2}$$

$$\upsilon(b_1, b'_R)\psi(b_2, \alpha(\alpha^{-1}(a)_R)) = \psi(b_1, \alpha(a)\upsilon(b_2, \beta(b')), \tag{D3}$$

$$\varphi(a_1, b_R)\tau(a_2, \alpha(\alpha^{-1}(a')_R)) = \tau(a_1, \alpha(a')\varphi(a_2, \beta(b)), \tag{D4}$$

$$\psi(b_1, a_1)(\alpha(\alpha^{-1}(a_2)_R) \otimes b_{2R}) = (\alpha(a_1) \otimes \beta(b_1))\psi(b_2, a_2), \tag{D5}$$

$$\varphi(a_1, b_1)(\alpha(a_2) \otimes \beta(b_2)) = (\alpha(\alpha^{-1}(a_1)_R) \otimes b_{1R})\varphi(a_2, b_2). \tag{D6}$$

Proof. By (CHA2), for all $a, a', a'' \in A$ and $b, b', b'' \in B$, we can obtain

$$\begin{aligned}&\sigma(a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b', \alpha(a'') \otimes \beta(b''))\\ &\qquad = \sigma(\alpha(a) \otimes \beta(b), a''_1 \otimes b''_1)\sigma(\alpha(a') \otimes \beta(b'), a''_2 \otimes b''_2).\end{aligned} \tag{12}$$

Letting $a = 1_A$ and $b' = b'' = 1_B$ in (12) yields (D1) by (11). Similarly, setting $a = a'' = 1_A$ and $b' = 1_B$ in (12), we get (D2) by (11).

By (CHA3), for all $a, a', a'' \in A$ and $b, b', b'' \in B$, we have

$$\sigma\big(\alpha(a) \otimes \beta(b), a'\alpha^{-1}(a'')_R \otimes \beta^{-1}(b'_R)b''\big) = \sigma(a_1 \otimes b_1, \alpha(a'') \otimes \beta(b''))\sigma(a_2 \otimes b_2, \alpha(a') \otimes \beta(b')). \tag{13}$$

(D3) can be obtained by letting $a = a' = 1_A$ and $b'' = 1_B$ in (13) and by (11). Likewise, one gets (D4) by putting $a' = 1_A$ and $b = b'' = 1_B$ in (13) and using (11).

By (CHA4), for all $a, a' \in A$ and $b, b' \in B$, we have

$$\sigma(a_1 \otimes b_1, a'_1 \otimes b'_1)(a_2\alpha^{-1}(a'_2)_R \otimes \beta^{-1}(b_{2R})b'_2) = (a'_1\alpha^{-1}(a_1)_R \otimes \beta^{-1}(b'_{1R})b_1)\sigma(a_2 \otimes b_2, a'_2 \otimes b'_2). \tag{14}$$

Letting $a = 1_A$ and $b' = 1_B$ in (14), we get (D5); and (D6) is derived by letting $a' = 1_A$ and $b = 1_B$ in (14). ∎

LEMMA 4.6. *Given the cobraiding σ on an R-smash product Hom-Hopf algebra $(A \natural_R B, \alpha \otimes \beta)$, consider the induced maps τ, υ, φ and ψ. Then*

(1) *(A, α, τ) and (B, β, υ) are cobraided Hom-Hopf algebras,*
(2) *(A, B, φ) and (B, A, ψ) are Hom-skew pairings.*

Proof. (1) Setting $b = b' = b'' = 1_B$ in (12) and (13), we get (CHA2) and (CHA3) for τ, respectively. (CHA4) can be derived by letting $b = b' = 1_B$ in (14); then by Lemmas 4.2 and 4.3, (A, α, τ) is a cobraided Hom-Hopf algebra. Similarly, we can prove that (B, β, υ) is a cobraided Hom-Hopf algebra.

(2) Letting $a'' = 1_A$ and $b = b' = 1_B$ in (12), and $a' = a'' = 1_A$ and $b = 1_B$ in (13), one can obtain (SP2) and (SP3) for φ, respectively. Then (A, B, φ) is a Hom-skew pairing by Lemmas 4.2 and 4.3. The rest of (2) can be demonstrated similarly. ∎

LEMMA 4.7. *Let $(A \natural_R B, \alpha \otimes \beta)$ be an R-smash product Hom-Hopf algebra. Suppose there exist forms $\tau : A \otimes A \to K$, $\varphi : A \otimes B \to K$, $\psi : B \otimes A \to K$, and $\upsilon : B \otimes B \to K$ such that*

(1) *(A, α, τ) and (B, β, υ) are cobraided Hom-Hopf algebras,*
(2) *(A, B, φ) and (B, A, ψ) are Hom-skew pairings,*
(3) *the conditions* (D1)–(D6) *in Lemma* 4.5 *hold.*

Then $(A \natural_R B, \alpha \otimes \beta, \sigma)$ is a cobraided Hom-Hopf algebra with the cobraided structure given by

$$\sigma(\alpha(a) \otimes \beta(b), \alpha(a') \otimes \beta(b')) = \varphi(a_1, b'_1)\tau(a_2, a'_1)\upsilon(b_1, b'_2)\psi(b_2, a'_2)$$

for $a, a' \in A$ and $b, b' \in B$.

Proof. It is obvious that σ satisfies (CHA1) and (CHA5).

Next, we show that (CHA2) holds for σ. For all $a, a', a'' \in A$ and b, b', b'' $\in B$,

$$
\begin{aligned}
&\sigma((a\otimes b)(a'\otimes b'), \alpha(a'')\otimes\beta(b''))\\
&\quad = \sigma(a\alpha^{-1}(a')_R\otimes\beta^{-1}(b_R)b', \alpha(a'')\otimes\beta(b''))\\
&\quad = \varphi(\alpha^{-1}(a\alpha^{-1}(a')_R)_1, b''_1)\tau(\alpha^{-1}(a\alpha^{-1}(a')_R)_2, a''_1)\\
&\qquad \times\upsilon(\beta^{-1}(\beta^{-1}(b_R)b')_1, b''_2)\psi(\beta^{-1}(\beta^{-1}(b_R)b')_2, a''_2)\\
&\overset{\text{(HA1),(HC1)}}{=} \varphi(\alpha^{-1}(a_1)\alpha^{-1}(\alpha^{-1}(a')_{R1}), b''_1)\tau(\alpha^{-1}(a_2)\alpha^{-1}(\alpha^{-1}(a')_{R2}), a''_1)\\
&\qquad \times\upsilon(\beta^{-2}(b_{R1})\beta^{-1}(b'_1), b''_2)\psi(\beta^{-2}(b_{R2})\beta^{-1}(b'_2), a''_2)\\
&\overset{\text{(CHA2),(SP2)}}{=} \varphi(a_1, \beta^{-1}(b''_{11}))\varphi(\alpha^{-1}(a')_{R1}, \beta^{-1}(b''_{12}))\tau(a_2, \alpha^{-1}(a''_{11}))\\
&\qquad \times\tau(\alpha^{-1}(a')_{R2}, \alpha^{-1}(a''_{12}))\upsilon(\beta^{-1}(b_{R1}), \beta^{-1}(b''_{21}))\upsilon(b'_1, \beta^{-1}(b''_{22}))\\
&\qquad \times\psi(\beta^{-1}(b_{R2}), \alpha^{-1}(a''_{21}))\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\quad = \varphi(a_1, \beta^{-1}(b''_{11}))\varphi(\alpha^{-1}(a')_{1R}, \beta^{-1}(b''_{12}))\tau(a_2, \alpha^{-1}(a''_{11}))\\
&\qquad \times\tau(\alpha^{-1}(a')_{2r}, \alpha^{-1}(a''_{12}))\times\upsilon(\beta^{-1}(b_{1R}), \beta^{-1}(b''_{21}))\upsilon(b'_1, \beta^{-1}(b''_{22}))\\
&\qquad \times\psi(\beta^{-1}(b_{2r}), \alpha^{-1}(a''_{21}))\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\overset{(3)}{=} \varphi(a_1, b''_1)\varphi(\alpha^{-1}(a')_{1R}, \beta^{-2}(b''_{211}))\tau(a_2, a''_1)\\
&\qquad \times\tau(\alpha^{-1}(a')_{2r}, \alpha^{-2}(a''_{211}))\upsilon(\beta^{-1}(b_{1R}), \beta^{-2}(b''_{212}))\upsilon(b'_1, \beta^{-1}(b''_{22}))\\
&\qquad \times\psi(\beta^{-1}(b_{2r}), \alpha^{-2}(a''_{212}))\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\overset{\text{(4),(HC1)}}{=} \varphi(a_1, b''_1)\varphi(\alpha(\alpha^{-1}(\alpha^{-1}(a')_1)_R), \beta^{-2}(b''_{21})_1)\tau(a_2, a''_1)\\
&\qquad \times\tau(\alpha(\alpha^{-1}(\alpha^{-1}(a')_2)_r), \alpha^{-2}(a''_{21})_1)\ \upsilon(\beta^{-1}(b_1)_R, \beta^{-2}(b''_{21})_2)\\
&\qquad \times\upsilon(b'_1, \beta^{-1}(b''_{22}))\psi(\beta^{-1}(b_2)_r, \alpha^{-2}(a''_{21})_2)\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\overset{\text{(D1),(D2)}}{=} \varphi(a_1, b''_1)\varphi(a'_1, \beta^{-2}(b''_{21})_2)\tau(a_2, a''_1)\tau(a'_2, \alpha^{-2}(a''_{21})_2)\\
&\qquad \times\upsilon(b_1, \beta^{-2}(b''_{21})_1)\upsilon(b'_1, \beta^{-1}(b''_{22}))\psi(b_2, \alpha^{-2}(a''_{21})_1)\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\overset{(3)}{=} \varphi(a_1, \beta^{-1}(b''_{11}))\varphi(a'_1, \beta^{-1}(b''_{21}))\tau(a_2, \alpha^{-1}(a''_{11}))\tau(a'_2, \alpha^{-1}(a''_{21}))\\
&\qquad \times\upsilon(b_1, \beta^{-1}(b''_{12}))\upsilon(b'_1, \beta^{-1}(b''_{22}))\psi(b_2, \alpha^{-1}(a''_{12}))\psi(b'_2, \alpha^{-1}(a''_{22}))\\
&\overset{\text{(HC1)}}{=} \varphi(a_1, \beta^{-1}(b''_1)_1)\tau(a_2, \alpha^{-1}(a''_1)_1)\upsilon(b'_1, \beta^{-1}(b''_2)_2)\psi(b'_2, \alpha^{-1}(a''_2)_2)\\
&\qquad \times\varphi(a'_1, \beta^{-1}(b''_2)_1)\tau(a'_2, \alpha^{-1}(a''_2)_1)\upsilon(b_1, \beta^{-1}(b''_1)_2)\psi(b_2, \alpha^{-1}(a''_1)_2)\\
&\quad = \sigma(\alpha(a)\otimes\beta(b), a''_1\otimes b''_1)\sigma(\alpha(a')\otimes\beta(b'), a''_2\otimes b''_2).
\end{aligned}
$$

(CHA3) for σ can be proved by a similar method.

Now we check (CHA4): for all $a, a' \in A$ and $b, b' \in B$,

$$
\begin{aligned}
&\sigma(a_1\otimes b_1, a'_1\otimes b'_1)(a_2\otimes b_2)(a'_2\otimes b'_2)\\
&\quad = u(\alpha^{-1}(a_1)_1, \beta^{-1}(b'_1)_1)\tau(\alpha^{-1}(a_1)_2, \alpha^{-1}(a'_1)_1)\upsilon(\beta^{-1}(b_1)_1, \beta^{-1}(b'_1)_2)\\
&\qquad \times\psi(\beta^{-1}(b_1)_2, \alpha^{-1}(a'_1)_2)(a_2\alpha^{-1}(a'_2)_R\otimes\beta^{-1}(b_{2R})b'_2)\\
&\quad \overset{(HC1)}{=} \varphi(\alpha^{-1}(a_{11}), \beta^{-1}(b'_{11}))\tau(\alpha^{-1}(a_{12}), \alpha^{-1}(a'_{11}))\upsilon(\beta^{-1}(b_{11}), \beta^{-1}(b'_{12}))\\
&\qquad \times\psi(\beta^{-1}(b_{12}), \alpha^{-1}(a'_{12}))(a_2\alpha^{-1}(a'_2)_R\otimes\beta^{-1}(b_{2R})b'_2)\\
&\qquad \times\psi(\beta^{-1}(b_1)_2, \alpha^{-1}(a'_1)_2)(a_2\alpha^{-1}(a'_2)_R\otimes\beta^{-1}(b_{2R})b'_2)\\
&\quad \overset{(2)}{=} \varphi(a_1, b'_1)\tau(\alpha^{-1}(a_{21}), a'_1)\upsilon(b_1, \beta^{-1}(b'_{21}))\psi(\beta^{-1}(b_{21}), \alpha^{-1}(a'_{21}))\\
&\qquad \times(\alpha^{-1}(a_{22})\alpha^{-2}(a'_{22})_R\otimes\beta^{-1}(\beta^{-1}(b_{22})_R)\beta^{-1}(b'_{22}))\\
&\quad \overset{(HC1)}{=} \varphi(a_1, b'_1)\tau(\alpha^{-1}(a_2)_1, a'_1)\upsilon(b_1, \beta^{-1}(b'_2)_1)\underline{\psi(\beta^{-1}(b_2)_1, \alpha^{-1}(a'_2)_1)}\\
&\qquad \times(\alpha^{-1}(a_2)_2\underline{\alpha^{-1}(\alpha^{-1}(a'_2)_2)_R}\otimes\underline{\beta^{-1}(\beta^{-1}(b_2)_{2R})}\beta^{-1}(b'_2)_2)\\
&\quad \overset{(D5)}{=} \varphi(a_1, b'_1)\tau(\alpha^{-1}(a_2)_1, a'_1)\upsilon(b_1, \beta^{-1}(b'_2)_1)\psi(\beta^{-1}(b_2)_2, \alpha^{-1}(a'_2)_2)\\
&\qquad \times(\alpha^{-1}(a_2)_2\alpha^{-1}(a'_2)_1\otimes\beta^{-1}(b_2)_1\beta^{-1}(b'_2)_2)\\
&\quad \overset{(1),(HC1)}{=} \varphi(a_1, b'_1)\underline{\tau(\alpha^{-1}(a_2)_1, \alpha^{-1}(a'_1)_1)}\ \underline{\upsilon(\beta^{-1}(b_1)_1, \beta^{-1}(b'_2)_1)}\psi(b_2, a'_2)\\
&\qquad \times(\underline{\alpha^{-1}(a_2)_2\alpha^{-1}(a'_1)_2}\otimes\underline{\beta^{-1}(b_1)_2\beta^{-1}(b'_2)_2})\\
&\quad \overset{(CHA4)}{=} u(a_1, b'_1)\tau(\alpha^{-1}(a_2)_2, \alpha^{-1}(a'_1)_2)\upsilon(\beta^{-1}(b_1)_2, \beta^{-1}(b'_2)_2)\psi(b_2, a'_2)\\
&\qquad \times(\alpha^{-1}(a'_1)_1\alpha^{-1}(a_2)_1\otimes\beta^{-1}(b'_2)_1\beta^{-1}(b_1)_1)\\
&\quad \overset{(1),(HC1)}{=} \underline{\varphi(\alpha^{-1}(a_1)_1, \beta^{-1}(b'_1)_1)}\tau(a_2, \alpha^{-1}(a'_1)_2)\upsilon(\beta^{-1}(b_1)_2, b'_2)\psi(b_2, a'_2)\\
&\qquad \times(\alpha^{-1}(a'_1)_1\underline{\alpha^{-1}(a_1)_2}\otimes\underline{\beta^{-1}(b'_1)_2}\beta^{-1}(b_1)_1)\\
&\quad \overset{(D6)}{=} \varphi(\alpha^{-1}(a_1)_2, \beta^{-1}(b'_1)_2)\tau(a_2, \alpha^{-1}(a'_1)_2)\upsilon(\beta^{-1}(b_1)_2, b'_2)\psi(b_2, a'_2)\\
&\qquad \times(\alpha^{-1}(a'_1)_1\alpha^{-1}(\alpha^{-1}(a_1)_1)_R\otimes\beta^{-1}(\beta^{-1}(b'_1)_{1R})\beta^{-1}(b_1)_1)\\
&\quad \overset{(2),(3)}{=} (a'_1\alpha^{-1}(a_1)_R\otimes\beta^{-1}(b'_{1R})b_1)\varphi(\alpha^{-1}(a_2)_1, \beta^{-1}(b'_1)_2)\\
&\qquad \times\tau(\alpha^{-1}(a_2)_2, \alpha^{-1}(a'_2)_1)\upsilon(\beta^{-1}(b_2)_1, \beta^{-1}(b'_2)_2)\psi(\beta^{-1}(b_2)_2, \alpha^{-1}(a'_2)_2)\\
&\quad = (a'_1\otimes b'_1)(a_1\otimes b_1)\sigma(a_2\otimes b_2, a'_2\otimes b'_2).
\end{aligned}
$$

Therefore, $(A\natural_R B, \alpha\otimes\beta, \sigma)$ is a cobraided Hom-Hopf algebra. ∎

Lemmas 4.2–4.7 imply

THEOREM 4.8. *An R-smash product Hom-Hopf algebra* $(A \natural_R B, \alpha\otimes\beta)$ *is cobraided if and only if there exist forms* $\tau : A\otimes A \to K$, $\varphi : A\otimes B \to K$, $\psi : B\otimes A \to K$, *and* $\upsilon : B\otimes B \to K$ *such that* (A, α, τ) *and*

(B,β,υ) are cobraided Hom-Hopf algebras, (A,B,φ) and (B,A,ψ) are Hom-skew pairings, and the conditions (D1)–(D6) *of Lemma* 4.5 *hold. Moreover, the cobraided structure σ on $(A \natural_R B, \alpha\otimes\beta)$ has the decomposition*

$$\sigma(\alpha(a)\otimes\beta(b),\alpha(a')\otimes\beta(b')) = \varphi(a_1,b'_1)\tau(a_2,a'_1)\upsilon(b_1,b'_2)\psi(b_2,a'_2).$$

THEOREM 4.9. *A smash product Hom-Hopf algebra $(A\natural H,\alpha\otimes\beta)$ is cobraided if and only if there exist forms $\tau: A\otimes A\to K$, $\varphi: A\otimes H\to K$, $\psi: H\otimes A\to K$, and $\upsilon: H\otimes H\to K$ such that (A,α,τ) and (H,β,υ) are cobraided Hom-Hopf algebras, (A,H,φ) and (H,A,ψ) are Hom-skew pairings, and the conditions* (D1)′–(D6)′ *below hold: for all $a,a'\in A$ and $h,h'\in B$,*

(D1)′ $\varphi(\beta(h'_1)\triangleright a,h_1)\upsilon(h'_2,h_2) = \upsilon(\beta(h'),h_1)\varphi(\alpha(a),h_2)$,

(D2)′ $\tau(\beta(h_1)\triangleright a,a'_1)\psi(h_2,a'_2) = \psi(\beta(h),a'_1)\tau(\alpha(a),a'_2)$,

(D3)′ $\upsilon(h_1,h'_2)\psi(h_2,\beta(h'_1)\triangleright a) = \psi(h_1,\alpha(a)\upsilon(h_2,\beta(h'))$,

(D4)′ $\varphi(a_1,h_2)\tau(a_2,\beta(h_1)\triangleright a') = \tau(a_1,\alpha(a')\varphi(a_2,\beta(h))$,

(D5)′ $\psi(h_1,a_1)(\beta(h_{21})\triangleright a_2\otimes h_{22}) = (\alpha(a_1)\otimes\beta(h_1))\psi(h_2,a_2)$,

(D6)′ $\varphi(a_1,h_1)(\alpha(a_2)\otimes\beta(h_2)) = (\beta(h_{11})\triangleright a_1\otimes h_{12})\varphi(a_2,h_2)$.

Moreover, the cobraided structure σ' on $(A\natural H,\alpha\otimes\beta)$ has the decomposition

$$\sigma'(\alpha(a)\otimes\beta(h),\alpha(a')\otimes\beta(h')) = \varphi(a_1,h'_1)\tau(a_2,a'_1)\upsilon(h_1,h'_2)\psi(h_2,a'_2).$$

Proof. Let $R(h\otimes a) = h_1\triangleright a\otimes h_2$ for $a\in A$ and $h\in H$ in Theorem 4.8. ∎

5. Applications. In this section, we apply the main results of Sections 3 and 4 to a concrete example.

The following result is clear.

LEMMA 5.1. *Let $K\mathbb{Z}_2 = K\{1,a\}$ be a Hopf group algebra* (*see* [9]). *Then $(K\mathbb{Z}_2,\mathrm{id}_{K\mathbb{Z}_2},\upsilon)$ is a cobraided Hom-Hopf algebra, where υ is given by*

υ	1	a
1	1	1
a	1	-1

Let

$$T_{2,-1} = K\{1,g,x,gx \mid g^2=1,\ x^2=0,\ xg=-gx\}$$

be Taft's Hopf algebra (see [10]); its coalgebra structure and antipode are given by

$$\Delta(g)=g\otimes g,\quad \Delta(x)=x\otimes g+1\otimes x,\quad \Delta(gx)=gx\otimes 1+g\otimes gx;$$
$$\varepsilon(g)=1,\quad \varepsilon(x)=0,\quad \varepsilon(gx)=0;$$

and

$$S(g) = g, \quad S(x) = gx, \quad S(gx) = -x.$$

Define a linear map $\alpha : T_{2,-1} \to T_{2,-1}$ by

$$\alpha(1) = 1, \quad \alpha(g) = g, \quad \alpha(x) = kx, \quad \alpha(gx) = kgx$$

where $0 \neq k \in K$. Then α is an automorphism of Hopf algebras.

So we get a Hom-Hopf algebra

$$H_\alpha = (T_{2,-1}, \alpha \circ \mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}} \circ \alpha, \varepsilon_{T_{2,-1}}, \alpha)$$

(see [6]).

LEMMA 5.2. *Let* H_α *be the Hom-Hopf algebra defined above. Then* (H_α, α, τ) *is a cobraided Hom-Hopf algebra, where* τ *is given by*

τ	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	0	0
gx	0	0	0	0

Proof. A straightforward but tedious computation. ∎

THEOREM 5.3. *Let* $K\mathbb{Z}_2$ *be the Hopf group algebra and* H_α *the Hom-Hopf algebra defined above. Define a module action* $\triangleright : K\mathbb{Z}_2 \otimes H_\alpha \to H_\alpha$ *by*

$$\begin{aligned} 1_{K\mathbb{Z}_2} \triangleright 1_{H_\alpha} &= 1_{H_\alpha}, & a \triangleright 1_{H_\alpha} &= 1_{H_\alpha}, \\ 1_{K\mathbb{Z}_2} \triangleright g &= g, & a \triangleright g &= g, \\ 1_{K\mathbb{Z}_2} \triangleright x &= kx, & a \triangleright x &= -kx, \\ 1_{K\mathbb{Z}_2} \triangleright gx &= kgx, & a \triangleright gx &= -kgx, \end{aligned}$$

Then by a routine computation we find that H_α *is a* $K\mathbb{Z}_2$*-module Hom-algebra. Therefore, by Theorem* 3.3*,* $(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2})$ *is a smash product Hom-algebra.*

Furthermore, $(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2})$ with the tensor product Hom-coalgebra structure becomes a Hom-Hopf algebra, where the antipode $\bar{S}$ is given by

$$\begin{aligned} \bar{S}(1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}) &= 1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}, & \bar{S}(1_{H_\alpha} \otimes a) &= 1_{H_\alpha} \otimes a, \\ \bar{S}(g \otimes 1_{K\mathbb{Z}_2}) &= g \otimes 1_{K\mathbb{Z}_2}, & \bar{S}(g \otimes a) &= g \otimes a, \\ \bar{S}(x \otimes 1_{K\mathbb{Z}_2}) &= -gx \otimes 1_{K\mathbb{Z}_2}, & \bar{S}(x \otimes a) &= -gx \otimes a, \\ \bar{S}(gx \otimes 1_{K\mathbb{Z}_2}) &= x \otimes 1_{K\mathbb{Z}_2}, & \bar{S}(gx \otimes a) &= x \otimes a. \end{aligned}$$

LEMMA 5.4. *Let* $K\mathbb{Z}_2$ *be the Hopf group algebra and* H_α *the Hom-Hopf algebra defined above. Define two linear maps* $\varphi : H_\alpha \otimes K\mathbb{Z}_2 \to K$ *and* $\psi : K\mathbb{Z}_2 \otimes H_\alpha \to K$ *as follows:*

φ	1	a
1	1	1
g	1	-1
x	0	0
gx	0	0

ψ	1	g	x	gx
1	1	1	0	0
a	1	-1	0	0

Then $(H_\alpha, K\mathbb{Z}_2, \varphi)$ *and* $(K\mathbb{Z}_2, H_\alpha, \psi)$ *are Hom-skew pairings.*

Proof. Straightforward. ∎

THEOREM 5.5. *With the notations above, the smash product Hom-Hopf algebra* $(H_\alpha \natural K\mathbb{Z}_2, \alpha \otimes \mathrm{id}_{K\mathbb{Z}_2}, \sigma)$ *is a cobraided Hom-Hopf algebra with cobraiding* σ *given as follows*:

σ	$1\otimes 1$	$1\otimes a$	$g\otimes 1$	$g\otimes a$	$x\otimes 1$	$x\otimes a$	$gx\otimes 1$	$gx\otimes a$
$1\otimes 1$	1	1	1	1	0	0	0	0
$1\otimes a$	1	-1	-1	1	0	0	0	0
$g\otimes 1$	1	-1	-1	1	0	0	0	0
$g\otimes a$	1	1	1	1	0	0	0	0
$x\otimes 1$	0	0	0	0	0	0	0	0
$x\otimes a$	0	0	0	0	0	0	0	0
$gx\otimes 1$	0	0	0	0	0	0	0	0
$gx\otimes a$	0	0	0	0	0	0	0	0

Proof. It is easy to prove that the conditions (D1)′–(D6)′ hold. We finish the proof by using Lemmas 5.1, 5.2, 5.4 and Theorem 4.9. ∎

Acknowledgements. The authors are deeply indebted to the referee for his/her useful suggestions and some improvements to the original manuscript, and for calling our attention to [3]. This work was partially supported by the NNSF of China (No. 11101128) and the NSF of Henan Province (Nos. 12A110013, 122300410110, 2011GGJS-062).

REFERENCES

[1] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011), 2216–2240.

[2] S. Caenepeel, B. Ion, G. Militaru and S. L. Zhu, *The factorization problem and the smash biproduct of algebras and coalgebras*, Algebra Represent. Theory 3 (2000), 19–42.

[3] Y. Chen, Z. Wang and L. Zhang, *Integrals for monoidal Hom-Hopf algebras and their applications*, J. Math. Phys. 54 (2013), 073515.

[4] T. S. Ma and S. H. Wang, *General double quantum groups*, Comm. Algebra 38 (2010), 645–672.

[5] A. Makhlouf and S. D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. 2 (2008), 51–64.

[6] A. Makhlouf and S. D. Silvestrov, *Hom-algebras and hom-coalgebras*, J. Algebra Appl. 9 (2010), 553–589.
[7] R. K. Molnar, *Semi-direct products of Hopf algebras*, J. Algebra 47 (1977), 29–51.
[8] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Reg. Conf. Ser. Math. 82, Amer. Math. Soc., Providence, RI, 1993.
[9] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.
[10] E. J. Taft, *The order of the antipode of finite-dimensional Hopf algebra*, Proc. Nat. Acad. Sci. USA 68 (1971), 2631–2633.
[11] D. Yau, *Module Hom-algebras*, arXiv:0812.4695v1.
[12] D. Yau, *Hom-quantum groups II: cobraided Hom-bialgebras and Hom-quantum geometry*, arXiv:0907.1880.
[13] D. Yau, *Hom-quantum groups III: Representations and module Hom-algebras*, arXiv:0911.5402.

Tianshui Ma (corresponding author), Haiying Li
College of Mathematics and Information Science
Henan Normal University
453007 Xinxiang, China
E-mail: matianshui@yahoo.com
haiyingli2012@yahoo.com

Tao Yang
College of Science
Nanjing Agricultural University
210095 Nanjing, China
E-mail: tao.yang.seu@gmail.com

Received 26 April 2013;
revised 19 October 2013 (5924)