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BROWDER RIESZ–SCHAUDER THEORY FOR POLYNOMIALLY FINITE RANK LINEAR RELATIONS

BY

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Abstract. We describe the Browder Riesz–Schauder theory of compact operators in Banach spaces in the context of polynomially finite rank linear relations in Banach spaces.

1. Introduction. The well known Riesz–Schauder theory for operators (see, for example, [T]) proves that if T is a bounded compact operator on a complex Banach space X and $\lambda \in \mathbb{K} \setminus \{0\}$, then for each $n \in \mathbb{N}$, $(\lambda - T)^n$ is a Fredholm operator of index zero, the ascent and descent of $\lambda - T$ are both finite and equal, and $X = R(\lambda - T)^r \oplus N(\lambda - T)^r$ where r denotes the ascent of $\lambda - T$. In 2006, Jeribi and Moalla [JM] generalized the above theory to the case when $T \in \mathcal{PK}(X)$, that is, T is a bounded operator on X and there exists a nonzero complex polynomial p such that $p(T)$ is compact and $p(\lambda) \neq 0$.

The purpose of this paper is to investigate the validity of the results of the type mentioned above for multivalued linear operators.

In order to make the paper easily accessible, some definitions and results from the theory of linear relations are recalled in Section 2. In particular, certain algebraic properties concerning the ascent, descent, nullity and deficiency of a linear relation in a vector space and some elementary results about closed, continuous and open linear relations in normed spaces are presented. Section 3 contains the main results of this paper, namely, Theorem 3.8 and Corollary 3.10. We introduce the notion of a polynomially finite rank linear relation and we also give some purely algebraic properties of polynomials of a linear relation and one result concerning the product of Fredholm linear relations. These results are used to extend the Riesz–Schauder theory of operators to the context of polynomially finite rank linear relations.

Multivalued linear operators made their appearance in functional analysis in J. von Neumann's book [N], motivated by the need to consider adjoints of nondensely defined operators used in applications to the theory of

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generalized equations [C] and also by the need to consider the inverses of certain operators, used, for example, in the study of some Cauchy problems associated to parabolic type equations in Banach spaces [FY1]. Interesting works on multivalued linear operators include the treatment of degenerate boundary value problems (see, for example, [FY2] and [GM]), application of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and to problems of fuzzy theory (see, for example, [AMR] and [GL]), application of multivalued methods to the invariant subspace problem (see [GCH] and [S]), application of the spectral theory of linear relations to the study of operators, for example, the spectral theory of ordered pairs of operators and of linear bundles (see, for example, [BCH] and the references therein), and several papers on linear relations of semi-Fredholm type and other related classes (see, for example, [ACW] and [CR]).

2. Preliminaries and auxiliary results. In this section we collect some results of the theory of linear relations, in an attempt to make our paper as self-contained as possible.

Let us first recall some basic definitions, following the notation and terminology of [CR] and [SSW]. Let $X, Y, Z, \dots$ denote arbitrary vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *linear relation* or *multivalued linear operator* $T : X \to Y$ is a mapping from a subspace $D(T) \subset X$, called the *domain* of T, into the collection of nonempty subsets of Y such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. The class of such linear relations is denoted by $LR(X, Y)$. We write $LR(X, X) =: LR(X)$. If $T \in LR(X)$ we say that T is a linear relation *in* X; and T is a linear relation *on* X when T is everywhere defined.

If $T \in LR(X, Y)$ maps all points of its domain to singletons, then T is said to be *single valued* or simply an *operator*. We note that a linear relation T is single valued if and only if $T(0) = \{0\}$. Every element $T \in LR(X, Y)$ is uniquely determined by its *graph*, $G(T)$, which is defined by

$$G(T) := \{(x, y) \in X \times Y : x \in D(T),\ y \in Tx\}.$$

Let $T \in LR(X, Y)$. The *inverse* of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. The subspace $T^{-1}(0)$, denoted by $N(T)$, is called the *null space* of T and we say that T is *injective* if $N(T) = \{0\}$. The *range* of T is the subspace $R(T) := T(D(T))$, and T is called *surjective* if $R(T) = Y$. The quantities

$$\alpha(T) := \dim N(T) \quad \text{and} \quad \beta(T) := \dim Y/R(T)$$

are called the *nullity* and the *deficiency* of T respectively, and the *index* of T

is defined by

$$k(T) := \alpha(T) - \beta(T)$$

provided $\alpha(T)$ and $\beta(T)$ are not both infinite.

Let $S, T \in LR(X, Y)$ and $R \in LR(Y, Z)$. The sum $T + S$ is the linear relation given by $G(T+S) := \{(x, y+z) : (x, y) \in G(T), (x, z) \in G(S)\}$. The *composition* or *product* RT is the linear relation given by $G(RT) := \{(x, z) : (x, y) \in G(T), (y, z) \in G(R) \text{ for some } y \in Y\}$. The product of linear relations is clearly associative. Hence if $T \in LR(X)$, then $T^n, n \in \mathbb{Z}$, is defined as usual with $T^0 = I$ and $T^1 = T$. It is easily seen that $(T^{-1})^n = (T^n)^{-1}, n \in \mathbb{Z}$. If T is a linear relation in a complex vector space X, then the *resolvent set* of T is

$$\rho(T) := \{\lambda \in \mathbb{C} : \alpha(\lambda - T) = \beta(\lambda - T) = 0\}$$

where $\lambda - T := \lambda I_{D(T)} - T$.

As a direct consequence of the definitions, if $T \in LR(X, Y)$ then for all $n, m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} D(T^{n+m}) &\subset D(T^n), \qquad R(T^{n+m}) \subset R(T^n), \\ N(T^n) &\subset N(T^{n+m}), \qquad T^n(0) \subset T^{n+m}(0), \end{aligned}$$

and if $N(T^r) = N(T^{r+1})$ (resp. $R(T^r) = R(T^{r+1})$) for some $r \in \mathbb{N} \cup \{0\}$, then $N(T^n) = N(T^r)$ (resp. $R(T^n) = R(T^r)$) for all nonnegative integers $n \geq r$. These properties lead to the introduction of the *ascent* and *descent* of T by

$$\begin{aligned} a(T) &:= \min\{r \in \mathbb{N} \cup \{0\} : N(T^r) = N(T^{r+1})\}, \\ d(T) &:= \min\{r \in \mathbb{N} \cup \{0\} : R(T^r) = R(T^{r+1})\}, \end{aligned}$$

respectively, whenever these minima exist. If no such numbers exist, the ascent and descent of T are defined to be ∞. Observe that $a(T) = 0$ if and only if T is injective, and $d(T) = 0$ if and only if T is surjective.

Let $T \in LR(X, Y)$ and let M be a subspace of X. Then the *restriction* of T to M is the linear relation $T|_M$ given by $G(T|_M) := \{(m, y) : m \in M \cap D(T), y \in Tm\}$.

Assume now that X and Y are normed spaces and let $T \in LR(X, Y)$. Then Q_T denotes the quotient map from X onto $Y/\overline{T(0)}$. Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and the *norm* of T is defined by $\|T\| := \|Q_T T\|$. We remark that $\|\cdot\|$ is not a true norm since $\|T\| = 0$ does not imply $T = 0$.

We say that $T \in LR(X, Y)$ is *closed* if its graph is a closed subspace, *continuous* if $\|T\| < \infty$, *bounded* if it is continuous and everywhere defined, *open* if $\gamma(T) > 0$ where $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\}$, and *compact* if $Q_T T$ is compact single valued; finally, T is called a *Fredholm linear relation* if T is closed with closed range and has a finite index.

We now list a few purely algebraic properties, which will be useful in the following.

LEMMA 2.1 ([SSW, Lemma 6.2]). *Let T be a linear relation in a vector space X. Assume that there exists some $m \in \mathbb{N} \cup \{0\}$ such that $\alpha(T^n) \leq m$ (resp. $\beta(T^n) \leq m$) for all nonnegative integers n. Then $a(T) \leq m$ (resp. $d(T) \leq m$).*

Following [SSW, Lemma 3.2] we define the *singular chain manifold* of a linear relation T in a vector space X by

$$R_c(T) := \Big(\bigcup_{n \in \mathbb{N}} T^n(0) \Big) \cap \Big(\bigcup_{n \in \mathbb{N}} N(T^n) \Big),$$

and we say that T has a *trivial* singular chain manifold if $R_c(T) = \{0\}$.

In [SSW], the authors prove that many of the results for operators are valid in the context of linear relations under the only additional condition that the linear relation has a trivial singular chain manifold, as for example the following result, which will be used to obtain the main theorem of this paper (Theorem 3.8 below).

LEMMA 2.2 ([SSW, Theorems 6.5, 6.11, 7.4 and Corollary 6.7]). *Let T be a linear relation in a vector space X. Then*

(i) *If $D(T) = X$ and $d(T) < \infty$, then $\beta(T) \leq \alpha(T)$.*
(ii) *If $R_c(T) = \{0\}$ and $a(T) < \infty$, then $\alpha(T) \leq \beta(T)$.*
(iii) *If $R_c(T) = \{0\}$, $a(T) = r < \infty$ and $\alpha(T) = \beta(T) < \infty$, then $a(T) = d(T)$, $\alpha(T^n) = \beta(T^n)$ for all $n \in \mathbb{N}$ and $X = R(T^r) \oplus N(T^r)$.*
(iv) *If $R_c(T) = \{0\}$, $D(T) = X$ and $\dim R(T) < \infty$, then $\alpha(\lambda - T) = \beta(\lambda - T) < \infty$ and $a(\lambda - T) = d(\lambda - T) < \infty$ for all $\lambda \in \mathbb{K} \setminus \{0\}$.*

Corollary 2.4 below gives a sufficient condition for a linear relation to have a trivial singular chain manifold. First, we recall the following property:

LEMMA 2.3 ([SS, Lemma 6.1]). *Let T be a linear relation in a complex vector space and assume that $\rho(T) \neq \emptyset$. Then for all $n, m \in \mathbb{N}$ we have*

$$X = D(T^m) + R(T^n) \quad \textit{and} \quad \{0\} = T^n(0) \cap N(T^m).$$

Now, as an immediate consequence of Lemma 2.3, we get

COROLLARY 2.4. *If T is a linear relation in a complex vector space X such that $\rho(T) \neq \emptyset$, then $R_c(T) = \{0\}$.*

The following result concerning the nullity, deficiency and index of the product of linear relations is sometimes useful.

LEMMA 2.5 ([SS, Lemma 5.1 and Proposition 5.2]). *Let X, Y and Z be vector spaces and let $S \in LR(X, Y)$ and $T \in LR(Y, Z)$. Then:*

(i) $\alpha(TS) \le \alpha(T) + \alpha(S)$ *and* $\beta(TS) \le \beta(T) + \beta(S)$.

(ii) *Assume that* S *and* T *have finite indices. Then*

$$k(TS) = k(T) + k(S) + \dim Y/(R(S) + D(T)) - \dim\{S(0) \cap N(T)\}.$$

In particular, $k(T^n) = nk(T)$ *for each* $n \in \mathbb{N}$ *if* $k(T) < \infty$ *and* $\rho(T) \neq \emptyset$.

We shall make frequent use of the following result concerning closed, continuous and open linear relations.

LEMMA 2.6 ([CR, Chapters II and III]). *Let* X, Y *and* Z *be normed spaces and let* $S, T \in LR(X,Y)$ *and* $U \in LR(Y,Z)$. *Then:*

(i) *S is closed if and only if $Q_S S$ is a closed linear operator and $S(0)$ is a closed subspace. Furthermore, S is closed if and only if so is its inverse, and hence the null space of a closed linear relation is a closed subspace.*

(ii) *If S is continuous with $D(S)$ and $S(0)$ both closed, then S is closed.*

(iii) *If S is open and $N(S)$ is closed, then $N(S) = N(Q_S S)$ and $\gamma(S) = \gamma(Q_S S)$.*

(iv) *If T is closed and S is single valued and continuous with $\overline{D(T)} \subset D(S)$, then $T + S$ is closed.*

(v) *If S and T are continuous, then $T + S$ is continuous.*

(vi) *Assume that S is closed, U has closed range and $\alpha(U) < \infty$ and $\gamma(U) > 0$. Then US is closed.*

(vii) *If U and S are continuous with $S(0) \subset D(U)$ then US is continuous.*

(viii) *If X and Y are Banach spaces and S is closed, then $R(S)$ is closed if and only if S is open.*

3. On polynomially finite rank linear relations. The notion of polynomial of an operator can be naturally generalized to linear relations as follows:

DEFINITION 3.1. Given a linear relation T in a vector space X and a polynomial $p(\eta) := \sum_{i=0}^{m} a_i \eta^i$ of degree m with coefficients in $\mathbb{K}$ we define

$$p(T) := \sum_{i=0}^{m} a_i T^i.$$

It is an immediate consequence of the definitions of addition and composition of linear relations that $p(T)$ is a linear relation such that $D(p(T)) = D(T^m)$, and T is everywhere defined if and only if so is $p(T)$.

We also note that if T is a continuous linear relation in a normed space X such that $\dim R(T) < \infty$, then T is compact, since clearly $Q_T T B_X \subset \|T\| B_{R(Q_T T)}$ where $B_{R(Q_T T)}$ denotes the closed unit ball in $R(Q_T T)$, which is a compact subset of $Q_T X$. This last remark together with Lemma 2.2(iv) and all the properties established in Section 2 suggests investigating the validity of the results due to Jeribi and Moalla [JM] mentioned in the introduction when the set $\mathcal{PK}(X)$ is replaced by $\mathcal{PFR}(X)$, defined as follows:

Definition 3.2. Let X be a complex Banach space. A bounded linear relation T on X is said to be of *polynomially finite rank type*, denoted $T \in \mathcal{PFR}(X)$, if there exists a nonzero complex polynomial $p(\eta) = \sum_{i=0}^{m} a_i \eta^i$ such that $p(T) = \sum_{i=0}^{m} a_i T^i$ is a finite rank linear relation with a nonempty resolvent set.

The following entirely algebraic results are basic for the proof of the main Theorem 3.8 below.

Lemma 3.3 ([A, Theorem 2.3]). *Let p and q be two polynomials with coefficients in $\mathbb{C}$ and let T be a linear relation in a complex vector space X. Then*
$$(qp)(T) = q(T)p(T) = p(T)q(T).$$

Lemma 3.4. *Let $p(\eta) = \sum_{i=0}^{m} a_i \eta^i$ be a nonzero complex polynomial and let $\lambda \in \mathbb{C}$ be such that $p(\lambda) \neq 0$. If T is a linear relation in a complex vector space X, then*
$$p(\lambda) - p(T) = (\lambda - T)q(T) = q(T)(\lambda - T)$$
where $q(T)$ is the polynomial in T of degree $m-1$ given by
$$q(T) = \sum_{i=1}^{m} a_i \sum_{j=0}^{i-1} \lambda^j T^{i-1-j}.$$

Proof. We first show that
$$(*) \qquad \lambda^r - T^r = (\lambda - T)\sum_{i=0}^{r-1} \lambda^i T^{r-1-i}, \qquad 1 \le r \le m.$$

The proof is by induction. For $r = 1$ the property is trivial. Assume the assertion is valid for $r = s$. Then
$$\begin{aligned}
\lambda^{s+1} - T^{s+1} &= \lambda^s \lambda - T^s T = \lambda^s \lambda - T^s((T-\lambda) + \lambda) \\
&= \lambda\lambda^s - T^s(T-\lambda) - \lambda T^s \quad \text{([CR, Proposition I.4.2 (e)])} \\
&= \lambda(\lambda^s - T^s) - (T-\lambda)T^s \quad \text{(Lemma 3.3)} \\
&= \lambda(\lambda - T)\sum_{i=0}^{s-1} \lambda^i T^{s-1-i} + (\lambda - T)T^s \quad \text{(induction hypothesis)} \\
&= (\lambda - T)(\lambda \sum_{i=0}^{s-1} \lambda^i T^{s-1-i} + T^s) \quad \text{(again Lemma 3.3)} \\
&= (\lambda - T)\sum_{i=0}^{s} \lambda^i T^{s-i}.
\end{aligned}$$
Hence the desired equality $(*)$ holds.

Furthermore, since $p(\lambda) - p(T) = \sum_{i=1}^{m} a_i(\lambda^i - T^i)$, a combination of $(*)$ and Lemma 3.3 implies that

$$p(\lambda) - p(T) = (\lambda - T)q(T) = q(T)(\lambda - T)$$

where $q(T) := \sum_{i=1}^{m} a_i \sum_{j=0}^{i-1} \lambda^i T^{i-1-j}$. ■

Corollary 3.5. *Let X, T, λ, p and q satisfy the hypotheses in Lemma 3.4. If $p(T)$ has a nonempty resolvent set, then $R_c(T) = \{0\}$.*

Proof. Since $p(\lambda) - p(T) = (\lambda - T)q(T) = q(T)(\lambda - T)$ it is easy to see that for all $n \in \mathbb{N} \cup \{0\}$,

$$(p(\lambda) - p(T))^n = (\lambda - T)^n q(T)^n = q(T)^n (\lambda - T)^n.$$

Therefore,

$$N(\lambda - T)^n \subset N(p(\lambda) - p(T))^n \quad \text{and} \quad (\lambda - T)^n(0) \subset (p(\lambda) - p(T))^n(0).$$

These inclusions together with Lemma 2.3 imply that for all nonnegative integers n and m,

$$(\lambda - T)^n(0) \cap N(\lambda - T)^m = \{0\},$$

and hence $R_c(\lambda - T) = \{0\}$. Now, the required conclusion follows immediately by observing that if T is a linear relation in a vector space and $\lambda \in \mathbb{K}$ then $R_c(T) = R_c(\lambda - T)$ ([SSW, Lemma 7.1]). ■

The next result about the product of Fredholm linear relations will be needed to obtain our main Theorem 3.8.

Proposition 3.6. *Let S and T be two Fredholm linear relations in a Banach space X. Then ST is a Fredholm linear relation.*

Proof. We first observe that ST is closed by Lemma 2.6(vi)&(viii). Moreover, by Lemma 2.5(i), $\alpha(ST)$ and $\beta(ST)$ are finite. Hence, it only remains to see that $R(ST)$ is closed. To this end, let

$$S_0 := S|_{N(S)+R(T)}.$$

Then S_0 is closed since S is closed and $N(S)+R(T)$ is closed. Moreover, since S and S_0 have the same null space, it follows that $\gamma(S) \leq \gamma(S_0)$ and thus Lemma 2.6(viii) shows that $R(S_0)$ is a closed subspace. This last property implies that $R(ST)$ is closed since

$$\begin{aligned} R(S_0) &= S(N(S)+R(T)) = SN(S)+SR(T) && \text{([CR, Proposition I.3.1(c)])} \\ &= SS^{-1}(0) + R(ST) = S(0) + R(ST) && \text{([CR, Corollary I.2.10])} \\ &= R(ST). \ ■ \end{aligned}$$

Lemma 3.7. *Let X be a normed space and let M and N be subspaces of X such that $M \subset N$. Then:*

(i) *N is closed in X if and only if N/M is closed in X/M.*

(ii) *If N is closed, then $X/N = (X/M)/(N/M)$ where the equality is a canonical isometry.*

Proof. (i) Follows immediately from the definitions.
(ii) See [CR, Lemma IV.5.2]. ∎

We are now in a position to state the main theorem of this paper.

THEOREM 3.8. *Let X be a complex Banach space and let T be a bounded linear relation on X such that $T \in \mathcal{PFR}(X)$, that is, there exists a nonzero polynomial $p(\eta) = \sum_{i=0}^{m} a_i \eta^i$ such that $p(T)$ is a finite rank linear relation with a nonempty resolvent set. Let $\lambda \in \mathbb{C}$ with $p(\lambda) \neq 0$. Then, for each $n \in \mathbb{N}$, $(\lambda - T)^n$ is a bounded Fredholm linear relation on X with index zero. The ascent and descent of $\lambda - T$ are both finite and equal. If $r := a(\lambda - T)$ then*

$$X = R(\lambda - T)^r \oplus N(\lambda - T)^r.$$

Proof. We first note that by Lemma 2.6(v)&(vii) the linear relations $p(T)$, $p(\lambda) - p(T)$ and $(\lambda - T)^n$, $n \in \mathbb{N}$, are bounded; moreover, Lemma 2.6(ii) shows that $p(T)$ and $p(\lambda) - p(T)$ are both closed.

We infer from Lemma 3.4 that

(3.1) $p(\lambda) - p(T) = (\lambda - T)q(T) = q(T)(\lambda - T)$

and since $q(T)$ commutes with $\lambda - T$, it is easy to see that

(3.2) $(p(\lambda) - p(T))^n = q(T)^n(\lambda - T)^n = (\lambda - T)^n q(T)^n$, $n \in \mathbb{N}$.

We now show that

(3.3) $\lambda - T$ *is closed.*

Indeed, this follows from Lemma 2.6(ii) since $\lambda - T$ is continuous with $D(\lambda - T) = X$ and $(\lambda - T)(0)$ is a closed subspace because

$$\begin{aligned}(\lambda - T)(0) \subset (\lambda - T)q(T)(0) &= (p(\lambda) - p(T))(0) \quad \text{(by (3.1))}\\ &= p(T)(0) \subset R(p(T)),\end{aligned}$$

which is finite-dimensional by hypothesis.

(3.4) $p(\lambda) - p(T)$ *is a Fredholm linear relation on X.*

Since $p(\lambda) \neq 0$ and $p(T)$ is a finite rank linear relation with $\rho(p(T)) \neq \emptyset$ (and hence $R_c(p(T)) = \{0\}$ by Corollary 2.4), (3.4) follows immediately from Lemma 2.2(iv) and the fact that $p(\lambda) - p(T)$ is closed.

(3.5) $\lambda - T$ *is a Fredholm linear relation on X.*

The statements (3.1) and (3.4) imply that $\alpha(\lambda - T) < \infty$ and $R(p(\lambda) - p(T))$ is a closed finite-codimensional subspace contained in $R(\lambda - T)$; thus we deduce from Lemma 3.7 that $R(\lambda - T)$ is closed and $\beta(\lambda - T) < \infty$. Hence (3.5) holds.

(3.6) *For each $n \in \mathbb{N}$, $(\lambda - T)^n$ and $(p(\lambda) - p(T))^n$ are Fredholm.*

This follows from (3.4), (3.5) and Proposition 3.6.

(3.7) *The ascent and descent of $\lambda - T$ are both finite.*

As a direct consequence of Lemma 2.2(iv), the ascent and descent of $p(\lambda) - p(T)$ are finite and equal. Let $r := a(p(\lambda) - p(T)) = d(p(\lambda) - p(T))$, so that for all $n \geq r$ we have

$$N(p(\lambda) - p(T))^n = N(p(\lambda) - p(T))^r,$$
$$R(p(\lambda) - p(T))^n = R(p(\lambda) - p(T))^r.$$

These equalities together with (3.2) and (3.6) imply that for all $n \geq r$,

$$\alpha(\lambda - T)^n \leq \alpha(p(\lambda) - p(T))^r, \qquad \beta(\lambda - T)^n \leq \beta(p(\lambda) - p(T))^r,$$

and so we deduce from Lemma 2.1 that the ascent and descent of $\lambda - T$ are finite, as desired.

On the other hand, it follows from Corollary 3.5 that $R_c(T) = \{0\}$, equivalently $R_c(\lambda - T) = \{0\}$ (see, [SSW, Lemma 7.1]). Combining this with $D(\lambda - T) = X$, $d(\lambda - T) < \infty$ ((3.7)) and Lemma 2.2(i) we get $\beta(\lambda - T) \leq \alpha(\lambda - T)$, and a combination of the properties $R_c(\lambda - T) = \{0\}$, $a(\lambda - T) < \infty$ ((3.7)) and Lemma 2.2(ii) leads to $\alpha(\lambda - T) \leq \beta(\lambda - T)$, so that $\alpha(\lambda - T) = \beta(\lambda - T) < \infty$. In this situation we can conclude from (3.6) and Lemma 2.2(iii) that for each $n \in \mathbb{N}$, $(\lambda - T)^n$ has index zero and $a(\lambda - T) = d(\lambda - T) < \infty$; moreover, $X = R(\lambda - T)^s \oplus N(\lambda - T)^s$ if $s := a(\lambda - T)$. ∎

The following example shows that if T is a bounded compact linear relation on a Banach space X and $\lambda \in \mathbb{K} \setminus \{0\}$, then $\lambda - T$ may not be Fredholm.

Example 3.9. *Let X be an infinite-dimensional Banach space and let T be a linear relation whose graph is $X \times X$. Then T is a bounded compact closed linear relation such that $I + T$ is not Fredholm.*

Indeed, we infer immediately from the definition of the sum of linear relations that $I + T = T$.

Observing that for any linear relation U in a vector space E we have $D(U) \times E = G(U) + (\{0\} \times E)$, $\{0\} \times U(0) = G(U) \cap (\{0\} \times E)$ and $N(U) \times \{0\} = G(U) \cap (E \times \{0\})$ ([CR, Proposition I.3.1]), we deduce trivially that $D(T) = N(T) = T(0) = X$. Therefore T is not an operator (since $T(0) \neq \{0\}$), $Q_T T$ is the zero operator on X (since $D(T) = T(0) = X$), so that T is a bounded compact linear relation and by Lemma 2.6(ii), T is closed. Finally, it follows from the equalities $I + T = T$ and $N(T) = X$ that $I + T$ is not Fredholm.

We remark that in the above example, $\dim T(0) = \infty$ and $\rho(T) = \emptyset$.

Corollary 3.10. *Let X, T, λ and p satisfy the hypotheses in Theorem 3.8. Then there exists a bounded finite rank single valued F on X such that the linear relation $\lambda - T - F$ is injective and surjective.*

Proof. By Theorem 3.8, $\lambda - T$ is a bounded Fredholm linear relation with index zero. Let $m := \alpha(\lambda - T) = \beta(\lambda - T)$. If $m = 0$, then consider $F = 0$. Hence we can assume without loss of generality that $m \geq 1$. Then $X = R(\lambda - T) \oplus M$, where M is a finite-dimensional subspace generated by $y_1, \ldots, y_m$, and let $x_1, \ldots, x_m$ be a basis of $N(\lambda - T)$. Choose elements $x'_1, \ldots, x'_m$ in the dual space of X such that $x'_i(x_j) = \delta_{ij}$ $(1 \leq i, j \leq m)$. Define

$$F : x \in X \mapsto F(x) := \sum_{i=1}^{m} x'_i(x) y_i \in X.$$

Clearly, F is a bounded operator on X with $\dim R(F) \leq m$.

To verify that $\lambda - T - F$ is injective, let $x \in N(\lambda - T - F)$. Then $Fx \in (\lambda - T)x \subset R(\lambda - T)$. Hence $x'_i(x) = 0$ for $1 \leq i \leq m$, which implies that $Fx = 0$ so that $x \in N(\lambda - T)$, and hence $x = \sum_{i=1}^{m} \alpha_i x_i$ for some constants $\alpha_i \in \mathbb{K}$ and thus $0 = x'_i(x) = \alpha_i$ for $1 \leq i \leq m$, which gives $x = 0$, as desired.

To show that $\lambda - T - F$ is surjective, let $z \in X$. Then $z = \sum_{i=1}^{m} \alpha_i y_i + u$ for some constants $\alpha_i \in \mathbb{K}$ and $u \in R(\lambda - T)$ (since $X = R(\lambda - T) \oplus M$). Let $x \in X$ be such that $(x, u) \in G(\lambda - T)$ and $v := x - \sum_{i=1}^{m} (\alpha_i + x'_i(x)) x_i$. Then $(v, u) \in G(\lambda - T)$ since $(x, u) \in G(\lambda - T)$ and $x_i \in N(\lambda - T)$, so that $(\lambda - T)x_i = (\lambda - T)(0)$ and hence $(\lambda - T)v = (\lambda - T)x$.

Furthermore, $z = u - Fv$ since $z = \sum_{i=1}^{m} \alpha_i y_i + u$ and

$$x'_j(v) = x'_j(x) - \sum_{i=1}^{m} (\alpha_i + x'_i(x)) x'_j(x_i) = x'_j(x) - \alpha_j - x'_j(x) = -\alpha_j,$$

so that $F(v) = \sum_{i=1}^{m} x'_i(v) y_i = -\sum_{i=1}^{m} \alpha_i y_i$. Therefore $(v, u) = (v, z + Fv) \in G(\lambda - T)$ and consequently $(v, u) \in G(\lambda - T - F)$, as desired. ∎

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