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ANISOTROPIC PARABOLIC PROBLEMS WITH SLOWLY OR RAPIDLY GROWING TERMS

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Abstract. We consider an abstract parabolic problem in a framework of maximal monotone graphs, possibly multi-valued, with growth conditions formulated with the help of an x-dependent N-function. The main novelty of the paper consists in the lack of any growth restrictions on the N-function combined with its anisotropic character, namely we allow the dependence on all the directions of the gradient, not only on its absolute value. This leads to using the notion of modular convergence and studying in detail the question of density of compactly supported smooth functions with respect to modular convergence.

1. Introduction. We are interested in the phenomenon of anisotropic behaviour in a parabolic problem. Our approach allows for embracing very general growth conditions for the nonlinear term. We concentrate on an abstract parabolic problem. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with $\mathcal{C}^2$ boundary $\partial\Omega$, $(0,T)$ be the time interval with $T < \infty$, $Q := (0,T) \times \Omega$, and $\mathcal{A}$ be a maximal monotone graph satisfying the assumptions (A1)–(A5) below. Given f and u_0 we want to find $u : Q \to \mathbb{R}$ and $A : Q \to \mathbb{R}^d$ such that

$$u_t - \operatorname{div} A = f \quad \text{in } Q, \tag{1.1}$$

$$(\nabla u, A) \in \mathcal{A}(t,x) \quad \text{in } Q, \tag{1.2}$$

$$u(0,x) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

$$u(t,x) = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{1.4}$$

Our main objective is to obtain an existence result for the widest possible class of maximal monotone graphs. Hence various nonstandard possibilities are considered including anisotropic growth conditions, x-dependent growth conditions and also relations given by a maximal monotone graph. The last ones provide the possibility of generalizing discontinuous relations, namely considering A as a discontinuous function of ∇u, where the jumps of A are filled by intervals forming vertical parts of the graph $\mathcal{A}$. Most of these generalities arise in a function that will prescribe the growth/coercivity con-

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ditions. In contrast to the usual case of Leray–Lions type operators, where polynomial growth is assumed, e.g. $|A(\xi)| \le c(1+|\xi|)^{p-1}$, $A(\xi)\cdot\xi \ge C|\xi|^p$ for some nonnegative constants c, C and $p > 1$, we shall work with N-functions. By M being an *N-function* we mean that $M : \bar{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+$, $M(x,a)$ is measurable with respect to x for all $a \in \mathbb{R}^d$ and continuous with respect to a for a.a. $x \in \bar{\Omega}$, convex in a, has superlinear growth, $M(x,a) = 0$ iff $a = 0$ and

$$\lim_{|a|\to\infty} \inf_{x\in\Omega} \frac{M(x,a)}{|a|} = \infty.$$

Moreover the *conjugate function* M^* is defined as

$$M^*(x,b) = \sup_{a\in\mathbb{R}^d} (b\cdot a - M(x,a)).$$

The graph $\mathcal{A}$ is expected to satisfy for almost all $(t,x) \in Q$ the following assumptions:

(A1) $\mathcal{A}$ passes through the origin.

(A2) $\mathcal{A}$ is a *monotone graph*:

$$(A_1 - A_2)\cdot(\xi_1 - \xi_2) \ge 0 \quad \text{for all } (\xi_1, A_1), (\xi_2, A_2) \in \mathcal{A}(t,x).$$

(A3) $\mathcal{A}$ is a *maximal monotone graph*: If $(\xi_2, A_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and

$$(A_1 - A_2)\cdot(\xi_1 - \xi_2) \ge 0 \quad \text{for all } (\xi_1, A_1) \in \mathcal{A}(t,x)$$

then $(\xi_2, A_2) \in \mathcal{A}(t,x)$.

(A4) $\mathcal{A}$ is an *M-graph*: There are nonnegative $k \in L^1(Q)$, $c_* > 0$ and an N-function M such that

$$A\cdot\xi \ge -k(t,x) + c_*(M(x,\xi) + M^*(x,A))$$

for all $(\xi, A) \in \mathcal{A}(t,x)$.

(A5) Existence of a measurable selection: Either there is $\tilde{A} : Q\times\mathbb{R}^d \to \mathbb{R}^d$ such that $(\xi, \tilde{A}(t,x,\xi)) \in \mathcal{A}(t,x)$ for all $\xi \in \mathbb{R}^d$ and $\tilde{A}$ is measurable, or there is $\tilde{\xi} : Q \times \mathbb{R}^d \to \mathbb{R}^d$ such that $(\tilde{\xi}(t,x,A), A) \in \mathcal{A}(t,x)$ for all $A \in \mathbb{R}^d$ and $\tilde{\xi}$ is measurable.

Let us briefly refer again to the classical Leray–Lions operators. Within the setting presented above we would use the N-function $M(a) = |a|^p$ with the conjugate function $M^*(a) = |a|^{p'}$, with $1/p + 1/p' = 1$.

As we allow for x dependence, our framework covers also the variable exponent case, namely $M(a) = |a|^{p(x)}$. A further generalization is the anisotropic character and functions other than just polynomials, e.g. the following function is acceptable:

$$M(x,a) = a_1^{p_1(x)} \ln(|a|+1) + e^{a_2^{p_2(x)}} - 1 \quad \text{for } a = (a_1, a_2) \in \mathbb{R}^2.$$

All the functions having growth essentially different than polynomial (e.g. close to linear or exponential) yield additional analytical difficulties and significantly restrict good properties of corresponding function spaces (like separability, reflexivity, or density of compactly supported smooth functions). We shall now discuss this issue in more detail.

Let us recall some definitions. By a *generalized Musielak–Orlicz class* $\mathcal{L}_M(Q)$ we mean the set of all measurable functions $\xi : Q \to \mathbb{R}^d$ for which the *modular*

$$\rho_{M,Q}(\xi) = \int_Q M(x, \xi(t,x))\,dx\,dt$$

is finite. By $L_M(Q)$ we denote the *generalized Orlicz space* which is the set of all measurable functions $\xi : Q \to \mathbb{R}^d$ for which $\rho_{M,Q}(\alpha\xi) \to 0$ as $\alpha \to 0$. This is a Banach space with the norm

$$\|\xi\|_M = \sup\Big\{\int_Q \eta\cdot\xi\,dx\,dt : \eta \in L_{M^*}(Q), \int_Q M^*(x,\eta)\,dx\,dt \le 1\Big\}.$$

In the above definitions we used the notion of generalized Musielak–Orlicz spaces. In contrast to the classical Orlicz spaces we cover the case of x-dependent N-functions as well as functions that depend on the whole vector, not only on its absolute value (i.e. anisotropic functions). Moreover, by $E_M(Q)$ we denote the closure of the bounded functions in $L_M(Q)$. The space $L_{M^*}(Q)$ is the dual space of $E_M(Q)$. A sequence z^j is said to *converge modularly* to z in $L_M(Q)$, written $z^j \xrightarrow{M} z$, if there exists $\lambda > 0$ such that

$$\rho_{M,Q}\Big(\frac{z^j - z}{\lambda}\Big) \to 0.$$

The basic estimates which we will frequently use are the Hölder inequality

$$\int_Q \xi\eta\,dx\,dt \le c\|\xi\|_M\|\eta\|_{M^*} \tag{1.5}$$

and the Fenchel–Young inequality

$$|\xi\cdot\eta| \le M(x,\xi) + M^*(x,\eta). \tag{1.6}$$

The key feature of our considerations is the lack of the assumption of Δ_2-condition. We say that M satisfies the Δ_2-*condition* if there exists a constant $c > 0$ and a summable function h such that

$$M(x,2a) \le cM(x,a) + h(x) \tag{1.7}$$

for all $a \in \mathbb{R}^d$. If M satisfies (1.7) then $L_M(Q)$ is separable, and compactly supported smooth functions are dense in the strong topology. If additionally M^* satisfies (1.7) then $L_M(Q)$ is reflexive. Notice that none of these assumptions is made in the present paper. For this reason the notion of modular topology and the issue of density of compactly supported smooth functions with respect to the modular topology are crucial. The basic properties of anisotropic Musielak–Orlicz spaces mentioned above were discussed and proved in [12].

As density arguments become an essential tool, the dependence of an N-function on x becomes a significant constraint. The problem arises when we try to estimate uniformly a convolution operator. To handle this, we need some regularity with respect to the space variable. More precisely, we will assume that the function M has the following properties:

(M) there exists a constant $H > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq 1/2$ and for all $\xi \in \mathbb{R}^d$ such that $|\xi| \geq 1$,

$$\frac{M(x,\xi)}{M(y,\xi)} \leq |\xi|^{\frac{H}{\ln \frac{1}{|x-y|}}}. \tag{1.8}$$

Moreover, for every bounded measurable set G and every $z \in \mathbb{R}^d$,

$$\int_G M(x,z) < \infty. \tag{1.9}$$

Below we formulate a definition and then state an existence theorem which is the main result of the present paper. We shall use the following notation. By $\mathcal{C}_c^\infty(\Omega)$ we denote the space of infinitely differentiable compactly supported functions in Ω. For $p \leq 1 \leq \infty$ and $k \in \mathbb{N}$, we denote by $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ the Lebesgue spaces and by $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ the Sobolev spaces. $W_0^{k,p}(\Omega)$ denotes the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$, and $W^{-k,p'}(\Omega)$ with $1/p + 1/p' = 1$ denotes its dual space. Moreover, we write $\mathcal{C}_{\text{weak}}(0,T;L^2(\Omega))$ for the space of all $\varphi \in L^\infty(0,T;L^2(\Omega))$ which satisfy $(\varphi(t), v) \in \mathcal{C}([0,T])$ for all $v \in \mathcal{C}(\bar{\Omega})$.

Definition 1.1. Assume that $u_0 \in L^2(\Omega)$ and $f \in L^\infty(Q)$. We say that (u, A) is a *weak solution* to (1.1)–(1.4) if

$$u \in L^\infty(0,T;L^2(\Omega)), \qquad \nabla u \in L_M(Q), \qquad A \in L_{M^*}(Q), \tag{1.10}$$

$$u \in \mathcal{C}_{\text{weak}}(0,T;L^2(\Omega)), \tag{1.11}$$

the identity

$$\int_Q (-u\varphi_t + A \cdot \nabla\varphi)\,dx\,dt + \int_\Omega u_0(x)\varphi(0,x)\,dx = \int_Q f\varphi\,dx\,dt \tag{1.12}$$

is satisfied for all $\varphi \in \mathcal{C}_c^\infty((-\infty,T) \times \Omega)$ and

$$(\nabla u(t,x), A(t,x)) \in \mathcal{A}(t,x) \quad \text{for a.a. } (t,x) \in Q. \tag{1.13}$$

Theorem 1.2. *Let M be an N-function satisfying* (M) *and let A satisfy conditions* (A1)–(A5)*. Given $f \in L^\infty(Q)$ and $u_0 \in L^2(\Omega)$ there exists a weak solution to* (1.1)–(1.4)*.*

The current paper complements [22]. Here we also consider the existence of weak solutions to the parabolic problem including multivalued terms. However, the essential differerence consists in the properties of the N-function describing the growth of the graph $\mathcal{A}$. In [22] we concentrated on the case of a time-dependent N-function. This required a more delicate approximation theorem and excluded the possibility of anisotropic functions. The present study does not extend the results of the previous paper, but parallels them. We omit the dependence of an N-function on time, but add the possibility of anisotropic behaviour.

Anisotropic parabolic problems were also considered in [16], but in a much simpler situation of the equation and the N-function homogeneous in space. Anisotropic and space-inhomogeneous problems, in the slightly different case of systems describing the flow of non-Newtonian fluids, were considered in [14, 15, 17, 23]. Those authors assumed the Δ_2-condition on the conjugate N-function. The simplified problem of the generalized Stokes equation, without the Δ_2-condition for the conjugate N-function, was considered in [18].

Maximal monotone graphs were also applied to problems arising in fluid mechanics in [4, 11] in the L^p setting and in [3, 5] in the setting of Orlicz spaces. The latter, however, were restricted to classical Orlicz spaces satisfying the Δ_2-condition.

Most of the earlier results on existence of solutions to parabolic problems in nonstandard settings concern the case of classical Orlicz spaces: see e.g. [6] and later studies of Benkirane, Elmahi and Meskine [2, 7, 8]. All of them concern the case of an N-function depending only on $|\xi|$ without dependence on x.

The paper is organized as follows: Section 2 contains the proof of Theorem 1.2, Section 3 is devoted to density of compactly supported smooth functions with respect to modular convergence. In the appendix we include some facts, used in the main body of the paper.

2. Existence of solutions. In this section we prove Theorem 1.2. The construction of an approximate problem is in two steps. By (A5) there exists a measurable selection $\tilde{A} : Q \times \mathbb{R}^d \to \mathbb{R}^d$ of the graph $\mathcal{A}$. Obviously, each such selection defined on $\mathbb{R}^d$ is monotone and due to (A4) it satisfies

$$\tilde{A}(t,x,\xi)\cdot\xi \geq -k(x,t)+c_*(M(x,\xi)+M^*(x,\tilde{A}(t,x,\xi)) \quad \text{for all } \xi\in\mathbb{R}^d. \tag{2.1}$$

We mollify $\tilde{A}$ with a smoothing kernel and then construct a finite-dimensional problem by the Galerkin method. Indeed, let

$$\begin{aligned} &S \in \mathcal{C}_c^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} S(y)\,dy = 1, \qquad S(y) = S(-y), \\ &S_\varepsilon(y) := (1/\varepsilon^d)S(y/\varepsilon), \end{aligned} \tag{2.2}$$

with supp S in the unit ball $B(0,1) \subset \mathbb{R}^d$ and define

$$A^\varepsilon(t,x,\xi) := (\tilde{A} * S_\varepsilon)(t,x,\xi) = \int_{\mathbb{R}^d} \tilde{A}(t,x,\zeta) S_\varepsilon(\xi-\zeta)\,d\zeta. \tag{2.3}$$

Using the convexity of M and M^*, and the Jensen inequality allows us to conclude that the approximation A^ε satisfies a condition analogous to (2.1):

$$A^\varepsilon \cdot \nabla u \geq -k(t,x) + c_*(M(x,\nabla u) + M^*(x,A^\varepsilon)). \tag{2.4}$$

For the proof of an analogous estimate for approximation in the case of polynomial conditions see [11] and [19].

The assumption (A5) gives either the existence of a selection $\tilde{A}$ as above, or the existence of a selection $\tilde{\xi} : Q \times \mathbb{R}^d \to \mathbb{R}^d$ such that $(\tilde{\xi}(t,x,A), A)$ is in the graph $\mathcal{A}$ for all $A \in \mathbb{R}^d$. In the second case we would define

$$\xi^\varepsilon(A) := (\tilde{\xi} * S_\varepsilon)(t,x,A) + \varepsilon A. \tag{2.5}$$

Then the function $A \mapsto \xi^\varepsilon(A)$ is invertible. Note that since $\varepsilon A \cdot A \geq 0$ one can show that for the pair $(\xi^\varepsilon(A), A)$ an analogue of (2.4) holds, and consequently also for $(\xi, (\xi^\varepsilon)^{-1}(\xi))$. Thus we may define

$$A^\varepsilon := (\tilde{\xi} * S_\varepsilon + \varepsilon\,\mathrm{Id})^{-1}. \tag{2.6}$$

One then proceeds analogously to the previous situation. In the following, we present the proof for the case when there exists a selection $\tilde{\xi}$, and A^ε is given by (2.3).

Consider now the basis $\{\omega_i\}$ consisting of eigenvectors of the Laplace operator with the Dirichlet boundary condition, and let $u^{\varepsilon,n}$ be the solution to the finite-dimensional problem with the function A^ε, namely $u^{\varepsilon,n}(t,x) := \sum_{i=1}^n c_i^{\varepsilon,n}(t)\omega_i(x)$ which solves the system

$$\begin{aligned} &(u_t^{\varepsilon,n},\omega_i) + (A^\varepsilon(t,x,\nabla u^{\varepsilon,n}),\nabla\omega_i) = \langle f,\omega_i\rangle, \qquad i = 1,\dots,n,\\ &u^{\varepsilon,n}(0) = P^n u_0, \end{aligned} \tag{2.7}$$

where P^n is the orthogonal projection of $L^2(\Omega)$ on span $\{\omega_1,\dots,\omega_n\}$. Let $Q^s := (0,s) \times \Omega$ with $0 < s < T$. From (2.4) we conclude that

$$\begin{aligned} \sup_{s\in(0,T)} \|u^{\varepsilon,n}(s)\|^2_{L^2(\Omega)} + c_* \int_Q [M(x,\nabla u^{\varepsilon,n}) + M^*(x,A^\varepsilon(t,x,\nabla u^{\varepsilon,n}))]\,dx\,dt \\ \leq c\Big(\|u_0\|^2_{L^2(\Omega)} + \|f\|_{L^\infty(Q)} + \int_Q k\,dx\,dt\Big). \end{aligned} \tag{2.8}$$

As a consequence of (2.8) there exists a subsequence (not relabelled) such that

$$\begin{aligned} \nabla u^{\varepsilon,n} &\overset{*}{\rightharpoonup} \nabla u^n \qquad \text{weakly* in } L_M(Q),\\ A^\varepsilon(\cdot,\cdot,\nabla u^{\varepsilon,n}) &\overset{*}{\rightharpoonup} A^n \qquad \text{weakly* in } L_{M^*}(Q). \end{aligned} \tag{2.9}$$

Moreover, from (2.7) we deduce the boundedness of the sequence $u_t^{\varepsilon,n}$ in $L_{M^*}(Q)$ and hence up to a subsequence we have

$$u_t^{\varepsilon,n} \overset{*}{\rightharpoonup} u_t^n \quad \text{weakly}^* \text{ in } L_{M^*}(Q). \tag{2.10}$$

Further we observe that (2.7) implies that $\frac{d}{dt}c_i^{\varepsilon,n}(t)$ is bounded in the space $L_{M^*}([0,T])$, which implies uniform integrability in $L^1([0,T])$. Consequently, there exists a monotone, continuous $L : \mathbb{R}_+ \to \mathbb{R}_+$ with $L(0) = 0$ such that for all $s_1, s_2 \in (0,T)$,

$$\left| \int_{s_1}^{s_2} \frac{d}{dt} c_i^{\varepsilon,n}(t)\,dt \right| \le L(|s_1 - s_2|),$$

and thus the sequence $c_i^{\varepsilon,n}$ is uniformly equicontinuous,

$$|c_i^{\varepsilon,n}(s_1) - c_i^{\varepsilon,n}(s_2)| \le L(|s_1 - s_2|).$$

From (2.8) we infer that $c_i^{\varepsilon,n}(t)$ is bounded in $L^\infty([0,T])$ and hence by the Arzelà–Ascoli theorem there exists a uniformly convergent subsequence $\{c_i^{\varepsilon_k,n}\}$ in $\mathcal{C}([0,T])$; taking into account the regularity of the basis $\{\omega_i\}_{i=1}^n$ we conclude that

$$u^{\varepsilon,n} \to u^n \quad \text{strongly in } \mathcal{C}([0,T];\mathcal{C}^1(\overline{\Omega})). \tag{2.11}$$

The limit passage $\varepsilon \to 0$ is done at the level of finite-dimensional problem. It follows the lines of [5], but we recall the main steps. Using (2.9)–(2.11) we obtain the limit problem

$$\begin{aligned} &(u_t^n, \omega_i) + (A^n, \nabla\omega_i) = \langle f, \omega_i\rangle, \qquad i = 1,\dots,n,\\ &u^n(0) = P^n u_0. \end{aligned} \tag{2.12}$$

To complete the limit passage we need to prove that

$$(\nabla u^n, A^n) \in \mathcal{A}. \tag{2.13}$$

Following [5] and [22], with simple algebraic tricks and estimates which are not included in the present paper, we conclude that for all $B \in \mathbb{R}^d$ and for a.a. $(t,x) \in Q$,

$$(A^n - \tilde{A}(t,x,B)) \cdot (\nabla u^n - B) \ge 0. \tag{2.14}$$

Hence, using the equivalence of (i) and (ii) in Lemma A.8, we arrive at (2.13).

Before passing to the limit as $n \to \infty$ we notice that in the same manner as before we obtain estimates uniform with respect to n:

$$\begin{aligned} \sup_{s\in(0,T)} \|u^n(s)\|_{L^2(\Omega)}^2 &+ \int_Q [M(x,\nabla u^n) + M^*(x,A^n)]\,dx\,dt \\ &\le c(\|u_0\|_{L^2(\Omega)}^2 + \|f\|_{L^\infty(Q)} + \|k\|_{L^1(Q)}). \end{aligned} \tag{2.15}$$

Consequently, there exists a subsequence, labelled the same, such that

$$\begin{aligned}
\nabla u^n &\overset{*}{\rightharpoonup} \nabla u && \text{weakly}^* \text{ in } L_M(Q),\\
u^n &\rightharpoonup u && \text{weakly in } L^1(0,T;W^{1,1}(\Omega)),\\
A^n &\overset{*}{\rightharpoonup} A && \text{weakly}^* \text{ in } L_{M^*}(Q),\\
u^n &\overset{*}{\rightharpoonup} u && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),\\
u^n_t &\overset{*}{\rightharpoonup} u_t && \text{weakly}^* \text{ in } W^{-1,\infty}(0,T;L^2(\Omega)).
\end{aligned}\tag{2.16}$$

Using (2.16) we let $n \to \infty$ and deduce from (2.12) that

$$u_t - \operatorname{div} A = f \tag{2.17}$$

in the distributional sense. Again, to complete the limit passage, we need to show that $(\nabla u, A) \in \mathcal{A}(t,x)$. It, however, requires more care, in contrast to the previous passage at fixed n. The essence of the present step is to use the maximal monotonicity of the graph $\mathcal{A}$, in particular the property in Lemma A.7. As the assumptions (A.1)–(A.3) are obviously satisfied, we focus on (A.4). We need to establish a strong energy inequality. Since testing (2.17) with a solution is not possible, we first approximate it with respect to the space variable. By Theorem 3.1 there exists $v^j \in L^\infty(0,T;\mathcal{C}_c^\infty(\Omega))$ such that

$$\nabla v^j \xrightarrow{M} \nabla u \text{ modularly in } L_M(Q) \text{ and } v^j \to u \text{ strongly in } L^2(Q). \tag{2.18}$$

Hence we shall test with a function of the form

$$u^{j,\epsilon} = K^\epsilon * (K^\epsilon * v^j \mathbb{1}_{(s_0,s)}) \tag{2.19}$$

with $K \in \mathcal{C}_c^\infty(\mathbb{R})$, $K(\tau) = K(-\tau)$, $\int_{\mathbb{R}} K(\tau)\,d\tau = 1$ and defining $K^\epsilon(t) = (1/\epsilon)K(t/\epsilon)$, $\epsilon < \min\{s_0, T-s\}$. Thus

$$\int_{s_0}^{s}\int_{\Omega} (u * K^\epsilon)\cdot \partial_t(v^j * K^\epsilon)\,dx\,dt = \int_Q A\cdot \nabla u^{j,\epsilon}\,dx\,dt - \int_Q f u^{j,\epsilon}\,dx\,dt. \tag{2.20}$$

By (2.18) we can easily let $j \to \infty$. Indeed, the left-hand side of (2.20) can be handled since it is possible to rewrite it as $\int_Q ((\partial_t K^\epsilon) * K^\epsilon * u) v^j\,dx\,dt$ and hence the limit passage is obvious. Note that for all $0 < s_0 < s < T$,

$$\begin{aligned}
\int_{s_0}^{s}\int_{\Omega} (K^\epsilon * u)\cdot \partial_t(K^\epsilon * u)\,dx\,dt &= \int_{s_0}^{s} \frac{1}{2}\frac{d}{dt}\|K^\epsilon * u\|^2_{L^2(\Omega)}\,dt\\
&= \tfrac{1}{2}\|K^\epsilon * u(s)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|K^\epsilon * u(s_0)\|^2_{L^2(\Omega)}.
\end{aligned}\tag{2.21}$$

Letting $\epsilon \to 0$ yields for almost all s_0, s, namely for all Lebesgue points of the function $u(t)$, the identity

$$\lim_{\epsilon\to 0}\int_{s_0}^{s}\int_{\Omega} (u * K^\epsilon)\cdot \partial_t(u * K^\epsilon) = \tfrac{1}{2}\|u(s)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u(s_0)\|^2_{L^2(\Omega)}. \tag{2.22}$$

Consider now the term

$$\int_0^T\int_\Omega A\cdot(K^\epsilon*((K^\epsilon*\nabla u)\,\mathbb{1}_{(s_0,s)}))\,dx\,dt=\int_{s_0}^s\int_\Omega(K^\epsilon*A)\cdot(K^\epsilon*\nabla u)\,dx\,dt.$$

Both the sequences $\{K^\epsilon*A\}$ and $\{K^\epsilon*\nabla u\}$ converge in measure in Q by Proposition A.5. Moreover

$$\int_Q(M(x,\nabla u)+M^*(x,A))\,dx\,dt<\infty.$$

Hence by Proposition A.6 the sequences $\{M^*(x,K^\epsilon*A)\}$ and $\{M(x,K^\epsilon*\nabla u)\}$ are uniformly integrable and with the help of Lemma A.2 we have

$$K^\epsilon*\nabla u\xrightarrow{M}\nabla u\quad\text{modularly in } L_M(Q),$$
$$K^\epsilon*A\xrightarrow{M^*}A\quad\text{modularly in } L_{M^*}(Q).$$

Proposition A.4 allows us to conclude

$$\lim_{\epsilon\to0}\int_{s_0}^s\int_\Omega(K^\epsilon*A)\cdot(K^\epsilon*\nabla u)\,dx\,dt=\int_{s_0}^s\int_\Omega A\cdot\nabla u\,dx\,dt. \tag{2.23}$$

Letting $\epsilon\to0_+$ on the right-hand side is obvious. Hence we obtain

$$\tfrac12\|u(s)\|_2^2-\tfrac12\|u(s_0)\|_2^2+\int_{Q^s}A\cdot\nabla u\,dx\,dt=\int_{Q_s}fu\,dx\,dt \tag{2.24}$$

for almost all $0<s_0<s<T$. For further considerations we need to know whether this holds for $s_0=0$, hence let $s_0\to0$. Thus, we need to establish (1.11).

We shall observe that using the approximate equation we can estimate the sequence $\{du^n/dt\}$ uniformly (with respect to n) in the space $L^1(0,T;W^{-r,2}(\Omega))$, where $r>d/2+1$. Consider $\varphi\in L^\infty(0,T;W_0^{r,2}(\Omega))$, $\|\varphi\|_{L^\infty(0,T;W_0^{r,2})}\le1$ and observe that

$$\left\langle\frac{du^n}{dt},\varphi\right\rangle=\left\langle\frac{du^n}{dt},P^n\varphi\right\rangle=-\int_\Omega A^n\cdot\nabla(P^n\varphi)\,dx+\int_\Omega f\cdot P^n\varphi\,dx.$$

Since the orthogonal projection is continuous in $W_0^{r,2}(\Omega)$ and $W^{r-1,2}(\Omega)\subset L^\infty(\Omega)$, we estimate as follows:

$$\begin{aligned}\left|\int_0^T\int_\Omega A^n\cdot\nabla(P^n\varphi)\,dx\,dt\right|&\le\int_0^T\|A^n\|_{L^1(\Omega)}\|\nabla(P^n\varphi)\|_{L^\infty(\Omega)}\,dt\\&\le c\int_0^T\|A^n\|_{L^1(\Omega)}\|P^n\varphi\|_{W_0^{r,2}}\,dt\le c\|A^n\|_{L^1(Q)}\|\varphi\|_{L^\infty(0,T;W_0^{r,2})}.\end{aligned} \tag{2.25}$$

From (2.15) and Lemma A.3 we conclude that there exists a monotone, continuous function $L:\mathbb{R}_+\to\mathbb{R}_+$, with $L(0)=0$, independent of n, such that

$$\int_{s_1}^{s_2}\|A^n\|_{L^1(\Omega)}\le L(|s_1-s_2|)$$

for all $s_1,s_2\in[0,T]$. Consequently, (2.25) gives

$$\left|\int_{s_1}^{s_2}\left\langle\frac{du^n}{dt},\varphi\right\rangle dt\right|\le L(|s_1-s_2|)$$

for all φ with $\operatorname{supp}\varphi\subset(s_1,s_2)\subset[0,T]$ and $\|\varphi\|_{L^\infty(0,T;W_0^{r,2})}\le 1$. Since

$$\|u^n(s_1)-u^n(s_2)\|_{W^{-r,2}}=\sup_{\|\psi\|_{W_0^{r,2}}\le 1}\left|\left\langle\int_{s_1}^{s_2}\frac{du^n(t)}{dt},\psi\right\rangle\right| \tag{2.26}$$

we have

$$\sup_{n\in\mathbb{N}}\|u^n(s_1)-u^n(s_2)\|_{W^{-r,2}}\le L(|s_1-s_2|), \tag{2.27}$$

so the family of functions $u^n:[0,T]\to W^{-r,2}(\Omega)$ is equicontinuous. Moreover, it is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and hence $\{u^n\}$ is relatively compact in $\mathcal{C}([0,T];W^{-r,2}(\Omega))$ with limit $u\in\mathcal{C}([0,T];W^{-r,2}(\Omega))$. Thus there exists a sequence $\{s_0^i\}_i$ with $s_0^i\to 0^+$ as $i\to\infty$ such that

$$u(s_0^i)\xrightarrow{i\to\infty}u(0)\quad\text{in } W^{-r,2}(\Omega). \tag{2.28}$$

The limit above coincides with the weak limit of $\{u(s_0^i)\}$ in $L^2(\Omega)$, which yields

$$\liminf_{i\to\infty}\|u(s_0)\|_{L^2(\Omega)}\ge\|u_0\|_{L^2(\Omega)}. \tag{2.29}$$

We infer from (2.12) that for any Lebesgue point s of u,

$$\begin{aligned}\limsup_{n\to\infty}\int_{Q_s}A^n\cdot\nabla u^n\,dx\,dt &= \tfrac12\|u_0\|_2^2-\liminf_{k\to\infty}\tfrac12\|u^n(s)\|_2^2+\lim_{n\to\infty}\int_{Q_s}fu^n\,dx\,dt\\ &\le \tfrac12\|u_0\|_2^2-\tfrac12\|u(s)\|_2^2+\int_{Q_s}fu\,dx\,dt\\ &\le \liminf_{i\to\infty}\big(\tfrac12\|u(s_0^i)\|_2^2-\tfrac12\|u(s)\|_2^2\big)+\int_{Q_s}fu\,dx\,dt\\ &=\lim_{i\to\infty}\int_{s_0^i}^{s}\int_\Omega A\cdot\nabla u\,dx\,dt=\int_0^s\int_\Omega A\cdot\nabla u\,dx\,dt,\end{aligned} \tag{2.30}$$

which proves (A.4), and Lemma A.7 allows us to complete the proof.

3. Approximation. In this section we focus on the density of compactly supported smooth functions with respect to the modular topology. Fundamental studies in this direction are due to Gossez for classical Orlicz spaces and elliptic equations [9, 10]. Similar considerations for isotropic x-dependent N-functions are due to Benkirane et al. [1]; see also [13] for the anisotropic case with an application to elliptic problems. Note that the main idea of the current approximation is analogous to [13]. However, Gwiazda et al. approximate the truncated functions which are appropriate test functions in the elliptic equation considered. This is not the case for parabolic problems. Hence the approximation theorem below is under weaker assumptions and the dependence on time is taken into account. Since this result is essential for proving existence of weak solutions, we include the details for completeness.

THEOREM 3.1. *If* $u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ *and if* $\nabla u \in L_M(Q)$ *then there exists a sequence* $v^j \in L^\infty(0,T;\mathcal{C}_c^\infty(\Omega))$ *satisfying*

$$\nabla v^j \xrightarrow{M} \nabla u \text{ modularly in } L_M(Q) \text{ and } v^j \to u \text{ strongly in } L^2(Q). \tag{3.1}$$

Proof. For the Lipschitz domain Ω there exists a finite family $\{\Omega_i\}$ of star-shaped Lipschitz domains such that

$$\Omega = \bigcup_{i\in J} \Omega_i$$

(cf. [21]). We introduce a partition of unity θ_i with $0 \le \theta_i \le 1$, $\theta_i \in \mathcal{C}_c^\infty(\Omega_i)$, $\operatorname{supp}\theta_i = \Omega_i$, $\sum_{i\in J}\theta_i(x) = 1$ for $x \in \Omega$ and define the truncation operator $T_\ell(u)$ as follows:

$$T_\ell(u) = \begin{cases} u & \text{if } |u| \le \ell, \\ \ell & \text{if } u > \ell, \\ -\ell & \text{if } u < -\ell. \end{cases} \tag{3.2}$$

Define $Q_i := (0,T) \times \Omega_i$. Obviously

$$T_\ell(u) \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega)), \qquad \nabla T_\ell u \in L_M(Q)$$

and for each $i \in J$,

$$\theta_i \cdot T_\ell(u) \in L^\infty(Q_i) \cap L^1(0,T;W_0^{1,1}(\Omega_i)) \cap L^\infty(0,T;L^2(\Omega_i)).$$

Introducing the truncation of u was necessary to have

$$\nabla T_\ell(u)\cdot\theta_i + T_\ell(u)\cdot\nabla\theta_i = \nabla(T_\ell(u)\cdot\theta_i) \in L_M(Q_i).$$

Without loss of generality assume that all Ω_i are star-shaped domains with respect to the ball $B(0,R)$. We define, for $(t,x) \in (0,T)\times\Omega$,

$$\mathcal{S}_\delta(\theta_i T_\ell(u))(t,x) := \frac{1}{1-\delta/R}\int_Q S_\delta(x-y)\theta_i T_\ell(u)(t,(1-\delta/R)y)\,dy. \tag{3.3}$$

Our aim is to show that there exists a constant $\lambda > 0$ such that

$$\lim_{l\to\infty}\lim_{\delta\to 0_+} \varrho_{M,Q_i}\left(\frac{\nabla u - \nabla\mathcal{S}_\delta(\theta_i T_\ell(u))}{\lambda}\right) = 0. \tag{3.4}$$

For this purpose we introduce a sequence of simple functions

$$\xi_n(t,x) := \sum_{j=1}^{n} \alpha_j^n \mathbb{1}_{G_j}(t,x), \qquad \alpha_j^n \in \mathbb{R}, \bigcup_{j=1}^{n} G_j = Q,$$

which converges to $\nabla(\theta_i \cdot T_\ell(u))$ modularly in $L_M(Q)$. Moreover, let $\lambda_0, \lambda_1, \lambda_2$ be some constants which we specify later such that

$$\begin{aligned}\varrho_{M,Q_i}&\left(\frac{\nabla u - \nabla\mathcal{S}_\delta(\theta_i T_\ell(u))}{\lambda}\right) \\ &\le \frac{\lambda_0}{\lambda}\rho_{M,Q_i}\left(\frac{\mathcal{S}_\delta\nabla(\theta_i T_\ell(u)) - \mathcal{S}_\delta\xi_n}{\lambda_0}\right) + \frac{\lambda_0}{\lambda}\rho_{M,Q_i}\left(\frac{\nabla(\theta_i T_\ell(u)) - \xi_n}{\lambda_0}\right) \\ &\quad + \frac{\lambda_1}{\lambda}\rho_{M,Q_i}\left(\frac{\mathcal{S}_\delta\xi_n - \xi_n}{\lambda_1}\right) + \frac{\lambda_2}{\lambda}\varrho_{M,Q_i}\left(\frac{\nabla u - \nabla(T_\ell(u)\theta_i)}{\lambda_2}\right) \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.5}$$

Consider first I_3. The existence of a sequence ξ_n is provided by Lemma A.1. Let $B_\delta := \{y \in \Omega : |y| < \delta\}$. Then

$$\mathcal{S}_\delta\xi_n - \xi_n = \int_{B_\delta} S_\delta(y) \sum_{j=1}^{n}\big(\alpha_j^n \mathbb{1}_{G_j}(t,(1-\delta/R)(x-y)) - \alpha_j^n \mathbb{1}_{G_j}(t,x)\big)\,dy \tag{3.6}$$

and the Jensen inequality and Fubini theorem yield

$$\begin{aligned}\rho_{M,Q_i}&\left(\frac{\mathcal{S}_\delta\xi_n(t,x) - \xi_n}{\lambda_1}\right) \\ &= \int_Q M\bigg(x, \frac{1}{\lambda_1}\int_{B_1} S(y)\sum_{j=1}^{n}\big(\alpha_j^n \mathbb{1}_{G_j}(t,(1-\delta/R)(x-\delta y)) - \alpha_j^n \mathbb{1}_{G_j}(t,x)\big)\,dy\bigg)\,dt\,dx \\ &\le \int_{B_1} S(y)\bigg(\int_Q M\bigg(x, \frac{1}{\lambda_1}\sum_{j=1}^{n}\alpha_j^n\big(\mathbb{1}_{G_j}(t,(1-\delta/R)(x-\delta y)) - \mathbb{1}_{G_j}(t,x)\big)\bigg)\,dt\,dx\bigg)\,dy.\end{aligned} \tag{3.7}$$

Note that $\big\{\frac{1}{\lambda_1}\sum_{j=1}^{n}\alpha_j^n\big(\mathbb{1}_{G_j}(t,(1-\delta/R)(x-\delta y)) - \mathbb{1}_{G_j}(t,x)\big)\,dt\,dx\big)\big\}_{\delta>0}$ converges a.e. in Q to zero as $\delta \to 0_+$ and

$$\begin{aligned}M\bigg(x, \frac{1}{\lambda_1}\sum_{j=1}^{n}\alpha_j^n\big(\mathbb{1}_{G_j}(t,(1-\delta/R)(x-\delta y)) - \mathbb{1}_{G_j}(t,x)\big)\bigg) \\ \le \sup_{|z|=1} M\bigg(x, \frac{1}{\lambda_1}\sum_{j=1}^{n}\alpha_j^n z\bigg).\end{aligned} \tag{3.8}$$

Assumption (1.9) implies that the right-hand side of (3.8) is integrable, hence the Lebesgue dominated convergence theorem shows that I_3 vanishes as $\delta \to 0_+$. Lemma 3.2 allows us to estimate I_1 on each Ω_i as follows:

$$(3.9) \qquad I_1 = \frac{\lambda_0}{\lambda}\rho_{M,Q_i}\bigg(\frac{\mathcal{S}_\delta(\nabla(\theta_i T_\ell(u)) - \xi_n)}{\lambda_0}\bigg) \le c\rho_{M,Q_i}\bigg(\frac{\nabla(\theta_i T_\ell(u)) - \xi_n}{\lambda_0}\bigg)$$

and hence by Lemma A.1 there exists a constant λ_0 such that

$$\lim_{n\to\infty}(I_1 + I_2) = 0.$$

Moreover, as $\ell \to \infty$ we observe that

$$T_\ell(u) \to u \quad \text{strongly in } L^1(0,T;W_0^{1,1}(\Omega))$$

and hence also, at least for a subsequence, almost everywhere. To find a uniform estimate we observe that $M(x,\nabla T_\ell(u(t,x))) \le M(x,\nabla u(t,x))$ a.e. in Q. Indeed, $T_\ell(u)$ and u coincide for $|u| \le \ell$ and on the remaining two sets, where $T_\ell(u)$ is equal to ℓ or $-\ell$, we have $T_\ell(u) \in L^1(0,T;W_0^{1,1}(\Omega))$, then $\nabla T_\ell(u)$ is almost everywhere equal to zero. Consequently, $M(x,\nabla T_\ell(u(t,x)))$ is uniformly integrable, which combined with pointwise convergence yields

$$\nabla T_\ell(u) \to \nabla u \quad \text{modularly in } L_M(Q)$$

as $\ell \to \infty$, hence there exists a constant λ_2 such that $\lim_{\ell\to\infty} I_4 = 0$. Finally, choosing $\lambda > \max\{3\lambda_0, 3\lambda_1, 3\lambda_2\}$, letting first $\delta \to 0_+$, then $n \to \infty$ and $\ell \to \infty$ we arrive at (3.4).

Strong convergence in L^2 is straightforward, since an N-function $M(x,a) = |a|^2$ satisfies the Δ_2-condition, and strong and modular convergence coincide. ∎

LEMMA 3.2. *Let M be an N-function satisfying condition* (M), *let S and S_δ be given by* (2.2), *and assume that Ω is a star-shaped domain with respect to a ball $B(0,R)$ for some $R > 0$. Define a family of operators by*

$$(3.10) \qquad \mathcal{S}_\delta z(t,x) := (1-\delta/R)^{-1}\int_\Omega S_\delta(x-y)z(t,(1-\delta/R)y)\,dy.$$

Then there exists a constant $c > 0$ (independent of δ) such that

$$(3.11) \qquad \int_Q M(x,\mathcal{S}_\delta z(t,x))\,dx\,dt \le c\int_Q M(x,z(t,x))\,dx\,dt$$

for every $z \in L_M(Q) \cap L^\infty(0,T;L^1(\Omega))$.

Proof. Since Ω is a star-shaped domain with respect to $B(0,R)$, for each $\lambda \in (0,1)$ we have

$$(1-\lambda)x + \lambda y \in \Omega \quad \text{for each } x \in \Omega,\ y \in B(0,R).$$

Hence for $\delta < R$ we may choose $\lambda = \delta/R$ and conclude that

$$(1-\delta/R)\Omega + \delta B(0,1) \subset \Omega.$$

Let $\mathcal{S}_\delta z(t,x)$ be defined by (3.10). Since $\overline{(1-\delta/R)\Omega+\delta B(0,1)}\subset\Omega$, we have $\mathcal{S}_\delta z\in L^\infty(0,T;\mathcal{C}_c^\infty(\Omega))$. For every $\delta>0$ there exists $N=N(\delta)$ such that a family $\{D_{\delta,k}\}_{k=1}^N$ of closed cubes with disjoint interiors and edge length δ covers Ω, i.e. $\Omega\subset\bigcup_{k=1}^N D_{\delta,k}$. Hence

$$\int_0^T\int_\Omega M(x,\mathcal{S}_\delta z(t,x))\,dx=\sum_{k=1}^N\int_0^T\int_{D_{\delta,k}\cap\Omega}M(x,\mathcal{S}_\delta z(t,x))\,dx\,dt. \tag{3.12}$$

For each δ,k we denote by $G_{\delta,k}$ a cube with edge length 2δ and concentric with $D_{\delta,k}$. Note that if $x\in D_{\delta,k}$, then there exist 2^d cubes $G_{\delta,k}$ such that $x\in G_{\delta,k}$. Define

$$m_k^\delta(\xi):=\inf_{(t,x)\in((0,T)\times G_{\delta,k})\cap Q}M(x,\xi)\le\inf_{(t,x)\in((0,T)\times D_{\delta,k})\cap Q}M(x,\xi) \tag{3.13}$$

and

$$\alpha_k(t,x,\delta):=\frac{M(x,\mathcal{S}_\delta z(t,x))}{m_k^\delta(\mathcal{S}_\delta z(t,x))}. \tag{3.14}$$

Then

$$\int_0^T\int_\Omega M(x,\mathcal{S}_\delta z(t,x))\,dx\,dt=\sum_{k=1}^N\int_0^T\int_{D_{\delta,k}\cap\Omega}\alpha_k(t,x,\delta)m_k^\delta(\mathcal{S}_\delta z(t,x))\,dx\,dt. \tag{3.15}$$

We aim to estimate $\alpha_k(t,x,\delta)$, and the main tool will be (1.8), the regularity of M with respect to x. Let (t_k,x_k) be the point where the infimum of $M(x,\xi)$ in $(0,T)\times G_{\delta,k}$ is attained. Then

$$\alpha_k(t,x,\delta)=\frac{M(x,\mathcal{S}_\delta z(t,x))}{M(x_k,\mathcal{S}_\delta z(t,x))}\le|\mathcal{S}_\delta z(t,x)|^{\frac{H}{\ln\frac{1}{|x-x_k|}}}. \tag{3.16}$$

Without loss of generality one can assume that $\|z\|_{L^\infty(0,T;L^1(\Omega))}\le1$. By Hölder's inequality (1.5) we obtain, for $\delta<R$,

(3.17)

$$\begin{aligned}|\mathcal{S}_\delta z(t,x)|&\le\left|\frac{1}{\delta^d}\left(1-\frac{\delta}{R}\right)^{-1}\sup_{B(0,1)}|S(y)|\int_\Omega\mathbb{1}_{B(0,\delta)}(y)z\left(t,\left(1-\frac{\delta}{R}\right)y\right)dy\right|\\&\le\frac{2}{\delta^d}\sup_{B(0,1)}|S(y)|\,\|z\|_{L^\infty(0,T;L^1(\Omega))}\le\frac{c}{\delta^d}.\end{aligned}$$

Since $x\in D_{\delta,k}$ and $x_k\in G_{\delta,k}$ we have $|x-x_k|\le\delta\sqrt{d}$ and for sufficiently small δ, e.g. $\delta<1/(2\sqrt{d})$ with the use of (3.17) we obtain

$$\begin{aligned}|\mathcal{S}_\delta z(t,x)|^{\frac{H}{\ln\frac{1}{\delta\sqrt{d}}}}&\le(c\delta^{-d})^{\frac{H}{\ln\frac{1}{\delta\sqrt{d}}}}\le c^{\frac{H}{\ln 2}}\cdot d^{\frac{dH}{\ln 4}}(e^{\ln\delta\sqrt{d}})^{\frac{dH}{\ln\delta\sqrt{d}}}\\&\le d^{\frac{dH}{\ln 4}}c^{\frac{H}{\ln 2}}e^{dH}=:C.\end{aligned} \tag{3.18}$$

Consequently,

$$|\alpha_k(t,x,\delta)| \le C. \tag{3.19}$$

Define $\tilde{M}(x,\xi) := \max_k m_k^\delta(\xi)$ where the maximum is taken with respect to all the sets $(0,T)\times G_{\delta,k}$. Obviously, $\tilde{M}(x,\xi) \le M(x,\xi)$ for all $(t,x)\in Q$. Using the uniform estimate (3.19) and the Jensen inequality we have

$$\begin{aligned}\int_Q M(x,\mathcal{S}_\delta z(t,x))\,dx\,dy &\le C\sum_{k=1}^N \int_0^T \int_{D_{\delta,k}} m_k^\delta(\mathcal{S}_\delta z(t,x))\,dx\,dt \\ &\le C\sum_{k=1}^N \int_{B(0,\delta)} |S_\delta(y)|\,dy \int_0^T \int_{(1-\delta/R)G_{\delta,k}} m_k^\delta(z(t,x))\,dx\,dt \\ &\le 2^d C\int_Q \tilde{M}(x,z(t,x))\,dx\,dt \le 2^d C\int_Q M(x,z(t,x))\,dx\,dt,\end{aligned} \tag{3.20}$$

which completes the proof. ∎

Appendix A. Auxiliary facts

LEMMA A.1. *Let $\mathbb{S}$ be the set of all simple, integrable functions on Q, and let* (1.9) *hold. Then $\mathbb{S}$ is dense with respect to the modular topology in $L_M(Q)$.*

For the proof in the isotropic case see [20, Theorem 7.6]. The anisotropic case follows exactly the same lines.

Below we formulate some facts concerning convergence in generalized Musielak–Orlicz spaces. For the proofs see [14].

LEMMA A.2. *Let $z^j : Q \to \mathbb{R}^d$ be a measurable sequence. Then $z^j \xrightarrow{M} z$ in $L_M(Q)$ modularly if and only if $z^j \to z$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(x,\lambda z^j)\}$ is uniformly integrable in $L^1(Q)$, i.e.,*

$$\lim_{R\to\infty}\Big(\sup_{j\in\mathbb{N}} \int_{\{(t,x):\,|M(x,\lambda z^j)|\ge R\}} M(x,\lambda z^j)\,dx\,dt\Big) = 0.$$

LEMMA A.3. *Let M be an N-function with $\int_Q M(x,z^j)\,dx\,dt \le c$ for all $j\in\mathbb{N}$. Then the sequence $\{z^j\}$ is uniformly integrable in $L^1(Q)$.*

PROPOSITION A.4. *Let M be an N-function and M^* its complementary function. Suppose that the sequences $\psi^j : Q \to \mathbb{R}^d$ and $\phi^j : Q \to \mathbb{R}^d$ are uniformly bounded in $L_M(Q)$ and $L_{M^*}(Q)$ respectively. Moreover, $\psi^j \xrightarrow{M} \psi$ modularly in $L_M(Q)$ and $\phi^j \xrightarrow{M^*} \phi$ modularly in $L_{M^*}(Q)$. Then $\psi^j\cdot\phi^j \to \psi\cdot\phi$ strongly in $L^1(Q)$.*

Proposition A.5. *Let K^j be a standard mollifier, i.e., $K \in \mathcal{C}^\infty(\mathbb{R})$, K has a compact support and $\int_{\mathbb{R}} K(\tau)\,d\tau = 1$, $K(t) = K(-t)$. Define $K^j(t) = jK(jt)$. Moreover, let $*$ denote convolution in the variable t. Then for any function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in L^1(Q)$,*

$$(\varrho^j * \psi)(t,x) \to \psi(t,x) \quad \textit{in measure.}$$

Proposition A.6. *Let K^j be defined as in Proposition* A.5. *Given an N-function M and a function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in \mathcal{L}_M(Q)$, the sequence $\{M(\varrho^j * \psi)\}$ is uniformly integrable.*

The next lemma is the main tool for showing that the limits of approximate sequences are in the graph $\mathcal{A}$ provided that the graph is maximal monotone. This lemma was formulated in [3] (see also [22]).

Lemma A.7. *Let $\mathcal{A}$ be a maximal monotone M-graph. Assume that there are sequences $\{A^n\}_{n=1}^\infty$ and $\{\nabla u^n\}_{n=1}^\infty$ defined on Q such that:*

$$(\nabla u^n(t,x), A^n(t,x)) \in \mathcal{A}(t,x) \quad \textit{a.e. in } Q, \tag{A.1}$$

$$\nabla u^n \overset{*}{\rightharpoonup} \nabla u \quad \textit{weakly}^* \textit{ in } L_M(Q), \tag{A.2}$$

$$A^n \overset{*}{\rightharpoonup} A \quad \textit{weakly}^* \textit{ in } L_{M^*}(Q), \tag{A.3}$$

$$\limsup_{n\to\infty} \int_Q A^n \cdot \nabla u^n \,dx\,dt \le \int_Q A \cdot \nabla u \,dx\,dt. \tag{A.4}$$

Then

$$(\nabla u(t,x), A(t,x)) \in \mathcal{A}(t,x) \quad \textit{a.e. in } Q.$$

Finally we summarize some properties of selections.

Lemma A.8. *Let $\mathcal{A}(t,x)$ be a maximal monotone M-graph satisfying* (A1)–(A5) *with a measurable selection $\tilde{A} : Q \times \mathbb{R}^d \to \mathbb{R}^d$. Then:*

(a1) $\operatorname{Dom} \tilde{A}(t,x,\cdot) = \mathbb{R}^d$ *a.e. in Q;*

(a2) *$\tilde{A}$ is monotone, i.e. for every $\xi_1, \xi_2 \in \mathbb{R}^d$ and a.a. $(t,x) \in Q$,*

$$(\tilde{A}(t,x,\xi_1) - \tilde{A}(t,x,\xi_2)) \cdot (\xi_1 - \xi_2) \ge 0; \tag{A.5}$$

(a3) *there are nonnegative $k \in L^1(Q)$, $c_* > 0$ and an N-function M such that for all $\nabla u \in \mathbb{R}^d$,*

$$\tilde{A} \cdot \nabla u \ge -k(t,x) + c_*(M(x,\nabla u) + M^*(x,\tilde{A})). \tag{A.6}$$

Moreover, let U be a dense set in $\mathbb{R}^d$ and $(B, \tilde{A}(t,x,B)) \in \mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$ and for all $B \in U$. Let also $(\nabla u, A) \in \mathbb{R}^d \times \mathbb{R}^d$. Then the following conditions are equivalent:

$$\begin{aligned}&\text{(i) } (A - \tilde{A}(t,x,B)) \cdot (\nabla u - B) \ge 0 \textit{ for all } (B, \tilde{A}(t,x,B)) \in \mathcal{A}(t,x),\\ &\text{(ii) } (\nabla u, A) \in \mathcal{A}(t,x).\end{aligned} \tag{A.7}$$

For the proof see [5].

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