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BLOW-UP FOR THE FOCUSING ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH CONFINING HARMONIC POTENTIAL

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Abstract. The focusing nonlinear Schrödinger equation (NLS) with confining harmonic potential

$$\mathrm{i}\partial_t u + \tfrac{1}{2}\Delta u - \tfrac{1}{2}|x|^2 u = -|u|^{4/(d-2)}u, \quad x \in \mathbb{R}^d,$$

is considered. By modifying a variational technique, we shall give a sufficient condition under which the corresponding solution blows up.

1. Introduction. The NLS with confining harmonic potential

$$\mathrm{i}\partial_t u + \tfrac{1}{2}\Delta u - \tfrac{1}{2}|x|^2 u = \mu |u|^p u, \quad x \in \mathbb{R}^d, \tag{1.1}$$

has been used to model the Bose–Einstein condensation (see for example [2, 3]). The most physically relevant case is $p = 2$, $d = 3$. Here u is a complex-valued function defined on some spatial-time slab $I \times \mathbb{R}^d$, $d \geq 3$, $4/d \leq p \leq 4/(d-2)$, $\mu = \pm 1$, with $\mu = 1$ being the *defocusing case* and $\mu = -1$ the *focusing case*. There are many mathematical works on the Cauchy problem for this equation (see, e.g., [1, 6, 10, 11, 12, 13]).

Set the initial datum

$$u(0,x) = u_0(x). \tag{1.2}$$

The natural choice of the initial space is

$$\Sigma := \{\varphi \in \dot{H}^1;\ x\varphi \in L^2\}$$

endowed with the norm

$$\|\varphi\|_{\Sigma}^2 = \|\nabla\varphi\|_2^2 + \|x\varphi\|_2^2.$$

It is easily seen that $\Sigma \hookrightarrow L^2$ by the standard uncertainty principle

$$\|f\|_{L^2}^2 \leq \frac{2}{d}\|\nabla f\|_{L^2}\|xf\|_{L^2}.$$

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For $u_0 \in \Sigma$, the solution u obeys two conservation laws, i.e.,

(1.3) Mass conservation: $$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx \equiv M(u_0),$$

(1.4) Energy conservation:

$$E(u(t)) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|x|^2|u|^2 - \frac{2}{p+2}|u|^{p+2} \right) dx \equiv E(u_0).$$

By saying that $u : [0,T) \times \mathbb{R}^d \to \mathbb{C}$ is a solution of (1.1)–(1.2), we mean $u \in C_t([0,T); \Sigma)$ where T is the maximal existence time, and u satisfies the Duhamel formula

$$u(t) = e^{i\frac{1}{2}t(\Delta - |x|^2)}u_0 + i\int_0^t e^{i\frac{1}{2}(t-\tau)(\Delta-|x|^2)}|u|^p u(\tau) \, d\tau, \qquad \forall t \in [0,T).$$

If $T = \infty$, we say u *globally exists*. We say that u *blows up in finite time* if $T < \infty$. For $4/d \le p < 4/(d-2)$, global existence of the solution for the Cauchy problem of (1.1) with $\mu = 1$ is a consequence of energy conservation, while blow-up occurs for the focusing case ($\mu = -1$). The latter case has been proven in [12], where the sufficient condition for blow-up is $M(u_0)+E(u_0) < M(\varphi) + E(\varphi)$, $\frac{d(M(u)+E(u))}{du}\big|_{u_0} < 0$ and the virial quantity less than 0. Here φ is the solution to the elliptic equation

(1.5) $$-\tfrac{1}{2}\Delta\varphi + \tfrac{1}{2}|x|^2\varphi + \varphi = |\varphi|^p\varphi, \qquad 4/d \le p < 4/(d-2).$$

Concerning $p = 4/(d-2)$, the *energy-critical case*, it has been shown in [6, 13] that the problem (1.1)–(1.2) for radial solutions with $\mu = 1$ is globally well-posed.

We call (1.1) the energy-critical NLS when $p = 4/(d-2)$, since if we abandon the harmonic potential for a moment, then (1.1) and the $\dot{H}^1$-norm of the initial data are both preserved by the scaling

$$u_\lambda(t,x) = \lambda^{(d-2)/2}u(\lambda^2 t, \lambda x).$$

Later, we shall use the transform

$$\tilde{u}_\lambda(t,x) = \lambda^{(d-2)/2}u(t, \lambda x).$$

Blow-up for the energy-critical case with $\mu = -1$ is expected to exist similarly to the focusing energy-critical NLS without harmonic potential ([7, 4]) and the focusing subcritical case. Recall that from [7, 4] finite time blow-up occurs provided the initial datum u_0 satisfies $E(u_0) < E(W)$ and $\|\nabla u_0\|_{L^2} \ge \|\nabla W\|_{L^2}$. Here E is the corresponding energy functional, and W is the solution to the elliptic equation

(1.6) $$-\Delta W = |W|^{4/(d-2)}W.$$

Note that there is no non-trivial solution to the equation

$$-\tfrac{1}{2}\varDelta\varPhi + \tfrac{1}{2}|x|^2\varPhi = |\varPhi|^{4/(d-2)}\varPhi.$$

So the energy constraint in our case cannot be given by the corresponding energy of the ground state. In this paper, by employing the variational idea of [5] (see also [9]), we shall prove that the energy constraint can be represented by $\|\nabla W\|_{L^2}$ and derive a sufficient condition for the solution of (1.1)–(1.2) to blow up in finite time. The key ingredient is to reduce the minimization problem for the non-coercive energy functional to minimization of a positive functional. Inspiration comes from an interesting observation. For ease of exposition, we define some functionals:

$$\begin{aligned}\mathcal{H}(\phi) &= \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}|x|^2|\phi|^2 - \frac{2}{2^*}|\phi|^{2^*}\right)dx,\\ \mathcal{K}(\phi) &= \int_{\mathbb{R}^d}(|\nabla\phi|^2 + |x|^2|\phi|^2 - 2|\phi|^{2^*})\,dx,\\ \mathcal{Q}(\phi) &= \int_{\mathbb{R}^d}(2|\nabla\phi|^2 - 2|x|^2|\phi|^2 - 4|\phi|^{2^*})\,dx,\\ \mathcal{K}_0(\phi) &= \int_{\mathbb{R}^d}(|\nabla\phi|^2 - 2|\phi|^{2^*})\,dx.\end{aligned}\tag{1.7}$$

Define

$$m_c := \inf\{\mathcal{H}(\phi);\ 0 \neq \phi \in \varSigma,\ \mathcal{K}(\phi) = 0\}.\tag{1.8}$$

Observe that $\mathcal{Q}(u)$ is the second derivative of the virial quantity $\|xu\|_2^2$. Moreover,

$$\mathcal{Q}(u) = 2\mathcal{K}(u) - 4\|xu\|_2^2,$$

which implies that if $\mathcal{K}(u) < 0$, so does $\mathcal{Q}(u)$. This key observation allows us to consider blow-up in the set

$$\mathbb{K} = \{\phi \in \varSigma;\ \mathcal{H}(\phi) < m_c,\ \mathcal{K}(\phi) < 0\}.$$

Otherwise, one should add the constraint

$$\{\phi \in \varSigma;\ \mathcal{H}(\phi) < m_c,\ \mathcal{K}(\phi) < 0,\ \mathcal{Q}(\phi) < 0\}$$

as in [12] for subcritical powers. We shall prove that $m_c > 0$ and exactly $m_c = \frac{2^{1-d/2}}{d}\|\nabla W\|_2^2$.

We now present our main result.

THEOREM 1. *Let* $u_0 \in \varSigma, p = 4/(d-2)$ *and let* u *be the corresponding solution to* (1.1)–(1.2). *Assume* $u_0 \in \mathbb{K}$. *Then* u *blows up in finite time.*

In Section 2, we shall find the value of m_c, derive some properties of u in $\mathbb{K}$, and then prove Theorem 1.

Notation. Throughout, we always denote $2^* = \frac{2d}{d-2}$; $\dot{H}^1$ is the Sobolev space with norm defined by $\|\cdot\|_{\dot{H}^1} = \|\mathcal{F}^{-1}|\xi|\mathcal{F}\cdot\|_{L^2}$, where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse.

2. The value of m_c and proof of Theorem 1. In this section, we investigate the minimization problem (1.8), show properties of solutions in $\mathbb{K}$, and finally prove Theorem 1.

To find the value of m_c in (1.8), we define

$$\mathcal{J}(\phi) := \mathcal{H}(\phi) - \frac{1}{2}\mathcal{K}(\phi) = \frac{2}{d}\int_{\mathbb{R}^d} |\phi|^{2^*}\,dx.$$

LEMMA 1. $m_c = \inf\{\mathcal{J}(\phi);\, 0 \neq \phi \in \Sigma,\, \mathcal{K}(\phi) \leq 0\}$.

Proof. Denote the above infimum by m'. We first prove $m_c \leq m'$. Denote

$$\mathbb{A} = \{\phi;\, 0 \neq \phi \in \Sigma,\, \mathcal{K}(\phi) = 0\}, \qquad \mathbb{B} = \{\phi;\, 0 \neq \phi \in \Sigma,\, \mathcal{K}(\phi) \leq 0\}.$$

For each $\phi \in \mathbb{B}$, $\mathcal{K}(\phi) \leq 0$. Thus, $\mathcal{H}(\phi) \leq \mathcal{J}(\phi)$. Set

$$\phi^\lambda(x) = \lambda^{(d-2)/2}\phi(\lambda x).$$

Since $\lim_{\lambda\to 0}\mathcal{K}(\phi^\lambda) = \infty$, there exists a $\lambda_0 \in (0,1]$ such that $\mathcal{K}(\phi^{\lambda_0}) = 0$, that is, $\phi^{\lambda_0} \in \mathbb{A}$. Therefore, we get

$$m_c \leq \mathcal{J}(\phi^{\lambda_0}) = \mathcal{H}(\phi^{\lambda_0}) \leq \mathcal{H}(\phi) \leq \mathcal{J}(\phi).$$

Thus, $m_c \leq m'$.

Conversely, given $\phi \in \mathbb{A}$, we have $\phi \in \mathbb{B}$ and $\mathcal{H}(\phi) = \mathcal{J}(\phi)$. Thus, $m' \leq m_c$. ■

The next lemma says that the infimum of $\mathcal{J}(\phi)$ on the set $\{\mathcal{K}(\phi) \leq 0\}$ is the same as that on the set $\{\mathcal{K}_0(\phi) < 0\}$.

LEMMA 2. $m_c = \inf\{\mathcal{J}(\phi);\, 0 \neq \phi \in \Sigma,\, \mathcal{K}_0(\phi) < 0\}$.

Proof. Denote the above infimum by $\bar{m}$. By the definition, $\mathcal{K}_0(\phi) < \mathcal{K}(\phi)$ for all $\phi \neq 0$. Hence, $\bar{m} \leq m_c$.

On the other hand, for all ϕ with $\mathcal{K}_0(\phi) < 0$, we have

$$\lim_{\lambda\to\infty}\mathcal{K}(\phi^\lambda) = \mathcal{K}_0(\phi) < 0.$$

Thus, there exists a $\tilde{\lambda} \in (1,\infty)$ such that $\mathcal{K}(\phi^{\tilde{\lambda}}) \leq 0$. So, $m_c \leq \bar{m}$. ■

From Lemmas 1 and 2, one can derive the value of m_c.

PROPOSITION 1. *Let m_c be defined as in* (1.8). *Then*

$$m_c = \frac{2^{1-d/2}}{d}\|\nabla W\|_2^2,$$

where W satisfies the equation $-\Delta W = W^{\frac{d+2}{d-2}}$.

Proof. Let $\bar{m}$ be as in the proof of Lemma 2. It is obvious that

$$\bar{m} \geq \inf_{0\neq\phi\in\Sigma} \frac{2}{d}\Big(\int_{\mathbb{R}^d} |\phi|^{2^*}\,dx\Big)\left[\frac{\int_{\mathbb{R}^d}|\nabla\phi|^2}{2\int_{\mathbb{R}^d}|\phi|^{2^*}}\right]^{\frac{2^*}{2^*-2}} =: \tilde{m}.$$

Next, we shall show by homogeneity and scaling $\phi \mapsto \mu\phi$ that $\bar{m} \leq \tilde{m}$. Indeed, for all $0 < \varepsilon$ (< 1), there exists $0 \neq \phi \in \Sigma$ such that

$$\tilde{m} + \varepsilon > \frac{2}{d}\Big(\int_{\mathbb{R}^d} |\phi|^{2^*}\,dx\Big)\left[\frac{\int_{\mathbb{R}^d}|\nabla\phi|^2}{2\int_{\mathbb{R}^d}|\phi|^{2^*}}\right]^{\frac{2^*}{2^*-2}} \tag{2.1}$$

$$= \frac{2}{d}\Big(\int_{\mathbb{R}^d} |\mu\phi|^{2^*}\,dx\Big)\left[\frac{\int_{\mathbb{R}^d}|\nabla(\mu\phi)|^2}{2\int_{\mathbb{R}^d}|\mu\phi|^{2^*}}\right]^{\frac{2^*}{2^*-2}}, \qquad \forall\mu > 0.$$

Taking

$$\mu = \frac{1}{(1-\varepsilon/\bar{m})^{1/2^*}}\left[\frac{\int_{\mathbb{R}^d}|\nabla\phi|^2}{2\int_{\mathbb{R}^d}|\phi|^{2^*}}\right]^{1/(2^*-2)},$$

we then have

$$\left[\frac{\int_{\mathbb{R}^d}|\nabla(\mu\phi)|^2}{2\int_{\mathbb{R}^d}|\mu\phi|^{2^*}}\right]^{\frac{2^*}{2^*-2}} = 1 - \frac{\varepsilon}{\bar{m}}, \qquad \mathcal{K}(\mu\phi) < 0.$$

Thus, by (2.1) and Lemma 2, we obtain

$$\tilde{m} + \varepsilon > \frac{2}{d}(1-\varepsilon/\bar{m})\int_{\mathbb{R}^d} |\mu\phi|^{2^*} \geq \bar{m} - \varepsilon.$$

This implies that $\tilde{m} \geq \bar{m}$. Hence

$$\bar{m} = \inf_{0\neq\phi\in\Sigma} \frac{2}{d}\Big(\int_{\mathbb{R}^d} |\phi|^{2^*}\,dx\Big)\left[\frac{\int_{\mathbb{R}^d}|\nabla\phi|^2}{2\int_{\mathbb{R}^d}|\phi|^{2^*}}\right]^{\frac{2^*}{2^*-2}}$$

$$= \inf_{0\neq\phi\in\Sigma} \frac{2^{1-d/2}}{d}\left[\frac{\|\nabla\phi\|_2}{\|\phi\|_{2^*}}\right]^d = \frac{2^{1-d/2}}{d}C_d^{-d},$$

where C_d is the sharp constant in the Sobolev inequality

$$\|\psi\|_{L^{2^*}} \leq C_d\|\nabla\psi\|_{L^2},$$

which is attained at W that is the solution of the equation

$$-\Delta\varphi = \varphi^{\frac{d+2}{d-2}}.$$

By a direct calculation, we obtain

$$m_c = \frac{2^{1-d/2}}{d}\|\nabla W\|_2^2. \quad \blacksquare$$

Proof of Theorem 1. To prove the theorem, we first establish some properties of the solution for $u_0 \in \mathbb{K}$. The argument for the theorem uses the standard convexity method (see [8]).

LEMMA 3. *Let $u_0 \in \Sigma$ and let u be the corresponding solution to* (1.1)–(1.2) *with maximal life-span I. If $u_0 \in \mathbb{K}$, then $u(t) \in \mathbb{K}$ for all $t \in I$.*

Proof. Suppose for contradiction that there exists $t_1 \in I$ such that $\mathcal{K}(u(t_1)) \geq 0$. Then by the continuity of the flow, there exists $0 < t_2 \leq t_1$ such that $\mathcal{K}(u(t_2)) = 0$. Hence by the definition of m_c, $E(u(t_2)) \geq m_c$. But the energy of the solution is conserved, which is a contradiction. ∎

LEMMA 4 (Coercivity). *Assume $u_0 \in \mathbb{K}$. Then $\mathcal{K}(u(t)) \leq 2(E(u) - m_c)$ for all $t \in I$.*

Proof. By Lemma 3, $u(t) \in \mathbb{K}$ for all $t \in I$. Set $u^\lambda(t,x) = \lambda^{(d-2)/2} u(t, \lambda x)$. Note that

$$\lim_{\lambda \to 0} \mathcal{K}(u^\lambda) = \infty. \tag{2.2}$$

Since $\mathcal{K}(u) < 0$, it follows from (2.2) that there exists $\tilde{\lambda} \in (0,1)$ such that $\mathcal{K}(u^{\tilde{\lambda}}) = 0$. This implies $E(u^{\tilde{\lambda}}) \geq m_c$. Thus,

$$\mathcal{K}(u) = \mathcal{K}(u) - \mathcal{K}(u^{\tilde{\lambda}}) = 2(E(u) - E(u^{\tilde{\lambda}})) \leq 2(E(u) - m_c). \ ∎$$

Proof of Theorem 1. Suppose for contradiction that u is global. Define the virial quantity

$$V(u)(t) = \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \, dx.$$

By a direct computation, we have

$$\frac{d}{dt} V(t) = 2 \operatorname{Im} \int_{\mathbb{R}^d} \bar{u} x \cdot \nabla u \, dx,$$

$$\frac{d^2}{dt^2} V(t) = \int_{\mathbb{R}^d} (2|\nabla u|^2 - 2|x|^2 |u(t,x)|^2 - 4|u(t,x)|^{2^*}) \, dx = 2\mathcal{K}(u) - 4V.$$

By an ODE technique and Lemma 4, for $0 \leq t \leq \pi/2$ we obtain

$$\begin{aligned} V(t) &= V(0)\cos(2t) + \tfrac{1}{2}\dot{V}(0)\sin(2t) + \int_0^t \mathcal{K}(u(s)) \sin[2(t-s)] \, ds \\ &\leq V(0)\cos(2t) + \tfrac{1}{2}\dot{V}(0)\sin(2t) + (m_c - E)(\cos(2t) - 1) \\ &\leq V(0)\cos(2t) + \tfrac{1}{2}\dot{V}(0)\sin(2t). \end{aligned}$$

It is easily seen that $V(t)$ becomes negative after $t = \pi/4$ in both cases $\dot{V}(0) \leq 0$ and $\dot{V}(0) \geq 0$. But this is impossible. Thus, u must blow up in finite time. ∎

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