COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 1

DOUBLE SINE SERIES WITH NONNEGATIVE COEFFICIENTS AND LIPSCHITZ CLASSES

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Abstract. Denote by $f_{\rm ss}(x,y)$ the sum of a double sine series with nonnegative coefficients. We present necessary and sufficient coefficient conditions in order that $f_{\rm ss}$ belongs to the two-dimensional multiplicative Lipschitz class $\mathrm{Lip}(\alpha,\beta)$ for some $0<\alpha\le 1$ and $0<\beta\le 1$. Our theorems are extensions of the corresponding theorems by Boas for single sine series.

1. Known results: single sine series. We give a brief summary of the known results for single sine series. Given a sequence $\{a_i : i=1,2,\ldots\}$ of nonnegative numbers such that

$$\sum_{i=1}^{\infty} a_i < \infty, \tag{1.1}$$

the sum of the sine series

$$\sum_{i=1}^{\infty} a_i \sin ix =: f_{\rm s}(x) \tag{1.2}$$

is a continuous function, by uniform convergence.

We recall (see [4, pp. 43–44]) that a periodic function φ belongs to the *Lipschitz class* $\mathrm{Lip}\,\alpha$ for some $\alpha>0$ if there exists a constant $C=C(\varphi)$ such that for all x and h we have

$$|\varphi(x+h)-\varphi(x)| \le C|h|^{\alpha}.$$

The following theorems by Boas [1] give necessary and sufficient conditions for a sine series to belong to $\mathrm{Lip}\,\alpha$, where $0<\alpha\le 1$.

THEOREM A. *Let* $\{a_i : i=1,2,\ldots\}$ *be a sequence of nonnegative numbers such that condition* (1.1) *is satisfied and let* $f_{\rm s}$ *be defined by* (1.2). *If*

2000 *Mathematics Subject Classification*: 42B05, 42A16.

Key words and phrases: double sine series with nonnegative coefficients, multiplicative Lipschitz classes, coefficient conditions.

This research was supported by the Hungarian National Foundation for Scientific Research under Grants TS 044 782 and T 046 192.

$0 < \alpha < 1$, *then* $f_{\rm s} \in \operatorname{Lip}\alpha$ *if and only if*

$$\sum_{i=m}^{\infty} a_i = O(m^{-\alpha}), \qquad m = 1, 2, \ldots; \tag{1.3}$$

or equivalently

$$\sum_{i=1}^{m} i a_i = O(m^{1-\alpha}), \qquad m = 1, 2, \ldots. \tag{1.4}$$

THEOREM B. *Let* $\{a_i : i = 1, 2, \ldots\}$ *and* $f_{\rm s}$ *be as in Theorem* A. *If* $\alpha = 1$, *then* $f_{\rm s} \in \operatorname{Lip} 1$ *if and only if*

$$\sum_{i=1}^{m} i a_i = O(1), \qquad m = 1, 2, \ldots. \tag{1.5}$$

Condition (1.5) formally coincides with (1.4) when applied for $\alpha = 1$. However, (1.5) is no longer equivalent to (1.3) for $\alpha = 1$.

We note that Theorem A remains valid if we replace $f_{\rm s}$ by $f_{\rm c}$, where $f_{\rm c}$ is the sum of the cosine series

$$\sum_{i=1}^{\infty} a_i \cos ix =: f_{\rm c}(x), \tag{1.6}$$

where $\{a_i : i = 1, 2, \ldots\}$ is a sequence of nonnegative numbers such that condition (1.1) is satisfied. On the other hand, Theorem B is no longer true for (1.6).

2. New results. From now on, we consider a double sequence $\{a_{ij} : i, j = 1, 2, \ldots\}$ of nonnegative numbers such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} < \infty. \tag{2.1}$$

The double sine series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \sin ix \sin jy =: f_{\rm ss}(x, y) \tag{2.2}$$

converges uniformly, and its sum $f_{\rm ss}$ is a continuous function.

Next, we give the definition of the two-dimensional multiplicative Lipschitz classes $\operatorname{Lip}(\alpha, \beta)$, where $\alpha, \beta > 0$. The definition is due to Móricz [3].

A function $\varphi(x, y)$ periodic in each variable is said to belong to the *two-dimensional Lipschitz class* $\operatorname{Lip}(\alpha, \beta)$ for some $\alpha, \beta > 0$ if there exists a constant $C = C(\varphi)$ such that for all x, y, h and k, we have

$$|\varphi(x + h, y + k) - \varphi(x + h, y) - \varphi(x, y + k) + \varphi(x, y)| \le C|h|^{\alpha}|k|^{\beta}. \tag{2.3}$$

Motivated by the one-variable case, Theorems 1–3 below are the extensions of Theorems A and B to double sine series, in which $\{a_{ij} : i, j = 1, 2, \ldots\}$ is a double sequence of nonnegative numbers such that condition (2.1) is satisfied and f_{ss} is defined by (2.2).

THEOREM 1. *If* $0 < \alpha, \beta < 1$, *then* $f_{\text{ss}} \in \text{Lip}(\alpha, \beta)$ *if and only if*

$$\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} a_{ij} = O(m^{-\alpha} n^{-\beta}), \qquad m, n = 1, 2, \ldots; \tag{2.4}$$

or equivalently

$$\sum_{i=1}^{m} \sum_{j=1}^{n} ij a_{ij} = O(m^{1-\alpha} n^{1-\beta}), \qquad m, n = 1, 2, \ldots. \tag{2.5}$$

The equivalence of (2.4) and (2.5) follows from the Lemma below when applied for $\gamma = \delta = 1,\ \mu = 1 - \alpha,\ \nu = 1 - \beta$.

THEOREM 2. *If* $\alpha = \beta = 1$, *then* $f_{\text{ss}} \in \text{Lip}(1, 1)$ *if and only if*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} ij a_{ij} = O(1), \qquad m, n = 1, 2, \ldots. \tag{2.6}$$

We observe that (2.6) formally coincides with (2.5) when $\alpha = \beta = 1$, but it is not equivalent to (2.4) when $\alpha = \beta = 1$.

THEOREM 3. *If* $0 < \alpha < 1$ *and* $\beta = 1$, *then* $f_{\text{ss}} \in \text{Lip}(\alpha, 1)$ *if and only if*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} ij a_{ij} = O(m^{1-\alpha}), \qquad m, n = 1, 2, \ldots. \tag{2.7}$$

The proof of Theorem 3 combines the methods of proof of Theorems 1 and 2.

We note that the symmetric counterpart of Theorem 3 gives a criterion for f_{ss} to belong to $\text{Lip}(1, \beta)$ for $0 < \beta < 1$.

Analysis of the proof of Theorem 1 shows that the sums

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \sin ix \cos jy =: f_{\text{sc}}(x, y),$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \cos ix \sin jy =: f_{\text{cs}}(x, y)$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \cos ix \cos jy =: f_{\text{cc}}(x, y),$$

of sine-cosine, cosine-sine and double cosine series belong to $\mathrm{Lip}(\alpha, \beta)$ when $0 < \alpha, \beta < 1$ if and only if (2.4) or equivalently (2.5) is satisfied. On the other hand, an analogous reformulation of Theorems 2 and 3 is no longer true for them when $\alpha = \beta = 1$ or $0 < \alpha < 1$, $\beta = 1$ or $\alpha = 1$, $0 < \beta < 1$.

It is not difficult to check that in Theorem 1–3 it is enough to require the fulfilment of conditions (2.4)–(2.7) for large enough m and n, say $m > n_0$ and $n > n_0$, where n_0 is some positive integer.

The following auxiliary result plays a key role in the proofs of Theorem 1–3. This lemma is an extension of the corresponding one by Boas [1, Lemma 1] to double series of nonnegative numbers.

LEMMA. *Let $a_{ij} \geq 0$, $i, j = 1, 2, \ldots.$ If $\gamma > \mu \geq 0$, $\delta > \nu \geq 0$ and*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} i^\gamma j^\delta a_{ij} = O(m^\mu n^\nu), \qquad m, n = 1, 2, \ldots,$$

then

$$\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} a_{ij} = O(m^{\mu-\gamma} n^{\nu-\delta}), \qquad m, n = 1, 2, \ldots.$$

If $\gamma > \mu > 0$ and $\delta > \nu > 0$, then the converse implication is also valid.

This lemma was proved in [2, Lemma 1] in the particular case when $\mu - \gamma = \nu - \delta = -1$. In the more general case of the Lemma above, the proof is analogous.

3. Proof of Theorem 1

(i) *Sufficiency.* Assume that conditions (2.4) or equivalently (2.5) hold, that is, there exist constants K and K_1 such that

$$\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} a_{ij} \leq \frac{K}{m^\alpha n^\beta}, \qquad m, n = 1, 2, \ldots, \tag{3.1}$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} ij a_{ij} \leq K_1 m^{1-\alpha} n^{1-\beta}, \qquad m, n = 1, 2, \ldots. \tag{3.2}$$

We claim that then $f_{\mathrm{ss}} \in \mathrm{Lip}(\alpha, \beta)$. Clearly, we have

$$f_{\mathrm{ss}}(x + 2h, y + 2k) - f_{\mathrm{ss}}(x + 2h, y) - f_{\mathrm{ss}}(x, y + 2k) + f_{\mathrm{ss}}(x, y)$$
$$= 4 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \cos i(x + h) \sin ih \cos j(y + k) \sin jk.$$

It follows that

$$|f_{ss}(x+2h,y+2k)-f_{ss}(x+2h,y)-f_{ss}(x,y+2k)+f_{ss}(x,y)| \tag{3.3}$$
$$\leq 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}|\sin ih|\,|\sin jk|$$
$$= 4\Big\{\sum_{i=1}^{m}\sum_{j=1}^{n}+\sum_{i=m+1}^{\infty}\sum_{j=1}^{n}+\sum_{i=1}^{m}\sum_{j=n+1}^{\infty}+\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\Big\}a_{ij}|\sin ih|\,|\sin jk|$$
$$=: S_1+S_2+S_3+S_4, \qquad \text{where } m := [1/|h|],\ n := [1/|k|],$$

where $[\cdot]$ denotes the integer part.

First, we estimate S_1 as follows:

$$S_1 \leq 4\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}|ih|\,|jk| = 4|h|\,|k|\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij}.$$

By (3.2), we immediately get

$$S_1 \leq 4|h|\,|k|K_1 m^{1-\alpha}n^{1-\beta} \leq 4K_1|h|^{\alpha}|k|^{\beta}. \tag{3.4}$$

Second, we consider S_2. It is clear that

$$S_2 \leq 4|k|\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} ja_{ij}. \tag{3.5}$$

Let N be an arbitrary integer such that $1 \leq n < N$. A summation by parts with respect to j gives that

$$\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} ja_{ij} = \sum_{i=m+1}^{\infty}\Big\{\sum_{j_1=1}^{n}\sum_{j=j_1}^{N} a_{ij} - n\sum_{j=n+1}^{N} a_{ij}\Big\} \leq \sum_{i=m+1}^{\infty}\sum_{j_1=1}^{n}\sum_{j=j_1}^{N} a_{ij},$$

whence by (3.1), if we let N tend to ∞, (3.5) can be estimated as follows:

$$S_2 \leq 4|k|\sum_{j_1=1}^{n}\sum_{i=m+1}^{\infty}\sum_{j=j_1}^{\infty} a_{ij} \leq 4|k|\sum_{j_1=1}^{n}\frac{K}{(m+1)^{\alpha}j_1^{\beta}} \tag{3.6}$$
$$\leq 4K|h|^{\alpha}|k|\sum_{j_1=1}^{n}\frac{1}{j_1^{\beta}}.$$

If $0<\beta<1$, then

$$\sum_{j_1=1}^{n}\frac{1}{j_1^{\beta}} \leq \int_0^n \frac{1}{x^{\beta}}\,dx = \frac{n^{1-\beta}}{1-\beta}.$$

Hence by (3.6) we find that

$$S_2 \leq 4K|h|^{\alpha}|k|\,\frac{n^{1-\beta}}{1-\beta} \leq \frac{4K}{1-\beta}\,|h|^{\alpha}|k|^{\beta}. \tag{3.7}$$

Third, S_3 is the symmetric counterpart of S_2, and can be estimated analogously:

$$S_3 \le \frac{4K}{1-\alpha}\,|h|^\alpha |k|^\beta. \tag{3.8}$$

Fourth, the estimate of S_4 is quite simple. By (3.1), we have

$$S_4 \le 4 \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} \le 4\,\frac{K}{(m+1)^\alpha (n+1)^\beta} \le 4K|h|^\alpha |k|^\beta. \tag{3.9}$$

Combining (3.3), (3.4), (3.7)–(3.9) shows that $f_{\rm ss} \in \mathrm{Lip}(\alpha,\beta)$.

(ii) *Necessity.* Now we assume that $f_{\rm ss} \in \mathrm{Lip}(\alpha,\beta)$, that is, (2.3) holds for $\varphi = f_{\rm ss}$. Let $0 < h,\ k \le 1$ and put $x = y = 0$ in (2.3). We obtain

$$|f_{\rm ss}(h,k) - f_{\rm ss}(h,0) - f_{\rm ss}(0,k) + f_{\rm ss}(0,0)| = |f_{\rm ss}(h,k)| \le C h^\alpha k^\beta,$$

whence

$$|f_{\rm ss}(x,y)| = \Big|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix \sin jy\Big| \le C x^\alpha y^\beta, \qquad x > 0,\ y > 0. \tag{3.10}$$

By uniform convergence (due to (2.1)), the double series in the middle can be integrated term by term with respect to x over the interval $(0,h)$:

$$\Big|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin jy \int_0^h \sin ix\,dx\Big| \le C y^\beta \int_0^h x^\alpha\,dx,$$

which gives

$$\Big|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin jy\,\frac{1-\cos ih}{i}\Big| \le \frac{C}{\alpha+1}\,y^\beta\,h^{\alpha+1}.$$

Integrating again term by term, this time with respect to y over $(0,k)$, we find that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\,\frac{1-\cos ih}{i}\,\frac{1-\cos jk}{j} \\ = 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\,a_{ij}\sin^2\frac{ih}{2}\sin^2\frac{jk}{2} \le \frac{C}{(\alpha+1)(\beta+1)}\,h^{\alpha+1}k^{\beta+1}.\end{aligned} \tag{3.11}$$

By the known inequality

$$\sin t \ge \frac{2}{\pi}\,t, \qquad 0 \le t \le \frac{\pi}{2},$$

we obtain

$$\sin^2\frac{ih}{2} \ge \left(\frac{2}{\pi}\,\frac{ih}{2}\right)^2 = \frac{i^2h^2}{\pi^2}, \qquad i = 1,2,\dots,[1/h] =: m,$$

and

$$\sin^2 \frac{jk}{2} \ge \frac{j^2k^2}{\pi^2}, \qquad j = 1, 2, \dots, [1/k] =: n.$$

From (3.11) and the last two inequalities we get

$$\begin{aligned}(3.12)\qquad \frac{4h^2k^2}{\pi^4} \sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} &= 4\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{ij}\, a_{ij} \frac{i^2h^2}{\pi^2}\, \frac{j^2k^2}{\pi^2} \\ &\le 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij}\, a_{ij} \sin^2 \frac{ih}{2} \sin^2 \frac{jk}{2} \le \frac{C}{(\alpha+1)(\beta+1)}\, h^{\alpha+1}k^{\beta+1}.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} &\le \frac{C\pi^4}{4(\alpha+1)(\beta+1)}\, h^{\alpha-1}k^{\beta-1} \\ &\le \frac{C\pi^4}{4(\alpha+1)(\beta+1)}\, (m+1)^{1-\alpha}(n+1)^{1-\beta} \le \frac{2^{2-\alpha-\beta}C\pi^4}{4(\alpha+1)(\beta+1)}\, m^{1-\alpha}n^{1-\beta}.\end{aligned}$$

This is (2.5), which was to be proved.

The proof of Theorem 1 is complete.

4. Proof of Theorem 2

(iii) *Sufficiency.* Suppose that (2.6) holds: there exists a constant K such that

$$(4.1)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} \le K, \qquad m = 1, 2, \dots.$$

By the Lemma of Section 2 with $\gamma = \delta = 1$, $\mu = \nu = 0$, there exists a constant K_1 such that

$$(4.2)\qquad \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} \le \frac{K_1}{mn}, \qquad m, n = 1, 2, \dots.$$

We claim that $f_{\mathrm{ss}} \in \mathrm{Lip}(1,1)$.

We start with the same estimate (3.3) as in part (i) of the proof of Theorem 1:

$$\begin{aligned}(4.3)\qquad |f_{\mathrm{ss}}(x+2h, y+2k) - f_{\mathrm{ss}}(x+2h, y) - f_{\mathrm{ss}}(x, y+2k) + f_{\mathrm{ss}}(x,y)| \\ \le S_1 + S_2 + S_3 + S_4,\end{aligned}$$

where the S_i are defined in (3.3).

We can estimate S_1 and S_4 as in part (i), except that this time we use (4.1) and (4.2) instead of (3.1) and (3.2). As a result, we obtain

$$(4.4)\qquad S_1 \le 4K|h|\,|k| \quad \text{and} \quad S_4 \le 4K_1|h|\,|k|.$$

On the other hand, S_2 and S_3 will be estimated in a different way. First, we deal with S_2. It is clear that

$$S_2 \le 4|k| \sum_{i=m+1}^{\infty} \sum_{j=1}^{n} j a_{ij} = 4|k| \sum_{j=1}^{n} j \sum_{i=m+1}^{\infty} a_{ij}. \tag{4.5}$$

In order to estimate the right-hand side, we consider the following partial sum:

$$\sum_{j=1}^{n} j \sum_{i=m}^{M} a_{ij} = \sum_{j=1}^{n} j \sum_{i=m}^{M} i^{-1}(i a_{ij}),$$

where M is an arbitrary integer for which $2 \le m < M$. A summation by parts, this time with respect to i, gives

$$\begin{aligned}
&\sum_{j=1}^{n} j \sum_{i=m}^{M} i^{-1}(i a_{ij}) \\
&\quad = \sum_{j=1}^{n} j \Big\{ -\frac{1}{m} \sum_{i=1}^{m-1} i a_{ij} + \sum_{i_1=m}^{M-1} \Big(\frac{1}{i_1} - \frac{1}{i_1+1} \Big) \sum_{i=1}^{i_1} i a_{ij} + \frac{1}{M} \sum_{i=1}^{M} i a_{ij} \Big\} \\
&\quad \le \sum_{i_1=m}^{M-1} \frac{1}{i_1^2} \sum_{i=1}^{i_1} \sum_{j=1}^{n} i j a_{ij} + \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{n} i j a_{ij}.
\end{aligned}$$

By (4.1) it follows that the last expression is not greater than

$$K \sum_{i_1=m}^{M-1} \frac{1}{i_1^2} + K\,\frac{1}{M} \le K\,\frac{1}{m-1} + K\,\frac{1}{M} \le K\,\frac{3}{m+1} + K\,\frac{1}{M},$$

whence, letting M tend to ∞, we obtain

$$\sum_{j=1}^{n} j \sum_{i=m}^{\infty} a_{ij} \le 3K\,\frac{1}{m+1}. \tag{4.6}$$

Putting together (4.5) and (4.6) yields

$$S_2 \le 4|k| 3K\,\frac{1}{m+1} \le 12K|h|\,|k|. \tag{4.7}$$

Since S_3 is a symmetric counterpart of S_2, an analogous reasoning yields

$$S_3 \le 12K|h|\,|k|. \tag{4.8}$$

Combining (4.3), (4.4), (4.7) and (4.8) gives that $f_{\rm ss} \in \mathrm{Lip}(1,1)$.

(iv) *Necessity.* Now we assume that $f_{\rm ss} \in \mathrm{Lip}(1,1)$. Clearly, inequality (3.10) in part (ii) of the proof of Theorem 1 holds in the case when $\alpha = \beta = 1$. Furthermore, we can also repeat the reasoning of part (ii) in this case (see (3.11) and (3.12)). As a result, we find that

$$\frac{4h^2k^2}{\pi^4}\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} \le \frac{C}{4}\,h^2k^2,$$

whence it follows that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} \le \frac{C\pi^4}{16}, \qquad m,n=1,2,\dots.$$

This is (2.6), which was to be proved.

The proof of Theorem 2 is complete.

5. Proof of Theorem 3. The proof is a combination of certain steps from the proofs of Theorems 1 and 2.

(v) *Sufficiency.* Assume that (2.7) holds. An application of the Lemma of Section 2 with $\gamma=\delta=1$, $\mu=1-\alpha$ and $\nu=0$ gives

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} = O\Big(\frac{1}{m^\alpha n}\Big), \qquad m,n=1,2,\dots. \tag{5.1}$$

To see that $f_{\rm ss}\in \mathrm{Lip}(\alpha,1)$, again we make use of estimate (3.3). Clearly, inequalities (3.4) for S_1 and (3.9) for S_4 hold in case $0<\alpha<1$ and $\beta=1$. To estimate S_2, we essentially repeat the reasoning from part (iii) of the proof of Theorem 2, while using (2.7). As a result, we obtain

$$S_2 \le 4|k|\,\frac{2^\alpha}{\alpha}\,2K\,\frac{1}{(m+1)^\alpha} \le 8\,\frac{2^\alpha}{\alpha}\,K|h|^\alpha|k| \tag{5.2}$$

(cf. (4.7)). To estimate S_3, we essentially repeat the reasoning of part (i) of the proof of Theorem 1, using (5.1). As a result, we have

$$S_3 \le \frac{4K_1}{1-\alpha}\,|h|^\alpha|k| \tag{5.3}$$

(cf. (3.8)). To sum up, by (3.3), (3.4) and (3.9) (the last two inequalities in the case when $0<\alpha<1$ and $\beta=1$), (5.2) and (5.3), we find that $f_{\rm ss}\in \mathrm{Lip}(\alpha,1)$.

(vi) *Necessity.* It is essentially a repetition of part (ii) of the proof of Theorem 1 in the case when $0<\alpha<1$ and $\beta=1$ (see also part (iv) of the proof of Theorem 2). We omit the details.

The proof of Theorem 3 is complete.

6. Concluding remark. The sufficiency part of Theorem B was proved by Boas [1] in a different way. Namely, Boas made use of the familiar theorem on termwise differentiation of an infinite series of differentiable functions when the differentiated series is uniformly convergent on a finite interval. Our method provides a new proof of the sufficiency part of Theorem B.

Acknowledgements. The author wishes to express her gratitude to Professor Ferenc Móricz for many insightful conversations and constant support.

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Received 22 November 2004;
revised 24 March 2005 (4530)