COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 1

ON THE CLASSIFICATION OF THE REAL FLEXIBLE DIVISION ALGEBRAS

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Abstract. We investigate the class of finite-dimensional real flexible division algebras.

We classify the commutative division algebras, completing an approach by Althoen and Kugler. We solve the isomorphism problem for scalar isotopes of quadratic division algebras, and classify the generalised pseudo-octonion algebras. In view of earlier results by Benkart, Britten and Osborn and Cuenca Mira *et al.*, this reduces the problem of classifying the real flexible division algebras to the normal form problem of the action of the group $\mathcal{G}_2$ by conjugation on the set of positive definite symmetric linear endomorphisms of $\mathbb{R}^7$. A method leading to the solution of this problem is demonstrated.

In addition, the automorphism groups of the real flexible division algebras are described.

1. Introduction. Let k be a field. A *k-algebra* is understood to be a vector space A over k, endowed with a bilinear multiplication map $A \times A \to A$, $(x, y) \mapsto xy$. The algebra A is said to be a *division algebra* if $A \neq \{0\}$ and the linear endomorphisms $L_a : A \to A$, $x \mapsto ax$, and $R_a : A \to A$, $x \mapsto xa$, are bijective for all $a \in A \setminus \{0\}$. In case A is finite-dimensional, this is equivalent to saying that A has no zero divisors, i.e. $xy = 0$ only if $x = 0$ or $y = 0$. Finite-dimensional real division algebras are known to have dimension either 1, 2, 4 or 8 (Hopf [10], Bott and Milnor [4], Kervaire [12]). All algebras considered in this paper are assumed to be finite-dimensional.

We denote by $\mathcal{A}^{\pm}$ the category of triples $(X, \bullet, [\,,])$, where X is a vector space over k, and $\bullet$ and $[\,,]$ are commutative and anti-commutative algebra structures on X respectively. Morphisms in $\mathcal{A}^{\pm}$ are those linear maps that respect both structures.

Given any k-algebra A, define $[x, y] = xy - yx$ and $x \bullet y = xy + yx$. The assignment $A \mapsto A^{\pm} = (A, \bullet, [\,,])$ defines a functor $?^{\pm}$ from the category of k-algebras to $\mathcal{A}^{\pm}$, acting on morphisms identically. If $\operatorname{char} k \neq 2$, then $?^{\pm}$ is an isomorphism of categories, with inverse $(X, \bullet, [\,,]) \mapsto (X, \mu)$, $\mu(x, y) = \frac{1}{2}(x \bullet y + [x, y])$.

2000 *Mathematics Subject Classification*: 15A21, 17A20, 17A35, 17A36, 17A45.

Key words and phrases: real division algebra, flexible algebra, scalar isotope, generalised pseudo-octonion, automorphism, group action.

Let $\mathcal{K}$ be a category, and $X, Y \in \mathcal{K}$. Then $[X]$ denotes the isomorphism class of X, and $\mathrm{Iso}(X,Y)$ denotes the set of isomorphisms from X to Y. A *cross-section* for $\mathcal{K}/\cong$ is a set $\mathcal{C} \subset \mathcal{K}$ with the property that for all $X \in \mathcal{K}$ there exists a unique $C \in \mathcal{C}$ such that $X \cong C$. A *classification* of $\mathcal{K}$ is an explicit description of a cross-section for $\mathcal{K}/\cong$.

Given a set X, $\mathbb{I} = \mathbb{I}_X$ denotes the identity map on X. We write $\mathbb{I}_n$ for the identity matrix of size $n \times n$. Given a linear operator T on a vector space V over k, and $\lambda \in k$, we write $\mathcal{E}_\lambda = \mathcal{E}_\lambda(T) = \ker(T - \lambda\mathbb{I}_V)$. If V is a Euclidean vector space, $\mathrm{Pds}(V)$ denotes the set of positive definite symmetric endomorphisms of V.

An algebra is called *flexible* if $x(yx) = (xy)x$ for all x and y. A classification of the real flexible division algebras will in a natural way generalise the famous theorems by Frobenius [9] and Zorn [17] stating that the associative and alternative (1) real division algebras are classified by the sets $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ respectively.

Recall that an algebra A is *quadratic* if it has an identity element $1 \neq 0$, and the set $\{1, x, x^2\}$ is linearly dependent for all $x \in A$. It is known that a real division algebra is quadratic if and only if it is power associative (2) (Dieterich [6]). Hence, in particular every real alternative division algebra is also quadratic.

From now on, unless otherwise stated, all algebras and vector spaces are assumed to be real.

In any quadratic algebra B, the subset

$$\mathrm{Im}\, B = \{b \in B \setminus \mathbb{R}1 \mid b^2 \in \mathbb{R}1\} \cup \{0\} \subset B$$

of purely imaginary elements is a linear subspace of B, and $B = \mathbb{R}1 \oplus \mathrm{Im}\, B$ (Frobenius [13]). We shall write $\alpha + v$ instead of $\alpha 1 + v$ when referring to elements in this decomposition.

Let $V = (V, \langle\,\rangle)$ be a finite-dimensional Euclidean space. A *dissident map* on V is a linear map $\eta : V \wedge V \to V$ with the property that the set $\{v, w, \eta(v \wedge w)\}$ is linearly independent whenever $\{v, w\}$ is. A dissident map η on V is called *flexible* if $\langle \eta(x \wedge y), x\rangle = 0$ for all $x, y \in V$. Note that this is equivalent to $\langle \eta(x \wedge y), z\rangle = \langle x, \eta(y \wedge z)\rangle$ for all $x, y, z \in V$.

A *dissident triple* (V, ξ, η) consists of a finite-dimensional Euclidean space V, a linear form $\xi : V \wedge V \to \mathbb{R}$ and a dissident map $\eta : V \wedge V \to V$. The class of dissident triples is given the structure of a category, denoted $\mathcal{D}$, by declaring as morphisms $(V, \xi, \eta) \to (V', \xi', \eta')$ those isometric (3) linear maps $\sigma : V \to V'$ which satisfy both $\xi = \xi'(\sigma \wedge \sigma)$ and $\sigma\eta = \eta'(\sigma \wedge \sigma)$.

(1) An algebra is *alternative* if any subalgebra generated by two elements is associative.

(2) That is, every subalgebra generated by a single element is associative.

(3) That is, satisfying $\langle x, y\rangle = \langle \sigma(x), \sigma(y)\rangle$ for all x and y.

Each dissident triple $(V,\xi,\eta)\in\mathcal{D}$ determines a quadratic division algebra $\mathcal{H}(V,\xi,\eta)=\mathbb{R}\times V$ with multiplication

$$(\alpha,v)(\beta,w)=(\alpha\beta-\langle v,w\rangle+\xi(v\wedge w),\alpha w+\beta v+\eta(v\wedge w)).$$

Defining $(\mathcal{H}\sigma)(\alpha,v)=(\alpha,\sigma(v))$ for morphisms $\sigma:(V,\xi,\eta)\to(V,\xi',\eta')$ makes $\mathcal{H}$ a functor from $\mathcal{D}$ to the category $\mathcal{Q}$ of quadratic division algebras ([4]).

Conversely, given a quadratic division algebra $B=\mathbb{R}1\oplus\operatorname{Im}B$, define linear maps $\varrho:B\to\mathbb{R}$ and $\iota:B\to\operatorname{Im}B$ such that $b=\varrho(b)1+\iota(b)$ for any $b\in B$. Given $x,y\in\operatorname{Im}B$, set $\langle x,y\rangle=-\frac{1}{2}\varrho(xy+yx)$, $\xi(x\wedge y)=\frac{1}{2}\varrho(xy-yx)$ and $\eta(x\wedge y)=\iota(xy)$. Now Osborn's theorem [14, p. 204] asserts that $V=(\operatorname{Im}B,\langle\,\rangle)$ is a Euclidean space, and that η is a dissident map on V. Therefore, the assignments $\mathcal{I}(B)=(V,\xi,\eta)$ and $\mathcal{I}\varphi:\operatorname{Im}B\to\operatorname{Im}C$, $(\mathcal{I}\varphi)(x)=\varphi(x)$ for $\varphi:B\to C$ define a functor $\mathcal{I}:\mathcal{Q}\to\mathcal{D}$.

The following proposition summarises the main results by Osborn ([14], cf. also [7]) in the language of categories.

Proposition 1.1. (i) *The functor $\mathcal{H}:\mathcal{D}\to\mathcal{Q}$ is an equivalence of categories, with quasi-inverse $\mathcal{I}:\mathcal{Q}\to\mathcal{D}$.*

(ii) *The quadratic division algebra $\mathcal{H}(V,\xi,\eta)$ is flexible if and only if $\xi=0$ and the dissident map η is flexible.*

The categories of flexible quadratic division algebras and the corresponding dissident triples will be denoted by $\mathcal{Q}^{\mathrm{fl}}$ and $\mathcal{D}^{\mathrm{fl}}$ respectively. We shall write only η as an abbreviation for $(V,0,\eta)\in\mathcal{D}^{\mathrm{fl}}$, and refer to $\mathcal{D}^{\mathrm{fl}}$ as the category of flexible dissident maps.

A *vector product* π on a Euclidean space V is a flexible dissident map on V with the additional property that if $u,v\in V$ is an orthonormal pair, then $\|\pi(u\wedge v)\|=1$. Vector products correspond to alternative division algebras under the functor $\mathcal{H}$. Hence there exist unique (up to isomorphism) vector products in dimensions 0, 1, 3 and 7 respectively. The automorphism group of the 7-dimensional vector product is the exceptional Lie group $\mathcal{G}_2$.

Let B be a quadratic algebra, and λ a nonzero real number. Then the *scalar isotope of B determined by* λ, denoted ${}_\lambda B=(B,\star)$, is defined by

$$(\alpha+v)\star(\beta+w)=(\alpha+\lambda v)(\beta+\lambda w),\qquad \alpha,\beta\in\mathbb{R},\ v,w\in\operatorname{Im}B.$$

Now ${}_\lambda B$ is flexible if B is flexible [2], and it is obviously a division algebra whenever B is.

Next, recall that $\mathfrak{su}_n\mathbb{C}$ denotes the simple real Lie algebra of $n\times n$ complex anti-hermitean matrices of trace zero, equipped with the anticommutative algebra structure $[\,,]$. For each $\delta\in\mathbb{R}\smallsetminus\{0\}$, the vector space

([4]) Morphisms of quadratic algebras are assumed to preserve the identity element.

$\mathfrak{su}_3\mathbb{C}$ with the multiplication

$$x * y = \delta[x, y] + \frac{i}{2}\left(xy + yx - \frac{2}{3}\operatorname{tr}(xy)\mathbb{I}_3\right) \tag{1}$$

is a flexible division algebra [3] of dimension 8, which we denote by O_δ. Benkart and Osborn call these algebras *generalised pseudo-octonions*, or *GP-algebras.*

We now recall the main theorem in [2] by Benkart, Britten and Osborn.

THEOREM 1.2 ([2, Theorem 1,4]). *If A is a finite-dimensional real algebra, then A is a flexible division algebra if and only if A has one of the following forms*:

(i) *A is a commutative division algebra of dimension* 1 *or* 2,
(ii) *A is isomorphic to a scalar isotope ${}_\lambda B$ of some quadratic real division algebra B which is flexible*,
(iii) *A is a generalised pseudo-octonion algebra.*

Although not explicitly stated in the theorem, it is proved in [2] that no algebra could meet more than one of the above conditions (i)–(iii).

A classification of all two-dimensional division algebras, and hence in particular of all commutative ones, is given in [11]. In Section 2 we give an alternative solution, based on the approach of Althoen and Kugler [1].

It is easily shown, using Proposition 1.1, that the set $\{\mathcal{H}(\lambda\pi_3)\}_{\lambda>0}$, i.e. $\mathbb{R} \times \mathbb{R}^3$ with the multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle, \alpha w + \beta v + \lambda\pi_3(v \wedge w)),$$

classifies the flexible quadratic division algebras of dimension 4. Here π_3 denotes a chosen vector product on $\mathbb{R}^3$.

In [5], Cuenca Mira *et al.* introduce vectorial isotopy, a method by which all flexible quadratic division algebras are constructed. Using the language of dissident maps, their main result can be formulated as follows ([5]).

PROPOSITION 1.3. *Let $\pi : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ be a vector product, and η any flexible dissident map on a Euclidean space V of dimension* 7. *Then*

(i) *For any $\varepsilon \in \mathrm{GL}(V)$, $\varepsilon^*\eta(\varepsilon \wedge \varepsilon)$ is a flexible dissident map.*
(ii) *$\eta \cong \delta^*\pi(\delta \wedge \delta) = \delta\pi(\delta \wedge \delta)$ for some $\delta \in \mathrm{Pds}(\mathbb{R}^7)$.*
(iii) *For $\delta_1, \delta_2 \in \mathrm{Pds}(\mathbb{R}^7)$, $\delta_1\pi(\delta_1 \wedge \delta_1) \cong \delta_2\pi(\delta_2 \wedge \delta_2)$ if and only if $\delta_1 = \sigma^{-1}\delta_2\sigma$ for some $\sigma \in \mathrm{Aut}(\pi)$.*

Hence, classifying the flexible dissident maps in 7-dimensional Euclidean space, and thereby the flexible quadratic division algebras of dimension 8,

([5]) By * we denote the adjoint operator.

completely and irredundantly, is equivalent to solving the normal form problem for the right group action

$$\mathrm{Pds}(\mathbb{R}^7) \times \mathcal{G}_2 \to \mathrm{Pds}(\mathbb{R}^7), \qquad (\delta, \sigma) \mapsto \delta \cdot \sigma = \sigma^{-1}\delta\sigma \tag{2}$$

The principal theorem of the present article, Theorem 1.4, classifies the category of all flexible division algebras up to the above normal form problem.

We denote by $e_1 = (1, 0)$ and $e_2 = (0, 1)$ the standard basis vectors in $\mathbb{R}^2$. We define the algebra $A_E(a, b)$ as the vector space $\mathbb{R}^2$ with multiplication in the standard basis given by Table 1, and $A_F(a, b)$ as $\mathbb{R}^2$ with multiplication given by Table 2.

Table 1. $A_E(a, b)$

$\cdot$	e_1	e_2
e_1	e_1	$ae_1 + be_2$
e_2	$ae_1 + be_2$	$-e_1$

Table 2. $A_F(a, b)$

$\cdot$	e_1	e_2
e_1	e_1	$ae_1 + be_2$
e_2	$ae_1 + be_2$	e_2

THEOREM 1.4. *Let $\mathcal{N} \in \mathrm{Pds}(\mathbb{R}^7)$ be a cross-section for $\mathrm{Pds}(\mathbb{R}^7)/\mathcal{G}_2$, the orbit set of the group action (2). Then the set*

$$\begin{aligned}
\{\mathbb{R}\} &\,\dot\cup\, \{A_E(a,b) \mid a > 0,\ b > (a^2+1)/2 \text{ or } a = 0,\ b \geq 1/2\} \\
&\dot\cup\, \{A_F(a,b) \mid a \geq 1/2;\ b \geq a;\ (a,b) \neq (1/2, 1/2)\} \\
&\dot\cup\, \{A_F(a,b) \mid a, b < 0;\ a \leq b \leq 1/2a - 1/2\} \\
&\dot\cup\, \{{}_\lambda\mathcal{H}(\mu\pi_3) \mid (\lambda, \mu) \in (\mathbb{R} \smallsetminus \{0\}) \times \mathbb{R}_{>0}\} \\
&\dot\cup\, \{{}_\lambda\mathcal{H}(\delta\pi_7(\delta \wedge \delta)) \mid (\lambda, \delta) \in (\mathbb{R} \smallsetminus \{0\}) \times \mathcal{N}\} \\
&\dot\cup\, \{O_\delta\}_{\delta > 0}
\end{aligned}$$

classifies the finite-dimensional real flexible division algebras.

The proof of Theorem 1.4 is given in Sections 2–3. The commutative division algebras are classified in Section 2. In Section 3, we show that ${}_\lambda A \cong {}_\mu B$ if and only if $\lambda = \mu$ and $A \cong B$ for quadratic division algebras A and B of dimension greater than 2. We also solve the irredundancy problem for the generalised pseudo-octonions by showing that $O_\gamma \cong O_\delta$ if and only if $\gamma = \pm\delta$. A method for finding a cross-section for $\mathrm{Pds}(\mathbb{R}^7)/\mathcal{G}_2$ is demonstrated in Section 5. The complete solution to this problem, which is rather technical, is postponed to a forthcoming publication.

As an additional information, we describe in Section 4 the automorphism groups of the flexible division algebras.

2. Commutative division algebras. In this section, we complete the classification of the commutative division algebras, based on the approach of Althoen and Kugler [1].

We shall use the following result by Segre.

Proposition 2.1 ([16, Theorem 1]). *Every finite-dimensional real or complex algebra with the property that $x^2 = 0$ implies $x = 0$ has at least one idempotent* ([6]).

The same result is proved in [1] for real division algebras of dimension 2.

Accordingly, if A is a real commutative division algebra of dimension 2, then it has a basis (u, v) such that the multiplication is given by Table 3, where $a, b, c, d \in \mathbb{R}$.

Table 3. Commutative division algebras

$\cdot$	u	v
u	u	$au + bv$
v	$au + bv$	$cu + dv$

Proposition 2.2 ([1, Theorem 3]). *An algebra given by Table 3 is a division algebra if and only if*

$$d^2 < 4b \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

First, we consider the case when A has exactly one idempotent.

Lemma 2.3. *Let A be a commutative division algebra of dimension 2, and $u \in A$ an idempotent. Then there exists an element $w \in A$ such that $w^2 = -u$.*

Proof. Let (u, v) be a basis for A such that the multiplication is given by Table 3. If $d = 0$, Proposition 2.2 implies that $4b^2c < 0$ and hence $c < 0$. Set $w = (1/\sqrt{-c})v$. Now suppose $d \neq 0$. Then, for an arbitrary element $w = \alpha u + \beta v$, $\beta \neq 0$, in $A \setminus \operatorname{span}\{u\}$ we have

$$w^2 = (\alpha u+\beta v)^2 = \alpha^2u^2+2\alpha\beta uv+\beta^2v^2 = (\alpha^2+2\alpha\beta a+\beta^2c)u+(2\alpha\beta b+\beta^2d)v.$$

Hence $w^2 = -u$ if and only if

$$\begin{cases} \alpha^2 + 2\alpha\beta a + \beta^2 c = -1, \\ 2\alpha\beta b + \beta^2 d = 0. \end{cases}$$

Using Proposition 2.2, we see that

$$\alpha = \sqrt{\frac{1}{4ab/d - 4b^2c/d^2 - 1}}, \qquad \beta = -2\alpha b/d$$

solves the system. ∎

The problem of determining the number of idempotents in a two-dimensional division algebra is considered in [1]. However, the proposition on page

([6]) Idempotents are understood to be non-zero.

629 there is stated incorrectly. An accurate version is given in [8, Prop. 3, p. 5], from which the following lemma is an immediate consequence.

LEMMA 2.4. *An algebra determined by Table* 3 *has exactly one idempotent if and only if either* $(2a-d)^2 < 4c(1-2b)$ *or* $d = 2a$, $b = 1/2$.

Taking into account Proposition 2.2, Lemma 2.3 and Lemma 2.4, we conclude that the basis (u, v) can be chosen in such a way that $c = -1$, $d = 0$ and either $a^2 + 1 < 2b$ or $a = 0$, $b = 1/2$. Moreover, if an algebra has such a basis, it is a commutative division algebra with exactly one idempotent.

Recall that $A_E(a, b)$ denotes $\mathbb{R}^2$ with multiplication given by Table 1.

PROPOSITION 2.5. (i) *The set* $\mathcal{E} = \{A_E(a,b) \mid a > 0,\ b > (a^2+1)/2$ *or* $a = 0,\ b \geq 1/2\}$ *classifies the two-dimensional commutative division algebras having exactly one idempotent.*

(ii) *Let* $a, b \in \mathbb{R}$, *and suppose that the algebra* $A_E(a,b)$ *has exactly one idempotent. Then* $\mathrm{Aut}(A_E(a,b)) = \{\mathbb{I}, (e_1, e_2) \mapsto (e_1, -e_2)\}$ *if* $a = 0$, *and* $\mathrm{Aut}(A_E(a,b)) = \{\mathbb{I}\}$ *otherwise.*

Proof. Suppose that $A = A_E(a,b)$ and $B = A_E(a',b')$ each have precisely one idempotent and let $\varphi : A \to B$ be an isomorphism. Then $\varphi(e_1) = e_1$. We have

$$(\varphi(e_2))^2 = \varphi(e_2^2) = -\varphi(e_1) = -e_1 = e_2^2,$$

so

$$0 = (\varphi(e_2))^2 - e_2^2 = (\varphi(e_2) - e_2)(\varphi(e_2) + e_2)$$

and hence $\varphi(e_2) = \pm e_2$.

If $\varphi(e_2) = e_2$, then $\varphi = \mathbb{I}$, and $(a', b') = (a, b)$.

If $\varphi(e_2) = -e_2$, then because

$$\varphi(e_1 e_2) = \varphi(a e_1 + b e_2) = a e_1 - b e_2,$$
$$\varphi(e_1)\varphi(e_2) = -e_1 e_2 = -a' e_1 - b' e_2,$$

it follows that $(a', b') = (-a, b)$. Conversely, it is clear that $A_E(a,b) \to A_E(-a,b)$, $(e_1, e_2) \mapsto (e_1, -e_2)$, is an isomorphism. In particular, assuming A and B to be division algebras, this yields the first statement of the proposition.

From the above it is also clear that $(e_1, e_2) \mapsto (e_1, -e_2)$ is an automorphism of $A_E(a,b)$ if and only if $a = 0$, which gives the second statement. ∎

Now, we consider the several idempotents case. Since all idempotents must be pairwise non-proportional, every algebra with at least two idempotents has a basis consisting of idempotents. Thus, in view of Proposition 2.2, every commutative division algebra of this type is isomorphic to $A_F(a,b)$ for some $(a,b) \in K = \{(a,b) \in \mathbb{R}^2 \mid ab > 1/4\}$.

LEMMA 2.6 (see also [1, pp. 629–630]). *The algebra $A_F(a,b)$, $(a,b) \in K$, has precisely two idempotents if $a = 1/2$ or $b = 1/2$, and precisely three idempotents otherwise.*

Proof. From Table 2 it is clear that the basis elements e_1 and e_2 are idempotents. Let $w = \alpha e_1 + \beta e_2$ be an arbitrary element in $A_F(a,b)$. We want to find solutions of the equation $w^2 = w$ for which $\alpha, \beta \neq 0$. Under this assumption,

$$w^2 = w \;\Leftrightarrow\; \begin{pmatrix} 1 & 2a \\ 2b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As $(a,b) \in K$ and therefore $1-4ab \neq 0$, the equation has the unique solution

$$(\alpha, \beta) = \left(\frac{2a-1}{4ab-1}, \frac{2b-1}{4ab-1} \right)$$

where $\alpha, \beta \neq 0$ precisely when $a, b \neq 1/2$. ∎

Define $K_n = \{(a,b) \in K \mid A_F(a,b) \text{ has } n \text{ idempotents}\}$. Any isomorphism of algebras A and B induces a bijection between their sets of idempotents. Conversely, if $A_F(a,b)$ with $(a,b) \in K_n$ has idempotents $u_1, \ldots, u_n$, where $(u_1, u_2) = (e_1, e_2)$, then every permutation $\sigma \in S_n$ gives rise to a linear automorphism $\varphi_\sigma : u_i \mapsto u_{\sigma(i)}$, $i \in \{1, 2\}$, of $A_F(a,b)$. The assignment $\sigma \mapsto \varphi_\sigma$ is injective. For $n \in \{2, 3\}$, we define a group action $K_n \times S_n \to K_n$ by

$$(a,b) \cdot \sigma = (a', b') \;\Leftrightarrow\; \varphi_\sigma : A_F(a,b) \to A_F(a',b') \text{ is an algebra isomorphism.}$$

Now the category of this group action is equivalent to the category of 2-dimensional commutative division algebras with precisely n idempotents. The equivalence is given by the maps $(a,b) \mapsto A_F(a,b)$ and $\sigma \mapsto \varphi_\sigma$.

This means that $x, y \in K_n$ parametrise isomorphic algebras if and only if they lie in the same orbit under S_n. Automorphisms of the algebra $A_F(x)$, $x \in K_n$, are those φ_σ for which σ stabilises x. Classifying the commutative division algebras having more than one idempotent is equivalent to finding normal forms for K_2 and K_3 under the actions of S_2 and S_3 respectively.

It is easily seen that the orbits in $K_2 = \{(a,b) \in K \mid a = 1/2 \text{ or } b = 1/2\}$ are the pairs of the form $\{(a,b), (b,a)\}$.

S_3 is generated by the the permutations $(1,2)(3)$ and $(1)(2,3)$. The first one sends each $(a,b) \in K_3 = K \setminus K_2$ to (b,a).

Consider the function

$$f_y(x) = \frac{x+y-1}{4xy-1}.$$

We see that $(a,b) \mapsto (f_b(a), b)$. Since

$$f_y'(x) = -\frac{(2y-1)^2}{(4xy-1)^2},$$

f is decreasing for $y \neq 1/2$. Moreover, $f_y(x) = x \Leftrightarrow x = (1 \pm (2y-1))/4y$, that is, $x = 1/2$ or $x = (1-y)/2y$. This shows that the subset

$$\left\{ (a,b) \in K_3 \,\middle|\, a,b > \frac{1}{2} \text{ or } a < \frac{1-b}{2b},\ b < \frac{1}{2} \right\}$$

of K_3 is mapped to

$$\left\{ (a,b) \in K_3 \,\middle|\, b > \frac{1}{2} > a \text{ or } a > \frac{1-b}{2b},\ b < \frac{1}{2} \right\}$$

and that $\{(a,b) \in K_3 \mid a > (1-b)/2b,\ b < 1/2\}$ is fixed by the permutation $(1)(2,3)$.

Using the fact that any other element in S_3 can be written as a product of these two permutations, it is straightforward to prove the following two propositions.

Proposition 2.7. (i) *The set* $\{A_F(1/2, b)\}_{b>1/2}$ *classifies the commutative division algebras having two idempotents.*

(ii) *The set*

$$\left\{ A_F(a,b) \,\middle|\, a > \frac{1}{2},\ b \geq a \right\} \cup \left\{ A_F(a,b) \,\middle|\, a,b < 0,\ a \leq b \leq \frac{1}{2a} - \frac{1}{2} \right\}$$

classifies the commutative division algebras having three idempotents.

The first part of Proposition 2.7 can also be derived from Theorem 3 and 5 of [1].

Proposition 2.8. (i) *If* $(a,b) \in K_2$, *then* $\mathrm{Aut}(A_F(a,b)) = \{\mathbb{I}\}$.

(ii) *If* $(a,b) \in K_3$, *then* $\mathrm{Aut}(A_F(a,b)) = \{\varphi_\sigma\}_{\sigma \in S}$, *where* $S \subset S_3$ *is given by*

$$S = \begin{cases} \{\mathbb{I}, (1,2)(3)\} & \textit{if } a = b \neq -1, \\ \{\mathbb{I}, (1)(2,3)\} & \textit{if } a = (1-b)/2b, \\ \{\mathbb{I}, (1,3)(2)\} & \textit{if } b = (1-a)/2a, \\ S_3 & \textit{if } a = b = -1, \\ \{\mathbb{I}\} & \textit{otherwise.} \end{cases}$$

It is worthwhile to note that S_3, the largest automorphism group occurring, corresponds to the algebra $A_F(-1,-1)$, while $\mathbb{C} \cong A_E(0,1)$ shares its automorphism group of order two with infinitely many other pairwise non-isomorphic algebras. This contrasts with the general experience from dimensions four and eight, where $\mathbb{H}$ and $\mathbb{O}$ seem to be the most symmetric objects.

3. Division algebras of dimension 4 and 8. In order to prove Theorem 1.4, we must consider the scalar isotopes of the flexible quadratic division algebras, and the generalised pseudo-octonions. Our result on scalar isotopy actually holds for arbitrary quadratic division algebras.

LEMMA 3.1. *Let* $A \in \mathcal{Q}$, $\dim A > 2$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$. *Then* $Z({}_\lambda A) = \operatorname{span}\{1_A\}$.

Proof. Let $x = \alpha + v \in \mathbb{R}1 \oplus \operatorname{Im} A$. Then $x \in Z({}_\lambda A)$ if and only if in the algebra A, $vw = wv$ for all $w \in \operatorname{Im} A$. From Proposition 1.1, it follows that $vw = wv$ only if v and w are linearly dependent. As $\dim(\operatorname{Im} A) > 1$ we get $v = 0$, and therefore $Z({}_\lambda A) = \operatorname{span}\{1_A\}$. ■

PROPOSITION 3.2. *Let A and B be quadratic division algebras of dimension greater than* 2, *and let* $\lambda, \mu \neq 0$. *Then* ${}_\lambda A \cong {}_\mu B$ *if and only if* $\lambda = \mu$ *and* $A \cong B$. *Moreover*, $\operatorname{Iso}({}_\lambda A, {}_\lambda B) = \operatorname{Iso}(A, B)$.

Proof. By Lemma 3.1, any isomorphism $\varphi : {}_\lambda A \to {}_\mu B$ maps 1_A to 1_B. We have $(\alpha + v) \star (\alpha + v) = \alpha^2 + 2\lambda\alpha v + \lambda^2 v^2$ for $\alpha + v \in \mathbb{R}1 \oplus \operatorname{Im} A$, so $(\alpha+v)\star(\alpha+v) \in \mathbb{R}1$ if and only if either $\alpha = 0$ or $v = 0$. The same obviously holds in ${}_\mu B$. Hence, for any $x \in A \setminus \mathbb{R}1$ we have

$$\varphi(x) \in \operatorname{Im} B \;\Leftrightarrow\; \varphi(x) \star \varphi(x) \in \mathbb{R}1_B \;\Leftrightarrow\; x \star x \in \mathbb{R}1_A \;\Leftrightarrow\; x \in \operatorname{Im} A.$$

Now, let $v, w \in \operatorname{Im} A \setminus \{0\}$. We have $\lambda\varphi(v) = \varphi(1_A \star v) = \varphi(1_A) \star \varphi(v) = \mu\varphi(v)$ and hence $\lambda = \mu$. Moreover, $\mu^2\varphi(v)\varphi(w) = \varphi(v) \star \varphi(w) = \varphi(v \star w) = \lambda^2\varphi(vw) = \mu^2\varphi(vw)$, which implies that $\varphi \in \operatorname{Iso}(A, B)$. An analogous calculation shows that $\operatorname{Iso}(A, B) \subset \operatorname{Iso}({}_\lambda A, {}_\lambda B)$. ■

COROLLARY 3.3. *If A is a quadratic division algebra of dimension* 4 *or* 8, *and* $\lambda \neq 0$, *then* $\operatorname{Aut}({}_\lambda A) = \operatorname{Aut}(A)$.

We now turn our attention to the GP-algebras. First recall the definition of the category $\mathcal{A}^\pm$ and the functor $?^\pm$ (Section 1). Next note that the complex Lie algebra $\mathfrak{sl}_3\mathbb{C}$ is obtained from $\mathfrak{su}_3\mathbb{C}$ by extending scalars to the complex numbers, that is, $\mathfrak{sl}_3\mathbb{C} = \mathfrak{su}_3\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$. Let $S_\delta = (\mathfrak{sl}_3\mathbb{C}, *)$, the multiplication "$*$" being defined as for O_δ. We write $S_\delta^\pm = (S_\delta, \bullet_*, [\,,]_*)$, where $\bullet$ and $[\,,]$ refer to the ordinary matrix multiplication.

The multiplication in the commutative algebra $S^+ = (S_\delta, \bullet_*)$ is independent of δ and is given by $x \bullet_* y = i(x \bullet y - \frac{2}{3}\operatorname{tr}(xy)\mathbb{I})$. The multiplication in $S_\delta^- = (S_\delta, [\,,]_*)$ is given by $[x, y]_* = 2\delta[x, y]$. Given a natural number n and $A \in \mathrm{GL}_n(\mathbb{R})$, we let $\kappa_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $X \mapsto A^{-1}XA$. Note that κ_A is indeed an automorphism of the matrix algebra $\mathbb{R}^{n\times n}$.

PROPOSITION 3.4. *Let* $\delta, \gamma \in \mathbb{R} \setminus \{0\}$. *Then* $O_\delta \cong O_\gamma \Leftrightarrow \delta = \pm\gamma$.

Proof. Let $\varphi : O_\delta \to O_\gamma$ be an isomorphism. Then φ induces an isomorphism $S_\delta \to S_\gamma$, also denoted by φ. Using the functor $?^\pm$, we see that $\varphi \in \mathrm{Iso}(S_\delta^-, S_\gamma^-)$ and $\varphi \in \mathrm{Aut}(S^+)$.

The homothety $h_{2\delta} : S_\delta^- \to \mathfrak{sl}_3\mathbb{C}$, $x \mapsto 2\delta x$, is an isomorphism of algebras. Hence $\mathrm{Aut}(\mathfrak{sl}_3\mathbb{C}) \to \mathrm{Iso}(S_\delta^-, S_\gamma^-)$, $\psi \mapsto \varphi = h_{2\gamma}^{-1}\psi h_{2\delta} = (\delta/\gamma)\psi$ is a bijection. We have $\mathrm{Aut}(\mathfrak{sl}_3\mathbb{C}) = \{\kappa_A, -\kappa_A(?^t) \mid A \in \mathrm{SL}_3(\mathbb{C})\}$.

If $\psi = \kappa_A$, then $\varphi = (\delta/\gamma)\kappa_A$. Hence

$$\begin{aligned}
\varphi(x) \bullet_* \varphi(y) &= \frac{\delta}{\gamma}\,\kappa_A(x) \bullet_* \frac{\delta}{\gamma}\,\kappa_A(y) \\
&= \frac{\delta^2}{\gamma^2}\, i\Big(\kappa_A(x) \bullet \kappa_A(y) - \frac{2}{3}\,\mathrm{tr}(\kappa_A(x)\kappa_A(y))\mathbb{I}\Big) \\
&= \frac{\delta^2}{\gamma^2}\, i\Big(\kappa_A(x{\bullet}y) - \frac{2}{3}\,\mathrm{tr}(xy)\kappa_A(\mathbb{I})\Big) = \frac{\delta^2}{\gamma^2}\,\kappa_A(x \bullet_* y), \\
\varphi(x \bullet_* y) &= \frac{\delta}{\gamma}\,\kappa_A(x \bullet_* y).
\end{aligned}$$

Because $\varphi \in \mathrm{Aut}(S^+)$, the relation $(\delta^2/\gamma^2)\kappa_A(x\bullet_* y) = (\delta/\gamma)\kappa_A(x\bullet_* y)$ must hold for all $x, y \in S^+$, which implies $\delta = \gamma$.

If $\psi = -\kappa_A(?^t)$, then $\varphi = -(\delta/\gamma)\kappa_A(?^t)$, and

$$\begin{aligned}
\varphi(x \bullet_* y) &= -\frac{\delta}{\gamma}\,\kappa_A\Big(i\Big(x{\bullet}y - \frac{2}{3}\,\mathrm{tr}(xy)\mathbb{I}\Big)\Big)^t = -\frac{\delta}{\gamma}\,\kappa_A\Big(i\Big(x^t{\bullet}y^t - \frac{2}{3}\,\mathrm{tr}(xy)\mathbb{I}\Big)\Big) \\
&= -\frac{\delta}{\gamma}\,\kappa_A\Big(i\Big(x^t{\bullet}y^t - \frac{2}{3}\,\mathrm{tr}(x^ty^t)\mathbb{I}\Big)\Big) = -\frac{\delta}{\gamma}\,\kappa_A(x^t \bullet_* y^t), \\
\varphi(x) \bullet_* \varphi(y) &= -\frac{\delta}{\gamma}\,\kappa_A(x^t) \bullet_* \Big(-\frac{\delta}{\gamma}\Big)\kappa_A(y^t) = \frac{\delta^2}{\gamma^2}\,\kappa_A(x^t) \bullet_* \kappa_A(y^t) \\
&= \frac{\delta^2}{\gamma^2}\,\kappa_A(x^t \bullet_* y^t).
\end{aligned}$$

Here, $\varphi \in \mathrm{Aut}(S^+)$ implies $\delta = -\gamma$.

We have seen that $O_\delta \cong O_\gamma$ only if $\delta = \pm\gamma$. Conversely, it is clear that $?^t : O_\delta \to O_{-\delta}$, $x \mapsto x^t$, is an isomorphism. ■

As a consequence of the proof, we get the following corollary.

COROLLARY 3.5. (i) $\mathrm{Aut}(O_\delta) = \{\kappa_A \mid A \in \mathrm{SU}_3\}$,
(ii) $\mathrm{Iso}(O_\delta, O_{-\delta}) = \{\kappa_A \circ (?^t) \mid A \in \mathrm{SU}_3\}$.

4. Automorphism groups of flexible division algebras. In order to describe the automorphism groups of all flexible division algebras, it remains to consider the quadratic ones.

Firstly, let π_3 be a vector product on $\mathbb{R}^3$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\eta = \lambda\pi$. For $\sigma \in \mathrm{O}_3(\mathbb{R})$ we now have

$$\sigma \in \mathrm{Aut}(\eta) \;\Leftrightarrow\; \sigma\eta = \eta(\sigma \wedge \sigma) \;\Leftrightarrow\; \lambda\sigma\pi_3 = \lambda\pi_3(\sigma \wedge \sigma) \;\Leftrightarrow\; \sigma \in \mathrm{Aut}(\pi_3).$$

Hence, by Proposition 1.1, $\mathrm{Aut}(\mathcal{H}(\eta)) \cong \mathrm{Aut}(\mathbb{H}) \cong \mathrm{SO}_3(\mathbb{R})$.

Secondly, consider $\eta = \delta\pi(\delta \wedge \delta)$, where $\delta \in \mathrm{Pds}(\mathbb{R}^7)$ and π is a vector product on $\mathbb{R}^7$. Let $\sigma \in \mathrm{O}_7(\mathbb{R})$. As $\delta = \delta^*$, we get

$$\begin{aligned}
\sigma \in \mathrm{Aut}(\eta) \;&\Leftrightarrow\; \sigma\eta = \eta(\sigma \wedge \sigma)\\
&\Leftrightarrow\; \sigma\delta\pi(\delta \wedge \delta) = \delta\pi((\delta\sigma) \wedge (\delta\sigma))\\
&\Leftrightarrow\; \delta\pi(\delta \wedge \delta) = (\delta\sigma)^*\pi((\delta\sigma) \wedge (\delta\sigma))\\
&\Leftrightarrow\; \pi = (\delta\sigma\delta^{-1})^*\pi((\delta\sigma\delta^{-1}) \wedge (\delta\sigma\delta^{-1})) \;\Leftrightarrow\; \delta\sigma\delta^{-1} \in \mathrm{Aut}(\pi).
\end{aligned}$$

Hence $\mathrm{Aut}(\eta) = \delta^{-1}\,\mathrm{Aut}(\pi)\delta \cap \mathrm{O}_7(\mathbb{R})$.

Lemma 4.1. *With the above notation,* $\delta^{-1}\,\mathrm{Aut}(\pi)\delta \cap \mathrm{O}_7(\mathbb{R}) = C_{\mathrm{Aut}(\pi)}(\delta)$.

Proof. Let $\underline{v} = \{v_1, \dots, v_7\}$ be an orthonormal eigenbasis for $\mathbb{R}^7$ with respect to δ, with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_7$. Let $f \in \mathrm{Aut}(\pi)$ be such that $\delta^{-1}f\delta \in \mathrm{O}_7(\mathbb{R})$; write $[f]_{\underline{v}} = (f^i_j)_{i,j}$. Then

$$1 = \|(\delta^{-1}f\delta)v_1\| = \|\lambda_1(\delta^{-1}f)v_1\| = \left\|\lambda_1 \sum_{i=1}^{7} \frac{1}{\lambda_i} f^i_1 v_i\right\| \;\Rightarrow\; fv_1 \in \mathcal{E}_{\lambda_1}(\delta).$$

Suppose $fv_k \in \mathcal{E}_{\lambda_k}(\delta)$ for all $k \leq n$. Then $fv_{n+1} \in [fv_1, \dots, fv_n]^\perp$, and thus $fv_{n+1} \in \mathcal{E}_{\lambda_{n+1}}(\delta)$. By induction, all eigenspaces of δ are invariant under f. Therefore, $f \in C_{\mathrm{Aut}(\pi)}(\delta)$, and $\delta^{-1}\,\mathrm{Aut}(\pi)\delta \cap \mathrm{O}_7(\mathbb{R}) \subset \delta^{-1}C_{\mathrm{Aut}(\pi)}(\delta)\delta \cap \mathrm{O}_7(\mathbb{R}) = C_{\mathrm{Aut}(\pi)}(\delta)$. The converse is trivial. ∎

Summarising Proposition 2.5, Proposition 2.8, Corollary 3.3, Corollary 3.5 and the above considerations, we obtain the following result concerning the automorphisms of flexible division algebras:

Theorem 4.2. (i) *Let $A = A_E(a,b)$ be a commutative division algebra with exactly one idempotent. Then* $\mathrm{Aut}(A) = \{\mathbb{I}, (e_1, e_2) \mapsto (e_1, -e_2)\}$ *if* $a = 0$, *and* $\mathrm{Aut}(A) = \{\mathbb{I}\}$ *otherwise.*

(ii) *If* $(a,b) \in K_2 = \{(a,b) \in \mathbb{R}^2 \mid b > a = 1/2$ *or* $a > b = 1/2\}$, *then* $\mathrm{Aut}(A_F(a,b)) = \{\mathbb{I}\}$.

(iii) *If* $(a,b) \in K_3 = \{(a,b) \in \mathbb{R}^2 \mid ab > 1/4,\ (a,b) \notin K_2\}$, *then* $\mathrm{Aut}(A_F(a,b)) = \{\varphi_\sigma\}_{\sigma \in S}$, *where $S \subset S_3$ is given by*

$$S = \begin{cases} \{\mathbb{I}, (1,2)(3)\} & \text{if } a = b \neq -1, \\ \{\mathbb{I}, (1)(2,3)\} & \text{if } a = (1-b)/2b, \\ \{\mathbb{I}, (1,3)(2)\} & \text{if } b = (1-a)/2a, \\ S_3 & \text{if } a = b = -1, \\ \{\mathbb{I}\} & \text{otherwise.} \end{cases}$$

(iv) $\mathrm{Aut}({}_\lambda A) \cong \mathrm{SO}_3(\mathbb{R})$ *if* $\lambda \neq 0$ *and* A *is a flexible quadratic division algebra of dimension* 4.

(v) $\mathrm{Aut}({}_\lambda A) \cong C_{\mathrm{Aut}(\pi)}(\delta)$ *if* $\lambda \neq 0$ *and* $A \cong \mathcal{H}(\delta^*\pi(\delta \wedge \delta))$, *where* $\delta \in \mathrm{Pds}(\mathbb{R}^7)$ *and* π *is a vector product on* $\mathbb{R}^7$.

(vi) $\mathrm{Aut}(O_\gamma) = \{\kappa_A \mid A \in \mathrm{SU}_3(\mathbb{R})\}$ *if* $\gamma \neq 0$.

5. The $\mathcal{G}_2$-action on $\mathrm{Pds}(\mathbb{R}^7)$**.** In this section, V denotes a fixed 7-dimensional Euclidean space, equipped with a vector product π. To abbreviate notation, we write xy instead of $\pi(x \wedge y)$.

Our approach to the normal form problem for the group action (2) is based on a handy description of the group $\mathcal{G}_2 = \mathrm{Aut}(\pi)$.

LEMMA 5.1. *Let* $u, v \in V$ *be orthonormal vectors. Then the following identities hold*:

(i) $u(uv) = -v$,
(ii) $v(uv) = u$.

In particular, π *induces a vector product on* $\mathrm{span}\{u, v, uv\}$. *If in addition* $z \in V$ *is a unit vector orthogonal to* u *and* v, *then*

(iii) $u(vz) = -(uv)z = (vu)z$.

Proof. Since π is in particular a flexible dissident map, we have $\langle ux, x\rangle = 0$ for all $x \in V$. If $x \in u^\perp$, then $\|ux\| = \|x\|$. This means that the linear map $L_u : u^\perp \to u^\perp$ is an isometry ([7]). Thus,

$$\langle x, u(uv)\rangle = \langle xu, uv\rangle = \langle ux, u(-v)\rangle = \langle x, -v\rangle$$

for all $x \in V$. The second identity follows from the first via anti-commutativity of π.

By (i), we also have $u(uz) = -z$. Moreover,

$$\begin{aligned} -2v = -\|u+z\|^2 v &= (u+z)((u+z)v) = (u+z)(uv+zv) \\ &= u(uv) + u(zv) + z(uv) + z(zv) \\ &= -v + u(zv) + z(uv) - v = -2v - u(vz) - (uv)z. \end{aligned}$$

Thus $u(vz) = -(uv)z$. ∎

([7]) Multiplication is considered in the algebra (V, π), that is, $L_u(x) = ux = \pi(u \wedge x)$.

A triple $(u, v, z) \in V^3$ is called a *Cayley triple* in V if $\{u, v, uv, z\}$ is an orthonormal set. We denote by $\mathcal{C}$ the set of all Cayley triples in V.

Given a Cayley triple $(u, v, z) \in \mathcal{C}$, let $U = \operatorname{span}\{u, v, uv\}$. For any $x, y \in U$, we have $\langle x, yz\rangle = \langle xy, z\rangle = 0$. Hence the vector space V decomposes into an orthogonal direct sum $V = U \oplus \operatorname{span}\{z\} \oplus Uz$. Because (by the same argument as for L_u in the proof of Lemma 5.1) R_z is an isometry on $z^{\perp}$, $(uz, vz, (uv)z)$ will be an orthonormal basis for Uz. Altogether this means that every Cayley triple $c = (u, v, z)$ determines an orthonormal basis $\underline{b}_c = (u, v, uv, z, uz, vz, (uv)z)$ for V.

Proposition 5.2 (cf. also [15, 11.16]). *The group $\mathcal{G}_2$ acts simply transitively on $\mathcal{C}$ by $g\cdot(u, v, z) = (g(u), g(v), g(z))$.*

Proof. Clearly, the above expression defines a group action. As every Cayley triple $c \in \mathcal{C}$ determines a basis $\underline{b}_c$ for V, and $\underline{b}_c \neq \underline{b}_{c'}$ if $c \neq c'$, the stabiliser of any $c \in \mathcal{C}$ is trivial.

For transitivity, we must show that the bases given by any two Cayley triples have the same multiplication table. Note that any permutation of a Cayley triple $c = (u, v, z)$ is again a Cayley triple, and that $(x, y, z), (xz, y, z) \in \mathcal{C}$ for all orthonormal pairs $x, y \in U$. Therefore, $(xz)z = z(zx) = -x$ and $(xz)(yz) = (y(xz))z = ((xy)z)z = -xy$. Using this, and Lemma 5.1, one readily constructs the multiplication table for $\underline{b}_c$:

$\cdot$	u	v	uv	z	uz	vz	$(uv)z$
u	0	uv	$-v$	uz	$-z$	$-(uv)z$	vz
v	$-uv$	0	u	vz	$(uv)z$	$-z$	$-uz$
uv	v	$-u$	0	$(uv)z$	$-vz$	uz	$-z$
z	$-uz$	$-vz$	$-(uv)z$	0	u	v	uv
uz	z	$-(uv)z$	vz	$-u$	0	$-uv$	v
vz	$(uv)z$	z	$-uz$	$-v$	uv	0	$-u$
$(uv)z$	$-vz$	uz	z	$-uv$	$-v$	u	0

This shows that the structure constants of (V, π) with respect to $\underline{b}_c$ are independent of the choice of $c \in \mathcal{C}$. ∎

Note that Proposition 5.2 implies that $\mathcal{G}_2$ acts transitively on the set of orthonormal pairs in V, and that the stabiliser of an orthonormal pair (u, v) acts simply transitively on the unit sphere in $\{u, v, uv\}^{\perp} \subset V$.

Given a linear operator α on V and $c \in \mathcal{C}$, $[\alpha]_c$ denotes the matrix of α with respect to the basis $\underline{b}_c$. Fixing a Cayley triple $s \in \mathcal{C}$, we obtain a bijection $\mathfrak{t} : \mathcal{G}_2 \to \mathcal{C},\ g \mapsto g\cdot s$. If we set $g_c = \mathfrak{t}^{-1}(c)$, then $[g_c^{-1}\delta g_c]_s = [\delta]_{g_c\cdot s} = [\delta]_c$.

The task now will be to describe a map $N : \operatorname{Pds}(V) \to \operatorname{Pds}(V)$ with the following properties:

(i) $[N(\delta)]_s = [\delta]_c$ for some $c \in \mathcal{C}$.
(ii) $N(\delta) = N(\delta')$ whenever $\delta = g^{-1}\delta' g$ for some $g \in \mathcal{G}_2$.

Then $N(\mathrm{Pds}(V))$ will be a cross-section for $\mathrm{Pds}(V)/\mathcal{G}_2$, and $N(\delta)$ the normal form of δ.

As $\mathcal{G}_2 \subset O(V)$, all properties of δ as a linear operator on a Euclidean space are preserved under conjugation with elements in $\mathcal{G}_2$. In particular, the set $\{(\lambda, \dim \mathcal{E}_\lambda(\delta)) \mid \dim \mathcal{E}_\lambda(\delta) > 0\}$ of eigenpairs of δ is an invariant for its orbit under $\mathcal{G}_2$. Hence the normal form problem may be solved separately for each possible set of eigenpairs. We distinguish 15 essentially distinct types of eigenpairs, determined by the number of eigenspaces, and their dimensions:

1: (7).
2: $(1,6)$, $(2,5)$, $(3,4)$.
3: $(1,1,5)$, $(1,2,4)$, $(1,3,3)$, $(2,2,3)$.
4: $(1,1,1,4)$, $(1,1,2,3)$, $(1,2,2,2)$.
5: $(1,1,1,1,3)$, $(1,1,1,2,2)$.
6: $(1,1,1,1,1,2)$.
7: $(1,1,1,1,1,1,1)$.

Four cases are trivial. Firstly, if δ is of type (7), there is only one eigenpair, $(\lambda, 7)$ where $\lambda \in \mathbb{R}_{>0}$. Any choice of a Cayley triple c will give rise to the matrix $[\delta]_c = \lambda \mathbb{I}_7$.

If δ is of type $(1,6)$, we choose $c = (u,v,z) \in \mathcal{C}$ such that u belongs to the eigenspace of dimension 1. In case $(2,5)$, any orthonormal basis (u,v) for the two-dimensional eigenspace may be extended to a Cayley triple $c = (u,v,z)$. Finally, if $\{(\lambda,1),(\mu,1),(\nu,5)\}$ is the set of eigenpairs of δ (this is the case $(1,1,5)$), then $c \in \mathcal{C}$ may be chosen such that $u \in \mathcal{E}_\lambda(\delta)$ and $v \in \mathcal{E}_\mu(\delta)$.

The matrices obtained will be

$$[\delta]_c = \begin{pmatrix} \lambda & \\ & \nu\mathbb{I}_6 \end{pmatrix} \qquad \text{for } (1,6),$$

$$[\delta]_c = \begin{pmatrix} \lambda\mathbb{I}_2 & \\ & \nu\mathbb{I}_5 \end{pmatrix} \qquad \text{for } (2,5),$$

$$[\delta]_c = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu\mathbb{I}_5 \end{pmatrix} \qquad \text{for } (1,1,5),$$

where $(\lambda,\mu,\nu) \in \mathbb{R}^3$.

In the remaining cases, the situation is more complicated. As an example, we will here consider the case $(1,2,4)$. Let $\{(\lambda,1),(\mu,2),(\nu,4)\}$ be the set of eigenpairs of δ. Take (u, v) to be any orthonormal pair in $\mathcal{E}_\mu$. Note that uv is uniquely determined up to sign by $\operatorname{span}\{u, v\} = \mathcal{E}_\mu$.

LEMMA 5.3. *There exists a unit vector $z \in \mathcal{E}_\nu(\delta)$ such that $(u, v, z) \in \mathcal{C}$ and $uz, vz \in \mathcal{E}_\nu(\delta)$.*

Proof. If $uv \in \mathcal{E}_\lambda$, then $\{u, v, uv\}^\perp = \mathcal{E}_\nu$ and any unit vector $z \in \mathcal{E}_\nu$ will do. Otherwise, let $V' = \{u, v, uv\}^\perp$, $V_0 = V' \cap \mathcal{E}_\nu$ and $V_1 = V' \cap V_0^\perp$. We have $\dim V_0 = 3$ and $\dim V_1 = 1$, and V' is invariant under the operator L_u. Because L_u is bijective on $u^\perp \supset V'$, the subspace $W_u = V_0 \cap L_u^{-1}(V_0)$ of V' has dimension at least 2. The same holds for L_v, and it follows that the intersection of W_u and $W_v = V_0 \cap L_v^{-1}(V_0)$ is a non-trivial subspace of V_0. Take z to be a unit vector in $W_u \cap W_v$. Then $z, uz, vz \in V_0 \subset \mathcal{E}_\nu$, and $(u, v, z) \in \mathcal{C}$. ■

In the case when $uv \in \mathcal{E}_\lambda$, any choice of z will yield the matrix

$$[\delta]_c = \begin{pmatrix} \mu\mathbb{I}_2 & & \\ & \lambda & \\ & & \nu\mathbb{I}_4 \end{pmatrix}.$$

If $uv \notin \mathcal{E}_\lambda$, choosing z as in Lemma 5.3 we have $\delta(uv) = a(uv) + b(uv)z$ and $\delta((uv)z) = b(uv) + d(uv)z$ for some $a, b, d \in \mathbb{R}$. We see that in this construction, $(uv)z$ is uniquely determined by δ up to sign. By possibly changing the sign of z, we can ensure that $(\nu - \lambda)b > 0$. This completely determines the scalars $a, b, d \in \mathbb{R}$.

We conclude that

$$[\delta]_c = N_\tau = \begin{pmatrix} \mu\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \nu\mathbb{I}_3 & 0 \\ 0 & b & 0 & d \end{pmatrix}$$

where

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \lambda & \\ & \nu \end{pmatrix} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \qquad \tau \in [0, \pi/2].$$

Hence $\{N_\tau\}_{\tau \in [0,\pi/2]}$ is a cross-section for the set of positive definite symmetric endomorphisms of type $(1, 2, 4)$ under the action of $\mathcal{G}_2$. We remark that $\tau = 0$ corresponds to the situation where $uv \in \mathcal{E}_\lambda$.

We hope that the above examples have convinced the reader about the fruitfulness of our method. The remaining ten cases can be dealt with similarly, although the problem then generally will be more complicated. A systematic treatment of these cases will be given in a forthcoming publication, thus completing the classification of all finite-dimensional real flexible division algebras.

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Received 26 November 2004;
revised 17 December 2004 (4534)