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ASSOCIATED PRIMES, INTEGRAL CLOSURES AND IDEAL TOPOLOGIES

BY

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Abstract. Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of a Noetherian ring R, and let N be a non-zero finitely generated R-module. The set $\overline{Q}^*(\mathfrak{a}, N)$ of quintasymptotic primes of $\mathfrak{a}$ with respect to N was originally introduced by McAdam. Also, it has been shown by Naghipour and Schenzel that the set $A_a^*(\mathfrak{b}, N) := \bigcup_{n\geq 1} \operatorname{Ass}_R R/(\mathfrak{b}^n)_a^{(N)}$ of associated primes is finite. The purpose of this paper is to show that the topology on N defined by $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle\mathfrak{b}\rangle\}_{n\geq 1}$ is finer than the topology defined by $\{(\mathfrak{b}^n)_a^{(N)}\}_{n\geq 1}$ if and only if $A_a^*(\mathfrak{b}, N)$ is disjoint from the quintasymptotic primes of $\mathfrak{a}$ with respect to N. Moreover, we show that if $\mathfrak{a}$ is generated by an asymptotic sequence on N, then $A_a^*(\mathfrak{a}, N) = \overline{Q}^*(\mathfrak{a}, N)$.

1. Introduction. Throughout this paper, all rings considered will be commutative and Noetherian and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by $\mathfrak{a}$. The important concepts of quintessential and quintasymptotic primes of $\mathfrak{a}$ were introduced by McAdam [6], and in [1], Ahn extended them to a finitely generated R-module N. We provide a brief review.

A prime ideal $\mathfrak{p}$ of R is called a *quintessential* (resp. *quintasymptotic*) *prime ideal* of $\mathfrak{a}$ with respect to N if there exists $\mathfrak{q} \in \operatorname{Ass}_{R_\mathfrak{p}^*} N_\mathfrak{p}^*$ (resp. $\mathfrak{q} \in \operatorname{mAss}_{R_\mathfrak{p}^*} N_\mathfrak{p}^*$) such that $\operatorname{Rad}(\mathfrak{a}R_\mathfrak{p}^* + \mathfrak{q}) = \mathfrak{p}R_\mathfrak{p}^*$. The set of quintessential (resp. quintasymptotic) primes of $\mathfrak{a}$ with respect to N is denoted by $Q(I, N)$ (resp. $\overline{Q}^*(\mathfrak{a}, N)$), and it is a finite set.

In [17], R. Y. Sharp *et al.* introduced the concept of integral closure of $\mathfrak{a}$ relative to N, and they showed that this concept has properties which reflect some of those of the usual concept of integral closure introduced by Northcott and Rees in [10]. In this paper, we shall denote the integral closure of $\mathfrak{a}$ with respect to N by $\mathfrak{a}_a^{(N)}$. On the other hand, in [9], it is shown that the sequence $\{\operatorname{Ass}_R R/(\mathfrak{a}^n)_a^{(N)}\}_{n\geq 1}$ of associated prime ideals is increasing

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and ultimately constant; we denote its ultimate constant value by $A_a^*(\mathfrak{a}, N)$. (The set $A_a^*(\mathfrak{a}, N)$ is called the set of *asymptotic primes of* $\mathfrak{a}$ with respect to N.) In the case $N = R$, $A_a^*(\mathfrak{a}, N)$ is the set $\widehat{A^*}(\mathfrak{a})$ of asymptotic primes of $\mathfrak{a}$ introduced by L. J. Ratliff, Jr. [12].

We now briefly summarize the results in this paper. Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R, and let N be a non-zero finitely generated R-module. Our purpose is to characterize the equivalence between the topologies on N defined by $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{b} \rangle\}_{n\geq 1}$ and $\{(\mathfrak{b}^n)_a^{(N)}\}_{n\geq 1}$ in terms of the quintasymptotic primes of $\mathfrak{a}$ with respect to N and $A_a^*(\mathfrak{b}, N)$. Then we show that, for any prime ideal $\mathfrak{p}$ of R containing $\mathfrak{a}$, the topologies induced by $\{(\mathfrak{a}^n R_\mathfrak{p})_a^{(N_\mathfrak{p})} :_{R_\mathfrak{p}} \langle \mathfrak{p} \rangle\}_{n\geq 1}$ and $\{(\mathfrak{a}^n R_\mathfrak{p})_a^{(N_\mathfrak{p})}\}_{n\geq 1}$ are equivalent if and only if $\mathfrak{p} \notin \overline{Q}^*(\mathfrak{a}, N)$. We also show that if $\mathfrak{a}$ is generated by an asymptotic sequence on N, then

$$A_a^*(\mathfrak{a}, N) = \overline{Q}^*(\mathfrak{a}, N).$$

We denote by $\mathcal{R}$ the *graded Rees ring* $R[u, \mathfrak{a}t] := \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}^n t^n$ of R with respect to $\mathfrak{a}$, where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $N[u, \mathfrak{a}t] := \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}^n N$ over $\mathcal{R}$ is denoted by $\mathcal{N}$; it is a finitely generated graded $\mathcal{R}$-module. If $(R, \mathfrak{m})$ is local, then R^* (resp. N^*) denotes the completion of R (resp. N) with respect to the $\mathfrak{m}$-adic topology. In particular, for every prime ideal $\mathfrak{p}$ of R, we denote by $R_\mathfrak{p}^*$ and $N_\mathfrak{p}^*$ the $\mathfrak{p}R_\mathfrak{p}$-adic completion of $R_\mathfrak{p}$ and $N_\mathfrak{p}$, respectively. For any ideal $\mathfrak{b}$ of R, the *radical* of $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Finally, for each R-module L, we denote by $\mathrm{mAss}_R L$ the set of minimal primes of $\mathrm{Ass}_R L$. For any unexplained notation and terminology we refer the reader to [3] or [8].

In the second section, we focus on the quintasymptotic and asymptotic primes of ideals with respect to N. In that section, among other things, we show that if $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals of R, then the topology on N defined by $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{b} \rangle\}_{n\geq 1}$ is finer than the topology defined by $\{(\mathfrak{b}^n)_a^{(N)}\}_{n\geq 1}$ if and only if the set of quintasymptotic primes of $\mathfrak{a}$ with respect to N is disjoint from that of asymptotic primes of $\mathfrak{b}$ with respect to N.

The main result of the third section is that if N is a non-zero finitely generated R-module and $\mathfrak{a}$ is an arbitrary ideal of R generated by an asymptotic sequence on N, then the sets of quintasymptotic and asymptotic primes of $\mathfrak{a}$ with respect to N are equal.

2. Asymptotic and quintasymptotic primes. Following [2], we shall use $A^*(\mathfrak{b}, N)$ to denote the ultimately constant values of $\mathrm{Ass}_R N/\mathfrak{b}^n N$ for all large n. The following lemma was proved by McAdam and Ratliff in [7] when $R = N$. It is easy to carry it over to modules (see [1]).

LEMMA 2.1. *Let $\mathfrak{a}$ be an ideal of R, and N a non-zero finitely generated R-module.*

(i) *Let $\mathfrak{p}$ be a prime ideal of R containing $\mathfrak{a}$, and S a multiplicatively closed subset of R such that $\mathfrak{p} \cap S = \emptyset$. Then $\mathfrak{p} \in Q(\mathfrak{a}, N)$ (resp. $\mathfrak{p} \in A^*(\mathfrak{b}, N)$) if and only if $\mathfrak{p}_S \in Q(\mathfrak{a}_S, N_S)$ (resp. $\mathfrak{p}_S \in A^*(\mathfrak{a}_S, N_S)$).*

(ii) *If T is a faithfully flat Noetherian extension of R, then $\mathfrak{p} \in Q(\mathfrak{a}, N)$ (resp. $\mathfrak{p} \in A^*(\mathfrak{b}, N)$) if and only if there exists $\mathfrak{q} \in Q(\mathfrak{a}T, N \otimes_R T)$ (resp. $\mathfrak{q} \in A^*(\mathfrak{a}T, N \otimes_R T)$) with $\mathfrak{q} \cap R = \mathfrak{p}$.*

The following lemma is known in the case $N = R$. The proof in [4, Ex. 8.2] can be easily carried over to modules, so we omit the proof.

LEMMA 2.2. *Let $(R, \mathfrak{m})$ be local and $\mathfrak{a}$ a proper ideal of R. Let N be a non-zero finitely generated R-module such that N is complete with respect to the $\mathfrak{m}$-adic topology. Then N is complete with respect to the $\mathfrak{a}$-adic topology.*

PROPOSITION 2.3. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R, and N a non-zero finitely generated R-module. Then the following conditions are equivalent:*

(i) $A^*(\mathfrak{b}, N) \cap Q(\mathfrak{a}, N) = \emptyset$.

(ii) *The topology on N induced by $\{\mathfrak{a}^n N :_N \langle \mathfrak{b} \rangle\}_{n \geq 1}$ is finer than the $\mathfrak{b}$-adic topology.*

Proof. In order to prove (i)⇒(ii), let $k \geq 1$. We need to show that there exists an integer $n \geq 1$ such that $\mathfrak{a}^n N :_N \langle \mathfrak{b} \rangle \subseteq \mathfrak{b}^k N$. Since

$$\mathrm{Ass}_R(\mathfrak{a}^n N :_N \langle \mathfrak{b} \rangle + \mathfrak{b}^k N / \mathfrak{b}^k N) \subseteq A^*(\mathfrak{b}, N),$$

in view of Lemma 2.1(i) it is enough to prove the claim in any localization $\mathfrak{p} \in A^*(\mathfrak{b}, N)$. Therefore we may assume that R is local at $\mathfrak{p} \in A^*(\mathfrak{b}, N)$. Recall that by hypothesis $\mathfrak{p} \notin Q(\mathfrak{a}, N)$. Also, by Lemma 2.1(ii), it is easy to see that $\mathfrak{p}R^* \in A^*(\mathfrak{b}R^*, N^*) \setminus Q(\mathfrak{a}R^*, N^*)$. Now, because $M^* \cap N = M$ for any submodule M of N, we may assume in addition that R is complete.

We use induction on $d := \dim N/\mathfrak{b}N$. Suppose $d = 0$, suppose there is an integer $k \geq 0$ such that $\mathfrak{a}^n N :_N \langle \mathfrak{b} \rangle \nsubseteq \mathfrak{b}^k N$ for all integers $n \geq 0$, and look for a contradiction. We have $\mathrm{Rad}(\mathfrak{b} + \mathrm{Ann}_R N) = \mathfrak{p}$ and so $\mathfrak{a}^n N :_N \langle \mathfrak{b} \rangle = \mathfrak{a}^n N :_N \langle \mathfrak{p} \rangle$. Now, let

$$E := (\mathfrak{b}^k N :_N \langle \mathfrak{p} \rangle)/\mathfrak{b}^k N \quad \text{and} \quad E_m = (\mathfrak{a}^m N :_N \langle \mathfrak{p} \rangle + \mathfrak{b}^k N)/\mathfrak{b}^k N$$

for all $m \geq k$. Then, as E is Artinian, it follows that $\bigcap_{m \geq k+1} E_m \neq 0$ (recall that $\mathfrak{a} \subseteq \mathfrak{b}$). Therefore there is an element $x_k \notin \mathfrak{b}^k N$ such that $x_k \in (\mathfrak{a}^m N :_N \langle \mathfrak{p} \rangle) + \mathfrak{b}^k N$ for every integer $m \geq 0$. The argument used to prove Chevalley's theorem [8, 30.1] can be applied to show that there exists a Cauchy sequence of elements $x_r \in N$ such that $x_r - x_k \in \mathfrak{b}^k N$ and $x_r \in (\mathfrak{a}^m N :_N \langle \mathfrak{p} \rangle) + \mathfrak{b}^r N$ for all integers $r \geq k$ and $m \geq 0$. Now, since N is complete with respect to the $\mathfrak{p}$-adic topology, by Lemma 2.2 there exists

$x \in N$ such that $\lim_{r\to\infty} x_r = x$. Hence $x - x_r \in \mathfrak{b}^k N$, and so $x \notin \mathfrak{b}^k N$. Moreover, we have $x - x_r \in \mathfrak{b}^r N$, and therefore $x \in (\mathfrak{a}^m N :_N \langle \mathfrak{p}\rangle) + \mathfrak{b}^r N$ for all integers $m \geq 1$ and $r \geq k$. Thus

$$x \in \bigcap_{r\geq k} \bigcap_{m\geq 1} (\mathfrak{a}^m N :_N \langle \mathfrak{p}\rangle + \mathfrak{b}^r N).$$

According to the Krull intersection theorem, $x \in \bigcap_{m\geq 1}(\mathfrak{a}^m N :_N \langle \mathfrak{p}\rangle)$. Now, using the Artin–Rees Lemma it is easy to see that there exists an integer $t \geq 1$ such that, for all integers $l \geq t$,

$$\mathfrak{a}^l N :_R Rx \subseteq \mathfrak{a}^{l-t} + (0 :_R rx) \subseteq \mathfrak{a}^{l-t} + z,$$

where z is an associated prime of N with $z \supseteq (0 :_R Rx)$. Then it readily follows that $\mathfrak{p} = \operatorname{Rad}(\mathfrak{a} + z)$, and so $\mathfrak{p} \in Q(\mathfrak{a}, N)$, which is a contradiction.

Now suppose $\dim N/\mathfrak{b}N > 0$. As above consider E_m. Then $\operatorname{Ass}_R E_m \subseteq A^*(\mathfrak{b}, N)$ and $\dim N_\mathfrak{q}/\mathfrak{b}N_\mathfrak{q} < d$ for all $\mathfrak{q} \in A^*(\mathfrak{b}, N)$. Hence by the inductive hypothesis the localizations of E_m at any $\mathfrak{q} \in A^*(\mathfrak{b}, N)$ with $\mathfrak{q} \neq \mathfrak{m}$ tend to zero. Therefore E_m has finite length for large m, and so $\operatorname{Ass}_R E_m \subseteq V(\mathfrak{p})$. Then the proof goes as before. We omit it.

In order to show the implication (ii)⇒(i), suppose $\mathfrak{p} \in A^*(\mathfrak{b}, N) \cap Q(\mathfrak{a}, N)$. Then, by Lemma 2.1, we can assume that $(R, \mathfrak{p})$ is local. Again from Lemma 2.1, it follows that $\mathfrak{p}R^* \in A^*(\mathfrak{b}R^*, N^*) \cap Q(\mathfrak{a}R^*, N^*)$. Hence we may also assume that R is complete. Note that statement (ii) is stable under localization and completion. Thus there exists a $z \in \operatorname{Ass}_R N$ such that $\mathfrak{p} = \operatorname{Rad}(\mathfrak{a} + z)$, and so we write $z = 0 :_R Rx$ for some $x \in N$. On the other hand, in view of the assumption and the Krull intersection theorem we have $\bigcap_{n\geq 1}(\mathfrak{a}^n N :_N \langle \mathfrak{b}\rangle) = 0$. So $\bigcap_{n\geq 1}(\mathfrak{a}^n N :_N \langle \mathfrak{p}\rangle) = 0$, and this is a contradiction since

$$x \in \bigcap_{n\geq 1} (\mathfrak{a}^n N :_N \langle z\rangle) = \bigcap_{n\geq 1} (\mathfrak{a}^n N :_N \langle \mathfrak{p}\rangle).$$

Hence $A^*(\mathfrak{b}, N) \cap Q(\mathfrak{a}, N) = \emptyset$, as desired. ∎

We are now ready to state and prove the main theorem of this section. The following remark will be needed in the proof.

Remark 2.4. Let $\mathfrak{c}$ be an ideal of R, and N a non-zero finitely generated R-module. Let $\pi : R \to R/\operatorname{Ann}_R N$ be the canonical ring homomorphism. Then it is readily checked that for any prime ideal $\mathfrak{p}$ of R (see [1, 3.6] and [6, 3.4(b)]),

(i) $\mathfrak{p} \in A_a^*(\mathfrak{c}, N)$ if and only if $\pi(\mathfrak{p}) \in \widehat{A}^*(\pi(\mathfrak{c}))$.
(ii) $\mathfrak{p} \in \overline{Q}^*(\mathfrak{c}, N)$ if and only if $\pi(\mathfrak{p}) \in \overline{Q}^*(\pi(\mathfrak{c}))$.

Theorem 2.5. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R, and N a non-zero finitely generated R-module. Then the following conditions are equivalent*:

(i) $A_a^*(\mathfrak{b},N) \cap \overline{Q}^*(\mathfrak{a},N) = \emptyset$.

(ii) *The topology induced by* $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{b} \rangle\}_{n\geq 1}$ *is finer than the topology defined by* $\{(\mathfrak{b}^n)_a^{(N)}\}_{n\geq 1}$.

Proof. (i)⇒(ii). Assume $A_a^*(\mathfrak{b},N) \cap \overline{Q}^*(\mathfrak{a},N) = \emptyset$ and let $l \geq 1$. We need to show that there exists an integer $m \geq 1$ such that $(\mathfrak{a}^m)_a^{(N)} :_R \langle \mathfrak{b} \rangle \subseteq (\mathfrak{b}^l)_a^{(N)}$. To do this, in view of Remark 2.4 and [17, 1.6], it is enough to show that

$$(\mathfrak{a}^m + \operatorname{Ann}_R N/\operatorname{Ann}_R N)_a :_{R/\operatorname{Ann}_R N} \langle \mathfrak{b} \rangle \subseteq (\mathfrak{b}^l + \operatorname{Ann}_R N/\operatorname{Ann}_R N)_a.$$

To ease notation, we will assume $R = R/\operatorname{Ann}_R N$. Then $\widehat{A}^*(\mathfrak{b}) \cap \overline{Q}^*(\mathfrak{a}) = \emptyset$ and we will show that

$$(\mathfrak{a}^m)_a :_R \langle \mathfrak{b} \rangle \subseteq (\mathfrak{b}^l)_a.$$

To this end, analogously to the proof of Proposition 2.3 and in view of [16, 2.3], [5, 3.15 and 3.16] and [6, 1.1], we may assume that $(R,\mathfrak{m})$ is a complete local ring such that $\mathfrak{m} \in \widehat{A}^*(\mathfrak{b})$ but $\mathfrak{m} \notin \overline{Q}^*(\mathfrak{a})$. Next, it is easy to see that $\mathfrak{m}/0_a \in \widehat{A}^*(\mathfrak{b} + 0_a/0_a)$ but $\mathfrak{m}/0_a \notin \overline{Q}^*(\mathfrak{b} + 0_a/0_a)$. Hence, without loss of generality we can assume that $(R,\mathfrak{m})$ is a reduced complete local ring. Then by [6, 2.1] we have $\mathfrak{m} \in A^*(\mathfrak{b})$ but $\mathfrak{m} \notin Q^*(\mathfrak{a})$. Thus by Proposition 2.3, there exists an integer $t \geq 1$ such that $\mathfrak{a}^t :_R \langle \mathfrak{b} \rangle \subseteq \mathfrak{b}^l$. Furthermore, as R is reduced complete local, a well known result of Rees [14, 1.4] shows that there exists an integer $m \geq 1$ such that $(\mathfrak{a}^m)_a \subseteq \mathfrak{a}^t$. Hence $(\mathfrak{a}^m)_a :_A \langle \mathfrak{b} \rangle \subseteq \mathfrak{b}^l \subseteq (\mathfrak{b}^l)_a$, as desired.

In order to show (ii)⇒(i), suppose the contrary and let $\mathfrak{p} \in A_a^*(\mathfrak{b},N) \cap \overline{Q}^*(\mathfrak{a},N)$. Then we may assume that $(R,\mathfrak{p})$ is local. Then there exists a $z^* \in \operatorname{mAss}_{R^*} N^*$ such that $\mathfrak{p}R^* = \operatorname{Rad}(\mathfrak{a}R^* + z^*)$. Furthermore, in view of [17, 1.6] and [6, 3.2(c)] we have

$$\bigcap_{n\geq 1} (\mathfrak{a}^n R^* + \operatorname{Ann}_{R^*} N^*/\operatorname{Ann}_{R^*} N^*)_a :_{R^*/\operatorname{Ann}_{R^*} N^*} \langle \mathfrak{b}R^* \rangle = \operatorname{Rad}(\operatorname{Ann}_{R^*} N^*),$$

so that

$$\bigcap_{n\geq 1} (\mathfrak{a}^n R^* + \operatorname{Ann}_{R^*} N^*/\operatorname{Ann}_{R^*} N^*)_a :_{R^*/\operatorname{Ann}_{R^*} N^*} \langle \mathfrak{p}R^* \rangle = \operatorname{Rad}(\operatorname{Ann}_{R^*} N^*)$$

and we obtain a contradiction to [6, 3.3]. ∎

COROLLARY 2.6. *Let* $(R,\mathfrak{m})$ *be local, and let* N *be a non-zero finitely generated* R*-module.*

(i) *It follows from Remark* 2.8(i) *that* $\mathfrak{m} \notin Q(\mathfrak{a},N)$ *if and only if the topology on* N *defined by* $\{\mathfrak{a}^n N :_N \langle \mathfrak{m} \rangle\}_{n\geq 1}$ *is equivalent to the* $\mathfrak{a}$*-adic topology.*

(ii) *It follows from Remark* 2.8(ii) *that* $\mathfrak{m} \notin \overline{Q}(\mathfrak{a}, N)$ *if and only if the topology defined by* $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{m} \rangle\}_{n \geq 1}$ *is equivalent to the topology induced by* $\{(\mathfrak{a}^n)_a^{(N)}\}_{n \geq 1}$.

COROLLARY 2.7. *Let* $\mathfrak{a}$ *and* $\mathfrak{p}$ *be ideals of* R *such that* $\mathfrak{a} \subseteq \mathfrak{p} \in \operatorname{Spec} R$. *Suppose that* N *is a non-zero finitely generated* R*-module. Then*:

(i) $\mathfrak{p} \notin Q(\mathfrak{a}, N)$ *if and only if the topology defined by* $\{\mathfrak{a}^n N_{\mathfrak{p}} :_{N_{\mathfrak{p}}} \langle \mathfrak{p} R_{\mathfrak{p}} \rangle\}_{n \geq 1}$ *on* $N_{\mathfrak{p}}$ *is equivalent to the* $\mathfrak{a} R_{\mathfrak{p}}$*-adic topology.*

(ii) $\mathfrak{p} \notin \overline{Q}(\mathfrak{a}, N)$ *if and only if the topology defined by* $\{(\mathfrak{a}^n R_{\mathfrak{p}})_a^{(N_{\mathfrak{p}})} :_{R_{\mathfrak{p}}} \langle \mathfrak{p} R_{\mathfrak{p}} \rangle\}_{n \geq 1}$ *is equivalent to the topology induced by* $\{(\mathfrak{a}^n R_{\mathfrak{p}})_a^{(N_{\mathfrak{p}})}\}_{n \geq 1}$.

We can derive the following results. They generalize the corresponding results of McAdam [6, 1.2 and 1.5] that extend Schenzel's original arguments in [13, (3.2) and (3.5)].

REMARK 2.8. Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R, and N a non-zero finitely generated R-module.

(i) An argument similar to that used in the proof of Proposition 2.3 shows that the $\mathfrak{a}$-adic topology on N is equivalent to the topology defined by $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{b} \rangle\}_{n \geq 1}$ if and only if $Q(\mathfrak{a}, N) \cap V(\mathfrak{b} + \operatorname{Ann}_R N) = \emptyset$.

(ii) An argument similar to that used in the proof of Theorem 2.5 shows that the topologies defined by $\{(\mathfrak{a}^n)_a^{(N)} :_R \langle \mathfrak{b} \rangle\}_{n \geq 1}$ and $\{(\mathfrak{a}^n)_a^{(N)}\}_{n \geq 1}$ are equivalent if and only if $V(\mathfrak{b} + \operatorname{Ann}_R N) \cap \overline{Q}^*(\mathfrak{a}, N) = \emptyset$.

3. Equality of asymptotic and quintasymptotic primes. The purpose of this section is to prove that for any ideal $\mathfrak{a}$ of R that can be generated by an asymptotic sequence on a non-zero finitely generated module N over R, the asymptotic and quintasymptotic primes of $\mathfrak{a}$ with respect to N coincide. We begin with

DEFINITION 3.1. Let N be a non-zero finitely generated R-module. A sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R is called an *asymptotic sequence on* N if

(i) For all $1 \leq i \leq n$, $x_i \notin \bigcup\{\mathfrak{p} \in A_a^*((x_1, \dots, x_{i-1}), N)\}$.
(ii) $N/\mathbf{x}N \neq 0$.

An asymptotic sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R (resp. contained in an ideal $\mathfrak{a}$) on N is *maximal* (resp. *maximal* in $\mathfrak{a}$) if $x_1, \dots, x_n, x_{n+1}$ is not an asymptotic sequence on N for any $x_{n+1} \in R$ (resp. $x_{n+1} \in \mathfrak{a}$). It is shown in Proposition 3.5 that all maximal asymptotic sequences on N in an ideal $\mathfrak{a}$ have the same length. This allows us to introduce the fundamental notion of *asymptotic grade* $\operatorname{agrade}(\mathfrak{a}, N)$. The concepts of asymptotic sequence and asymptotic grade were independently introduced by Rees [15] and Ratliff [11] in the case $N = R$. We refer the reader to the book [5] for some nice facts about asymptotic sequences.

The following result extends McAdam's results from commutative Noetherian rings to finitely generated modules (see [6, 0.1]).

PROPOSITION 3.2. *Let* $\mathfrak{a}$ *be an ideal of* R, *and* N *a non-zero finitely generated* R*-module. Then*

$$A_a^*(\mathfrak{a}, N) = \{\mathfrak{q} \cap R \mid \mathfrak{q} \in \overline{Q}^*(u\mathcal{R}, \mathcal{N})\}.$$

Proof. Let $\mathfrak{p} \in A_a^*(\mathfrak{a}, N)$. Then by Remark 2.8 and [5, 3.18] there exists $z \in \operatorname{mAss}_R N$ such that $z \subseteq \mathfrak{p}$ and $\mathfrak{p}/z \in \widehat{A}^*(\mathfrak{a} + z/z)$. Hence, in view of [1, 3.6], $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in \overline{Q}^*(u\mathcal{R}, \mathcal{N})$, and so $A_a^*(\mathfrak{a}, N) \subseteq \{\mathfrak{q} \cap R \mid \mathfrak{q} \in \overline{Q}^*(u\mathcal{R}, \mathcal{N})\}$.

A similar argument also works for the opposite inclusion. ∎

COROLLARY 3.3. *Under the assumptions of Proposition* 3.2,

$$\overline{Q}^*(\mathfrak{a}, N) \subseteq A_a^*(\mathfrak{a}, N).$$

LEMMA 3.4. *Let* N *be a non-zero finitely generated* R*-module and let* $\mathbf{x} = x_1, \ldots, x_n$ *be an asymptotic sequence on* N. *Then* $\operatorname{ht}_N(x_1, \ldots, x_i) = i$ *for each* $1 \le i \le n$.

Proof. It is enough to show that if $\mathfrak{p} \in \operatorname{mAss}_R N/(x_1, \ldots, x_i)N$, then $\operatorname{ht}_N \mathfrak{p} = i$. To do this, recall that

$$\operatorname{mAss}_R N/(x_1, \ldots, x_i)N \subseteq A_a^*((x_1, \ldots, x_i), N)$$

and $x_1, \ldots, x_i$ is an asymptotic sequence on N. Now, the assertion follows by induction. ∎

PROPOSITION 3.5. *Let* $(R, \mathfrak{m})$ *be local, and let* N *be a non-zero finitely generated* R*-module. Then for any ideal* $\mathfrak{a}$ *of* R,

$$\operatorname{agrade}(\mathfrak{a}, N) = \min\{\operatorname{ht}(\mathfrak{a}R^* + \mathfrak{q}/\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{mAss}_{R^*} N^*\}.$$

Proof. Let $\operatorname{agrade}(\mathfrak{a}, N) = n$, and $\mathbf{x} = x_1, \ldots, x_n$ be a maximal asymptotic sequence on N in $\mathfrak{a}$. Since the set $A_a^*(\mathbf{x}, N)$ is finite, it follows that $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in A_a^*(\mathbf{x}, N)$. There exists $\mathfrak{q} \in \operatorname{Spec} R^*$ such that $\mathfrak{q} \cap R = \mathfrak{p}$ and $\mathfrak{q} \in A_a^*(\mathbf{x}R^*, N^*)$. Furthermore, by Remark 2.8 and [5, 3.18], there exists $z \in \operatorname{mAss}_{R^*} N^*$ such that $z \subseteq \mathfrak{q}$ and $\mathfrak{q}/z \in A_a^*(\mathbf{x} + z/z)$. Then by [1, 4.9], $x_1 + z, \ldots, x_n + z$ is an asymptotic sequence in the complete domain R^*/z. Thus by Lemma 3.4, $\operatorname{ht} \mathfrak{q}/z = n$. As $\mathfrak{a}R^* + z \subseteq \mathfrak{q}$, this shows that $\operatorname{ht}(\mathfrak{a}R^* + z/z) \le n$. Now, the assertion follows easily from [1, 4.15]. ∎

Now we are ready to prove the main result of this section.

Theorem 3.6. *Let N be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an ideal of R generated by an asymptotic sequence $x_1, \ldots, x_n$ on N. Then*

$$A_a^*(\mathfrak{a}, N) = \overline{Q}^*(\mathfrak{a}, N).$$

Proof. In view of Corollary 3.3, it is sufficient to show that $A_a^*(\mathfrak{a}, N) \subseteq \overline{Q}^*(\mathfrak{a}, N)$. To do this, let $\mathfrak{p} \in A_a^*(\mathfrak{a}, N)$. Then $\mathfrak{p}R_\mathfrak{p} \in A_a^*(\mathfrak{a}R_\mathfrak{p}, N_\mathfrak{p})$, and so $\operatorname{agrade}(\mathfrak{p}R_\mathfrak{p}, N_\mathfrak{p}) = n$. Hence by Proposition 3.5, there exists $\mathfrak{q} \in \operatorname{mAss}_{R_\mathfrak{p}^*} N_\mathfrak{p}^*$ such that $\dim R_\mathfrak{p}^*/\mathfrak{q} = n$. Since $x_1 + \mathfrak{q}, \ldots, x_n + \mathfrak{q}$ is an asymptotic sequence in the complete domain $R^*/\mathfrak{q}$, it follows from Lemma 3.4 that $\operatorname{ht}(\mathfrak{a}R_\mathfrak{p}^* + \mathfrak{q}/\mathfrak{q}) = n$. Hence $\operatorname{Rad}(\mathfrak{a}R_\mathfrak{p}^* + \mathfrak{q}) = \mathfrak{p}R_\mathfrak{p}^*$, and so $\mathfrak{p} \in \overline{Q}^*(\mathfrak{a}, N)$, as required. ∎

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