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# STRONGLY BOUNDED AUTOMORPHISM GROUPS

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**Abstract.** A group $G$ is strongly bounded if every isometric action of $G$ on a metric space has bounded orbits. We show that the automorphism groups of typical countable structures with the small index property are strongly bounded. In particular we show that this is the case when $G$ is the automorphism group of the countable universal locally finite extension of a periodic abelian group.

**0. Introduction.** A group $G$ has *property* (FH) if every affine isometric action of $G$ on a Hilbert space has a fixed point [7]. For countable groups property (FH) is equivalent to Kazhdan's property (T) (see [7]). Recent papers [1] and [4] contain uncountable examples of (FH)-groups: $\mathrm{Sym}(\omega)$ and $H^\omega$, where $H$ is a non-trivial finite perfect group. The latter example was found by Koppelberg and Tits in [13] as the first example of an uncountable group with Serre's *property* (FA) (that any action on a simplicial tree without inversions has fixed points). Now it is known that (FH) implies (FA) (see [7]). On the other hand, property (FA) implies the following one: A group $G$ has *cofinality* $> \omega$ if $G$ cannot be presented as the union of a strictly increasing chain $\{H_n : n \in \omega\}$ of proper subgroups. We now define the main notion of the paper.

Definition 1 ([1], [4]). A group $G$ is *Cayley bounded* if for every generating subset $U \subset G$ there exists $n \in \omega$ such that every element of $G$ is a product of $n$ elements of $U \cup U^{-1} \cup \{1\}$. A group is *strongly bounded* if it is Cayley bounded and has cofinality $> \omega$.

It is shown in [4] that strongly bounded groups have property (FH). Moreover $G$ is strongly bounded if and only if every isometric action of $G$ on a metric space has bounded orbits ([4, Proposition 2.4]). The groups $\mathrm{Sym}(\omega)$ and $H^\omega$, where $H$ is a non-trivial finite perfect group, are strongly bounded. This fact for the latter group has been proved in [4] and thus $H^\omega$ has become the first example answering the question from [7] whether uncountable (FH)-groups exist.

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The case of $\mathrm{Sym}(\omega)$ has been considered in [1]. The proof there is based on some ideas from [16]. In our paper we suggest another approach, which covers many cases of this kind. We prove that if a countable structure $M$ has an amalgamation base (see [10]), then $\mathrm{Aut}(M)$ has a property very close to strong boundedness. As a corollary we will see that many well known automorphism groups are strongly bounded. We also extend the number of examples by the automorphism group of the countable universal locally finite group and some related groups.

The property which we study is slightly technical and is a topological version of the condition arising in the following (very useful) characterization of strongly bounded groups, which has been proved in [4, Proposition 2.7]:

> A group $G$ is strongly bounded if and only if for every presentation of $G$ as $G = \bigcup X_n$ for an increasing sequence $X_n$, $n \in \omega$, with $\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subseteq X_{n+1}$ there is a number $n$ such that $X_n = G$.

Further definitions are given in the next section.

When the paper was submitted, Y. de Cornulier informed the author that independently A. Kechris and Ch. Rosendal had obtained in [12] a stronger version of our Theorem 4. Using it they have proved that if $G$ is a closed oligomorphic subgroup of $\mathrm{Sym}(\omega)$ having ample generic elements, then $G$ is strongly bounded (Theorem 5.17 of [12]). This covers all possible oligomorphic applications of Theorem 4. Therefore we concentrate only on non-oligomorphic examples omitting the cases of the random graph and $\omega$-categorical $\omega$-stable structures.

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Notation. We denote by $\omega$ the set of all non-negative integer numbers; $\mathbb{Z}(m)$ = the cyclic group consisting of $m$ elements.

**1. Generic automorphisms and strong boundedness.** For a countable structure $M$ we study $\mathrm{Aut}(M)$ as a closed subgroup of $\mathrm{Sym}(\omega)$. Here we consider $\mathrm{Sym}(\omega)$ as a complete metric space by defining $d(g,h) = \sum\{2^{-n} : g(n) \neq h(n) \text{ or } g^{-1}(n) \neq h^{-1}(n)\}$. It is well known that all closed subgroups of $\mathrm{Sym}(\omega)$ are of the form $\mathrm{Aut}(M)$ for appropriate structures $M$ defined on $\omega$.

Definition 2 (see [10]). An $\mathrm{Aut}(M)$-invariant set $\Gamma$ of finite subsets of $M$ (including $\emptyset$) is an *amalgamation base* if:

- any tuple $\overline{\alpha}$ of finite maps extendible to a tuple of automorphisms has an extension to a tuple of automorphisms inducing a tuple of permutations on some $A \in \Gamma$ with $\mathrm{Dom}(\overline{\alpha}) \subseteq A$;
- for any $A, B, C \in \Gamma$ with $A \subseteq B$, $A \subseteq C$ there exist $\gamma \in \mathrm{Aut}(M/A)$ and $D \in \Gamma$ containing $C \cup \gamma(B)$ such that whenever $\alpha, \beta \in \mathrm{Aut}(M)$ induce permutations of $\gamma(B)$ and $C$ which agree on $A$ then there exists $\delta \in \mathrm{Aut}(M)$ extending both $\alpha|\gamma(B)$ and $\beta|C$ and inducing a permutation on $D$.

Theorems 2.9 and 5.3 of [10] prove that if $M$ has an amalgamation base as above, then $M$ has the *small index property*: any subgroup of $\mathrm{Aut}(M)$ of index less than the continuum is open.

It is worth noting that in [10] the corresponding definition is slightly wider. Nevertheless the formulation given in Definition 2 holds for all examples studied in [10]. We use this form because it allows us to find easily some additional properties of $\mathrm{Aut}(M)$. We now describe this in detail.

A tuple $\overline{g} = (g_1, \ldots, g_n) \in \mathrm{Aut}(M)^n$ is *$\Gamma$-generic* if the following two conditions hold:

(a) for any $A \in \Gamma$ the set of pointwise stabilizers $\{\mathrm{Aut}(M/B) : A \subseteq B \in \Gamma$ and $g_i(B) = B$ for all $i \leq n\}$ is a base of open neighbourhoods of 1 in $\mathrm{Aut}(M)$;

(b) for any $\overline{g}$-invariant $A \in \Gamma$ and any $B \in \Gamma$ extending $A$, if $\overline{h}$ is a tuple of permutations of $B$ which agree with $\overline{g}$ on $A$, then there is $\alpha \in \mathrm{Aut}(M/A)$ such that $\overline{g}^{\alpha}$ extends $\overline{h}$.

By Proposition 2.3 of [10], if two $\Gamma$-generic tuples $\overline{g}$ and $\overline{g}'$ induce the same permutation of a set $B \in \Gamma$, then there is $\alpha \in \mathrm{Aut}(M/B)$ conjugating $\overline{g}$ to $\overline{g}'$. In particular we see that all $\Gamma$-generic tuples are conjugate (because we assume $\emptyset \in \Gamma$).

Theorem 2.9 of [10] states that if $\Gamma$ is an amalgamation base of $M$, then the set of all $\Gamma$-generic tuples from $(\mathrm{Aut}(M))^n$ is comeagre in $(\mathrm{Aut}(M))^n$ in the product topology (in [10] the proof is given for $\omega$-categorical structures, but it works in general; see [2]).

As a result we obtain the following statement (probably folklore).

PROPOSITION 3. *If a countable structure $M$ has an amalgamation base $\Gamma$ then for any $n$ the space* $(\mathrm{Aut}(M))^n$ *has generic tuples* (*its* $\mathrm{Aut}(M)$*-orbit in the space* $(\mathrm{Aut}(M))^n$ *under the conjugacy action on coordinates is comeagre*) *and they are exactly the $\Gamma$-generic tuples.*

We are ready to formulate the main result of this section ([1]).

([1]) It is shown in [12] that the statement of the theorem holds under the weaker assumption that for every $n$ the space $(\mathrm{Aut}(M))^n$ has a generic tuple.

Theorem 4. *Assume that $M$ is countably infinite and has an amalgamation base. Let $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_i \subseteq \cdots$ be an increasing sequence such that* $\mathrm{Aut}(M) = \bigcup X_i$ *and for each $n$, $\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subseteq X_{n+1}$ and $X_n$ does not contain an open subgroup. Then $X_n = \mathrm{Aut}(M)$ for some $n$.*

*Proof.* Let $G = \mathrm{Aut}(M)$. Let $\Gamma$ be an amalgamation base of $M$. Assume that the sequence $X_n$ is strictly increasing. If for some tuple $\overline{a} \in M$ and some number $n \in \omega$ the set $G_{\overline{a}} \cap X_n$ is comeagre in the stabilizer $G_{\overline{a}} = \mathrm{Aut}(M/\overline{a})$ (thus has the Baire property and is non-meagre in $G_{\overline{a}}$), then Pettis' theorem ([11, Theorem 9.10]) implies that the set $(G_{\overline{a}} \cap X_n) \cdot (G_{\overline{a}} \cap X_n)^{-1}$ contains an open neighbourhood of 1. Since $X_n \cdot X_n \cdot X_n^{-1} \subseteq X_{n+2}$, we see that $X_{n+2} \cap G_{\overline{a}} = G_{\overline{a}}$, thus $X_{n+2}$ contains an open subgroup of $G$, a contradiction.

As the union of countably many meagre sets is meagre, we may assume that no $X_n$ is meagre in $G$. Now for any $\overline{a}$ and $n$ some $gG_{\overline{a}} \cap X_n$ is not meagre in $gG_{\overline{a}}$; thus $G_{\overline{a}} \cap X_{n+2}$ is not meagre in $G_{\overline{a}}$. Hence we may assume that $G_{\overline{a}} \cap X_n$ is not meagre in $G_{\overline{a}}$ for any $\overline{a}$ and $n$.

Our further argument is based on the proof of Theorem 6.1 of [10]. We build a binary tree $\{g_s : s \in 2^{<\omega}\} \subset G$ such that for any $s \in 2^{<\omega}$ we have $g_{s0} \in X_{|s|}$, $g_{s1} \notin X_{|s|+2}$ and the tuple $(g_\emptyset, g_{s(1)}, \ldots, g_s)$ is a $\Gamma$-generic tuple of automorphisms. Then for every $s \in 2^{<\omega}$ we define $\gamma_s \in G$ so that $g_{s1}^{\gamma_{s1}} = g_{s0}^{\gamma_{s0}} = g_{s0}^{\gamma_s}$ (at every step existence of such $\gamma_s$ will follow from Proposition 2.3 of [10]).

We start our construction from any generic $g_\emptyset$ and $\gamma_\emptyset := \mathrm{id}_M$. Enumerate $M = \{a_0, \ldots, a_n, \ldots\}$. Assume that all $\gamma_s$ with $|s| = n$ are already defined. Let $A = \{a_0, \ldots, a_n, \gamma_s^{-1}(a_0), \ldots, \gamma_s^{-1}(a_n)\}$. Since $(g_\emptyset, g_{s(1)}, \ldots, g_s)$ is a generic tuple of automorphisms, there is a finite $B_s \subset M$ such that $A \subseteq B_s$ and all $g_\emptyset, g_{s(1)}, \ldots, g_s$ belong to the $G$-stabilizer of the set $B_s$. By the Kuratowski–Ulam Theorem the set $C = \{g \in G : (g_\emptyset, g_{s(1)}, \ldots, g_s, g)$ is $\Gamma$-generic$\}$ is comeagre in $G$. Since $G_{B_s} \cap X_n$ is neither meagre nor comeagre, there are $g_{s0} \in C \cap G_{B_s} \cap X_n$ and $g_{s1} \in (C \cap G_{B_s}) \setminus X_{n+2}$. Since the tuples $(g_\emptyset, g_{s(1)}, \ldots, g_s, g_{s0})$ and $(g_\emptyset, g_{s(1)}, \ldots, g_s, g_{s1})$ are generic and agree on $B_s$, there is $f \in G_{B_s}$ such that

$$(g_\emptyset, g_{s(1)}, \ldots, g_s, g_{s1})^f = (g_\emptyset, g_{s(1)}, \ldots, g_s, g_{s0}).$$

Define $\gamma_{s0} := \gamma_s$ and $\gamma_{s1} := f \cdot \gamma_s$.

By the choice of $B_s$ for any $\sigma \in 2^\omega$ the sequence $\gamma_{\sigma|n}$, $n \in \omega$, is a Cauchy sequence. As a result it has a limit $\gamma_\sigma$. For $\sigma \neq \tau$ with $\sigma|n = s0$ and $\tau|n = s1$ we have $g_{s0}^{\gamma_\sigma} = g_{s0}^{\gamma_{s0}} = g_{s0}^{\gamma_s}$ and $g_{s1}^{\gamma_\tau} = g_{s1}^{\gamma_{s1}} = g_{s0}^{\gamma_s}$ (by the choice of $f$ at every step). Since $g_{s0} \in X_n$ and $g_{s1} \notin X_{n+2}$, we see that $\gamma_\sigma \gamma_\tau^{-1} \notin X_n$. On the other hand, as $G = \bigcup X_n$, we may assume that $X_{n-2}$ contains uncountably many elements $\gamma_\delta$. Thus we may assume that $\gamma_\sigma$ and $\gamma_\tau$ as above are in $X_{n-2}$. This together with the properties of the sequence $X_n$ gives a contradiction with the condition $\gamma_\sigma \gamma_\tau^{-1} \notin X_n$. ∎

**2. Universal locally finite extensions of abelian groups.** In 1959 Ph. Hall [6] discovered the countable universal locally finite group $U$. The group has become an attractive object both for algebraists [8], [9] and logicians [14], [15]. Below we give an extended version of the original definition.

Let $A$ denote a countable periodic abelian group. A locally finite group $G$ is called a *universal locally finite extension of* $A$ (see [9]) if $A \subseteq Z(G)$ and the following condition ($A$-injectivity) is satisfied: suppose $A \leq B \leq D$ with $A \subset Z(D)$ and $|D/A| < \infty$; then for any $A$-embedding $\phi : B \to G$ ($\phi(a) = a$ for $a \in A$) there exists an $A$-embedding $D \to G$ which extends $\phi$. It is known that such a group $G$ is unique if $|G| = \omega$ (see [9]); in this case it is denoted by $H(A)$. It is also known that any finite partial automorphism fixing $A$ pointwise extends to an inner automorphism of $H(A)$.

The following theorem is the main point of this section. In the formulation, we denote by $\mathrm{Aut}(H(A)/A)$ the set of all automorphisms of $H(A)$ which fix $A$ pointwise.

THEOREM 5. *The group* $\mathrm{Aut}(H(A)/A)$ *is strongly bounded.*

**2.1.** *Universal locally finite groups.* The proof of Theorem 5 is based on the following presentation of $H(A)$. Let $\mathbf{K}_A$ be the isomorphism closure of the class of all finite subgroups of $H(A)$ in the language extended by symbols for all elements of $A$ (and denoted by $L_A$). The subgroups are considered as partial models of $L_A$. Then $\mathbf{K}_A$ has the joint embedding property and the amalgamation property; $H(A)$ is the universal homogeneous structure for $\mathbf{K}_A$ (see [9]).

We now describe the amalgamation in $\mathbf{K}_A$ (following Lemmas 2 and 3 of [9]); this will also be applied in the proof. Let $L$, $K$ and $E = L \cap K$ be structures from $\mathbf{K}_A$. We may assume that they all have the same intersection with $A$ (otherwise extend the group, say $L$, to an appropriate subgroup of the direct sum of $L$ and $A$ with $L \cap A$ amalgamated). Let $S$ be a left transversal of $E$ in $L$ and $T$ be a left transversal of $E$ in $K$. The *permutation product of* $L$ *and* $K$ *relative to* $S$ *and* $T$ is the subgroup $P < \mathrm{Sym}(S \times T \times E)$ generated by $L$ and $K$ which are considered as permutation groups on $S \times T \times E$ with the following actions:

$$(s,t,e)l = (s',t,e'), \quad \text{where} \quad s \cdot e \cdot l = s' \cdot e',$$
$$(s,t,e)k = (s,t',e'), \quad \text{where} \quad t \cdot e \cdot k = t' \cdot e'.$$

It is easy to see that $P$ amalgamates $L$ and $K$ over $E$ under their representations in $P$. Now the group $H(A)$ is constructed by Fraissé's procedure as a tower of structures from $\mathbf{K}_A$: $G_1 \leq \cdots \leq G_n \leq \cdots$. At every step the group $G_{n+1}$ is obtained from $G_n$ by amalgamation with some embedding over $A$ of a subgroup $E \leq G_n$ into some structure from $\mathbf{K}_A$.

Let us verify that $G := \bigcup G_i$ has the property in the definition of $H(A)$. If $B$ is a subgroup of $G := \bigcup G_i$, $A \leq B < D$ and $|D/A|$ is finite, then find a finite $D_0 < D$ such that $D$ is the direct sum of $D_0$ and $A$ with $D_0 \cap A$ amalgamated. We may additionally assume that $B$ is generated by $D_0 \cap B$ and $A$. By the construction of $G$ the group $D_0$ is $A$-embeddable into $G$ over $B \cap D_0$. It is easy to see that the image $\sigma(D_0)$ and $A$ generate a subgroup of $H(A)$ which is $B$-isomorphic to $D$.

*Proof of Theorem 5.* The proof consists of two steps. The first one is the claim that $\mathrm{Aut}(H(A)/A)$ has an amalgamation base. The second one is a reduction of the statement of the theorem to Theorem 4.

First we prove that $\mathbf{K}_A$ (considered as the class of all subgroups of $H(A)$) is an amalgamation base. We want to verify the amalgamation property from the definition of amalgamation bases and the property that any $n$-tuple $\overline{\alpha}$ of finite partial automorphisms extends to an $n$-tuple $\overline{\gamma}$ of automorphisms of some $D \in \mathbf{K}_A$ (fixing $D \cap A$ pointwise).

To see the latter we may assume that the tuple $\overline{\alpha}$ is defined on a subgroup $K$ of $H(A)$. Then for such a pair $(K, \overline{\alpha})$, with $K \in \mathbf{K}_A$, any $\mathrm{Range}(\alpha_i)$ is a subgroup of $K$ (where $\alpha_i \in \overline{\alpha}$). Moreover defining $\overline{\alpha}$ on $K \cap A$ to be the identity we have

$$K \cap A = \mathrm{Range}(\alpha_i) \cap A = \mathrm{Dom}(\alpha_i) \cap A.$$

To find an appropriate pair $(D, \overline{\gamma})$ extending $(K, \overline{\alpha})$ we apply Lemma 1 from [6]: any isomorphism between two subgroups of $K$ is realized by an inner automorphism of $\mathrm{Sym}(K)$ under the regular representation of $K$ in $\mathrm{Sym}(K)$. Now the centralizer of $K \cap A$ in $\mathrm{Sym}(K)$ becomes the required extension $D$ of $K$, where $\overline{\alpha}$ is extended to a tuple of automorphisms fixing $K \cap A$ pointwise.

Let us verify that the amalgamation property from the definition of amalgamation bases holds for $\mathbf{K}_A$. Let $L$, $K$ and $E = L \cap K$ be structures from $\mathbf{K}_A$. We assume that these groups have the same intersection with $A$ (otherwise extend the group, say $L$, to an appropriate subgroup of the direct sum of $L$ and $A$ with $L \cap A$ amalgamated). Let $S$ be a left transvesal of $E$ in $L$ and $T$ be a left transversal of $E$ in $K$. Let $J$ be the permutation product of $L$ and $K$ relative to $S$ and $T$. We claim that taking an embedding of $J$ into $H(A)$ over $L$ and $A$ we obtain the required $\gamma(K)$ and the required common extension $D$ of $L$ and $\gamma(K)$.

To see this let $\alpha \in \mathrm{Aut}(L)$, $\beta \in \mathrm{Aut}(K)$ and let $\lambda \in \mathrm{Aut}(E)$ be induced by both $\alpha$ and $\beta$. The groups $\langle \alpha \rangle$ and $\langle \beta \rangle$ are homomorphic images of a common finite group $\langle \delta \rangle$, and $\alpha \in \mathrm{Aut}(L)$ and $\beta \in \mathrm{Aut}(K)$. Then consider the semidirect products $L \rtimes \langle \delta \rangle$ and $K \rtimes \langle \delta \rangle$. They have isomorphic subgroups generated by $E$ and $\langle \delta \rangle$. Since $\delta$ preserves $E$, we have $L \cap \langle E, \delta \rangle = E = K \cap \langle E, \delta \rangle$. We now see that $S$ is a left transversal of $\langle E, \delta \rangle$ in $L \rtimes \langle \delta \rangle$

and $T$ is a left transversal of $\langle E, \delta\rangle$ in $K \rtimes \langle\delta\rangle$ respectively. Amalgamating this diagram by the method described before the proof we obtain a common extension $\widehat{J}$ of $L\rtimes\langle\delta\rangle$ and $K\rtimes\langle\delta\rangle$, where both $\alpha$ on $L$ and $\beta$ on $K$ are realized by the inner automorphism induced by $\delta$. It is clear that $J$ is isomorphic to the subgroup generated by $L$ and $K$ in $\widehat{J}$. Therefore the class $\mathbf{K}_A$ is an amalgamation base. Below we apply this for the second step of the proof.

We use some arguments similar to Theorem 1.3 of [2]. Let

$$G := \mathrm{Aut}(H(A)/A) = \bigcup_{i\in\omega} X_i,$$

where $X_0 \subseteq X_1 \subseteq \cdots$ is a chain of subsets of $G$ satisfying de Cornulier's condition (as at the end of the Introduction). To apply Theorem 4 we only need to show that for all natural numbers $m$, $X_m$ contains no open subgroups. Suppose, to the contrary, that some $X_m$ contains an open subgroup $L < G$. Then there is a finite subgroup $F < H(A)$ such that the pointwise stabilizer $G_F$ is a subgroup of $L$. Take $F' < H(A)$ such that $\langle F, F'\rangle$ is isomorphic to $F \oplus_{F\cap A} F$ under an isomorphism fixing $F$ and mapping $F'$ onto another $F$. Then $F$ and $F'$ are of the same isomorphism type over $A$ in $H(A)$ and $F\cap F' \subseteq A$. By increasing $m$ if necessary we may assume that $X_m$ contains all automorphisms mapping $F$ onto $F'$ and fixing $A$. Then $G_{F'} \subseteq X_m^{-1}\cdot X_m\cdot X_m \subseteq X_{m+2}$ and for any $F''$ of the same $G_F$-orbit with $F'$ over $A$ we similarly have $G_{F''} \subseteq X_{m+4}$.

Let $\alpha\in G$. Let $\beta$ be an inner automorphism of $H(A)$ such that $\beta|_F = \alpha|_F$. Then $\beta^{-1}\alpha \in G_F$. Find a finite subgroup $F''$ as above such that the element of $H(A)$ which induces $\beta$, centralizes $F''$ (apply universality of $H(A)$). Then $\beta \in G_{F''}$ and $\alpha \in X_{m+4}\cdot X_{m+4} \subseteq X_{m+5}$. Hence $G = X_{m+5}$. ∎

The restriction map $\mathrm{Aut}(H(A)) \to \mathrm{Aut}(A)$ is a surjective homomorphism with the kernel $\mathrm{Aut}(H(A)/A)$. By Theorem 5 if $\mathrm{Aut}(H(A))$ acts on a tree $T$ (resp. a Hilbert space $\mathbf{H}$) by (affine) isometries, then $T^{\mathrm{Aut}(H(A)/A)} \neq \emptyset$ (resp. $\mathbf{H}^{\mathrm{Aut}(H(A)/A)} \neq \emptyset$) and therefore $\mathrm{Aut}(H(A))$ fixes a point if and only if $\mathrm{Aut}(A)$ fixes a point on $T^{\mathrm{Aut}(H(A)/A)}$ (resp. $\mathbf{H}^{\mathrm{Aut}(H(A)/A)}$). As a result we have the following theorem.

THEOREM 6. *Suppose that $A$ is a countable periodic abelian group. Then* $\mathrm{Aut}(H(A))$ *has property* (FA) (*resp.* (FH)) *if and only if* $\mathrm{Aut}(A)$ *does.* ∎

**2.2.** *Strongly bounded automorphism groups of countable abelian $p$-groups.* Let $A$ denote a countable periodic abelian group (as in Theorem 6). It is well known that $A$ decomposes into a direct sum $\sum_p D_p \oplus \sum_p U_p$, where each $D_p$ is a direct sum of Prüfer groups $\mathbb{Z}(p^\infty)$ and each $U_p$ is a reduced $p$-group (without divisible subgroups). Since each $D_p$ is a characteristic subgroup of $A$, $\prod_p \mathrm{Aut}(D_p)$ is a homomorphic image of $\mathrm{Aut}(A)$ (Chapter 16 of [5]).

It is clear that $\mathrm{Aut}(A)$ has property (FH) (resp. (FA)) if and only if so do the groups $\prod_p \mathrm{Aut}(D_p)$ and the corresponding kernel. In our next result we strengthen Theorem 6 by reducing the problem if $\mathrm{Aut}(H(A))$ satisfies (FH) to the kernel.

To find amalgamation bases of abelian $p$-groups we shall use a fact which can be deduced from the corresponding chapters of [5]; it is explicitly stated in Theorem 1.2 of [3] describing abelian groups with quantifier elimination:

> Let $C$ be $\mathbb{Z}(p^n)$, where $n \in (\omega \setminus \{0\}) \cup \{\infty\}$. Let $B = \bigoplus_{i<l} C$, where $l \in \omega \cup \{\omega\}$. Then any isomorphism between finite subgroups of $B$ extends to an automorphism of $B$.

For $n = \infty$ (this is the only case we are interested in) the group $B$ is the countable universal homogeneous structure with respect to the class $\mathbf{L}$ of all finite subgroups. This means that $\mathbf{L}$ has the joint embedding property and the amalgamation property (in the standard sense): the amalgamation of groups $H, K \in \mathbf{L}$ can be easily obtained by $\bigoplus$ with $K \cap H$ amalgamated.

THEOREM 7. *Let $Q$ be a set of prime numbers and $f$ be a function $Q \to (\omega \setminus \{0\}) \cup \{\infty\}$. For $p \in Q$ let $A_p(f)$ be the direct sum $\bigoplus_\omega(\mathbb{Z}(p^{f(p)}))$ of pairwise isomorphic cyclic (or Prüfer) groups and $A(f) = \bigoplus_{p\in Q} A_p(f)$. Then the group $\mathrm{Aut}(A(f))$ is strongly bounded.*

*Proof.* The group $\mathrm{Aut}(A(f))$ acts on the disjoint union $\widehat{A} = \bigcup_{p\in Q} A_p(f)$. We consider the set $\widehat{A}$ as a structure with unary predicates for all $A_p(f)$, where each $A_p(f)$ is considered as an abelian group. The automorphism group of that structure coincides with $\mathrm{Aut}(A(f))$.

Let $\mathbf{K}$ be the class of all substructures of $\widehat{A}$ of the form $K_{p_1} \cup \dots \cup K_{p_l}$ (considered as tuples $(K_{p_1}, \dots, K_{p_l})$), where $\{p_1, \dots, p_l\} \subseteq Q$ and each $K_{p_i}$ is a finite subgroup of $A_{p_i}(f)$. We want to show that $\mathbf{K}$ is an amalgamation base of $\widehat{A}$.

We assume that any partial automorphism of $\widehat{A}$ is defined on a disjoint union of subgroups of appropriate $A_p(f)$'s (it can be made so). Then for every pair $(\overline{K}, \overline{\alpha})$, where $\overline{K} \in \mathbf{K}$, any $\mathrm{Range}(\alpha_i)$ consists of a collection of subgroups of groups $K_{p_i}$ from $\overline{K}$ (where $\alpha_i \in \overline{\alpha}$). The condition that the set of all pairs $(\overline{D}, \overline{\alpha})$, where $\overline{D} \in \mathbf{K}$ and $\overline{\alpha}$ is an $n$-tuple of automorphisms of $\overline{D}$, is cofinal in the set of $n$-tuples of partial automorphisms easily follows from the fact stated before the theorem.

The amalgamation property from the definition of amalgamation bases can be shown as follows. For finite $H, K \in A_p(f)$ take the direct sum $H \oplus_{H\cap K} K$ of $H$ and $K$ with $H \cap K$ amalgamated. If $\alpha \in \mathrm{Aut}(H)$, $\beta \in \mathrm{Aut}(K)$ induce the same automorphism $\gamma$ of $H \cap K$, then define $\delta((h,k)) := (\alpha(h), \beta(k))$ for $(h,k) \in H \oplus_{H\cap K} K$. It is easily seen that $\delta$ is an automorphism. Extending this procedure to tuples we see that the class of substructures $\overline{K}$ of $\widehat{A}$ as above is an amalgamation base.

We now apply this to the statement of the theorem. The application is basically the same as in the proof of Theorem 5. ∎

Let $A$ be a countable periodic abelian group and $\sum_p D_p \oplus \sum_p U_p$ be a decomposition of $A$ into a direct sum of Prüfer groups $\mathbb{Z}(p^\infty)$ (occurring in $D_p$) and reduced $p$-groups $U_p$. As we have already noted, $\prod_p \mathrm{Aut}(D_p)$ is a homomorphic image of $\mathrm{Aut}(A)$. In the following proposition we consider the case of $\prod_p \mathrm{Aut}(D_p)$.

Proposition 8. *The group $\prod_p \mathrm{Aut}(D_p)$ has either property* (FH), (FA), *or strong boundedness if and only if every $D_p$ is an infinite direct sum of Prüfer $p$-groups.*

*Proof.* Assume that there is $p$ such that $D_p$ is a finite direct sum of Prüfer $p$-groups. Then it is well known that $\mathrm{Aut}(D_p)$ is of the form $\mathrm{GL}_n(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. By [5, Vol. 2, p. 318] the group of units of $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}(p-1) \times J_p$, where $J_p$ is the additive group of $p$-adic integers. Consider $J_p$ as a subgroup of the vector space $\mathbb{Q}_p$ over $\mathbb{Q}$. Since $\dim(\mathbb{Q}_p) = 2^\omega$, the group $J_p$ embeds into $\mathbb{R}$. This means that there exists a non-trivial homomorphism from $\mathrm{Aut}(D_p)$ into $J_p$; it can be realized by the determinant (considered as a homomorphism onto the group of units of $\mathbb{Z}_p$) and the projection onto $J_p$. To see that properties (FA), (FH), and strong boundedness do not hold, notice that $\mathrm{rank}(J_p) = 2^\omega$ and the group $J_p$ can be expressed as the union of a countable chain of proper subgroups (of cardinality $2^\omega$). Thus the cofinality of $\mathrm{Aut}(D_p)$ and consequently $\prod_p \mathrm{Aut}(D_p)$ is not greater than $\omega$.

For the other direction it is enough to apply Theorem 7 to $\prod_p \mathrm{Aut}(D_p)$ with infinite-dimensional groups $D_p$. ∎

We finish this section by a further remark concerning properties (FH) and (FA). Let $p$ be a prime number. We consider $U_p$. Let $\tau$ be the $p$-length of $U_p$. It is defined to be the minimal ordinal $\sigma$ with $p^\sigma U_p = 0$, where $p^0 U_p = U_p$, $p^{\gamma+1} U_p = p(p^\gamma U_p)$ and for limit ordinals $\varrho$, $p^\varrho U_p = \bigcap\{p^\gamma U_p : \gamma < \varrho\}$ (see [5]).

Let $P_\sigma := p^\sigma U_p[p]$, $\sigma \le \tau$. Consider the natural homomorphism (Section 114 of [5])

$$\varrho : \mathrm{Aut}(U_p) \to \prod_{\sigma<\tau} \mathrm{Aut}(P_\sigma/P_{\sigma+1}).$$

Since $U_p$ is countable the homomorphism is onto (Theorem 114.2 of [5]). The quotient $P_\sigma/P_{\sigma+1}$ is a vector space over $\mathrm{GF}(p)$. If its dimension is finite, we have the natural homomorphism (with vanishing SL) from $\mathrm{Aut}(P_\sigma/P_{\sigma+1})$ onto the multiplicative group $\mathrm{GF}(p)^*$.

PROPOSITION 9. *If* $\mathrm{Aut}(U_p)$ *has property* (FA) (*resp.* (FH)), *then the number of finite-dimensional spaces* $P_\sigma/P_{\sigma+1}$ *is finite.*

*Proof.* If this number is infinite, by the above we have a homomorphism from $\mathrm{Aut}(U_p)$ onto $(\mathrm{GF}(p)^*)^\omega$. By Theorem 36.1 of [5] a *basic subgroup* (see [5]) of $(\mathrm{GF}(p)^*)^\omega$ is an endomorphic image of the group. As a result we have a homomorphism onto a direct sum $\Lambda$ of infinitely many copies of a finite cyclic group. Since $\Lambda$ has cofinality $\le \omega$, the group $\mathrm{Aut}(U_p)$ does not have (FA). ∎

**3. Free groups.** We now give another application of Theorem 4. Consider the case of the automorphism group of the free group $\mathbf{F}_\omega$ of rank $\omega$. Let $\mathcal{B}$ consist of all subgroups of $\mathbf{F}_\omega$ generated by finite subsets of bases of $\mathbf{F}_\omega$. It is clear that $\mathcal{B}$ is $\mathrm{Aut}(\mathbf{F}_\omega)$-invariant and the pointwise stabilizer of any member of $\mathcal{B}$ is clopen in $\mathrm{Aut}(\mathbf{F}_\omega)$. It is shown in [2] that:

(a) any tuple $\overline{\alpha}$ of finite maps extendible to a tuple of automorphisms of $\mathbf{F}_\omega$ has an extension to a tuple of automorphisms inducing a tuple of permutations on some $A \in \mathcal{B}$ with $\mathrm{Dom}(\overline{\alpha}) \subseteq A$;

(b) for any $A, B, C \in \mathcal{B}$ with $A \le B$, $A \le C$, there exist $\gamma \in \mathrm{Aut}(\mathbf{F}_\omega/A)$ and some $D \in \mathcal{B}$ containing $C \cup \gamma(B)$ such that whenever $\alpha, \beta \in \mathrm{Aut}(\mathbf{F}_\omega)$ induce permutations of $\gamma(B)$ and $C$ which agree on $A$, then there exists $\delta \in \mathrm{Aut}(\mathbf{F}_\omega)$ extending both $\alpha|\gamma(B)$ and $\beta|C$ and inducing a permutation on $D$.

Although $\mathcal{B}$ consists of infinite subgroups, the pointwise stabilizer of each $F \in \mathcal{B}$ coincides with the stabilizer of some tuple $\overline{a} \in F$. This allows us to carry out all standard arguments. Lemma 1.2 of [2] states that the set of all $\mathcal{B}$-generic $n$-tuples (which are defined exactly as in Section 1) is comeagre in $\mathrm{Aut}(\mathbf{F}_\omega)$. As in Section 1 we deduce that $(\mathrm{Aut}(\mathbf{F}_\omega))^n$ has generic tuples of automorphisms and the set of generic tuples coincides with the set of $\mathcal{B}$-generic tuples. Applying Theorem 4 and arguments for Theorem 5 (as well as Theorem 1.3 of [2]) we obtain the following theorem.

THEOREM 10. *The group* $\mathrm{Aut}(\mathbf{F}_\omega)$ *is strongly bounded.*

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