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INDUCED ALMOST CONTINUOUS FUNCTIONS ON HYPERSPACES

BY

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Abstract. For a metric continuum X, let $C(X)$ (resp., 2^X) be the hyperspace of subcontinua (resp., nonempty closed subsets) of X. Let $f : X \to Y$ be an almost continuous function. Let $C(f) : C(X) \to C(Y)$ and $2^f : 2^X \to 2^Y$ be the induced functions given by $C(f)(A) = \mathrm{cl}_Y(f(A))$ and $2^f(A) = \mathrm{cl}_Y(f(A))$. In this paper, we prove that:

- If 2^f is almost continuous, then f is continuous.
- If $C(f)$ is almost continuous and X is locally connected, then f is continuous.
- If X is not locally connected, then there exists an almost continuous function $f : X \to [0, 1]$ such that $C(f)$ is almost continuous and f is not continuous.

Introduction. A *continuum* is a nonempty, nondegenerate, compact connected metric space. All the spaces considered in this paper are continua. Given a continuum X we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\}.$$

Both are considered with the Hausdorff metric D.

Given a (not necessarily continuous) function between continua $f : X \to Y$, we can consider its graph $\Lambda(f) = \{(p, f(p)) \in X \times Y : p \in X\}$ and the induced function $2^f : 2^X \to 2^Y$ given by $2^f(A) = \mathrm{cl}_Y(f(A))$ ($f(A)$ is the image of A under f). We are interested in functions $f : X \to Y$ for which the natural induced map $C(f) : C(X) \to C(Y)$ is defined. Thus we need to require that, for each $A \in C(X)$, $f(A)$ is connected; we call a function satisfying this condition *weakly Darboux* (in Real Analysis a *Darboux function* is a function such that the image of a connected set is a connected set).

Of course, every continuous function is weakly Darboux. It is known that almost continuous functions are weakly Darboux (see Lemma 1). Recall that f is *almost continuous* provided that, for each open subset U of $X \times Y$ such that $\Lambda(f) \subset U$, there exists a continuous function $g : X \to Y$ such that $\Lambda(g) \subset U$. We say that f is *proper almost continuous* if f is almost contin-

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uous but not continuous. A simple example of a proper almost continuous function is the function $h : [0,1] \to [-1,1]$ given by $h(t) = \sin(1/t)$ if $t > 0$, and $h(0) = 0$.

Almost continuous functions were introduced by J. Stallings in [3] where he used them to generalize some fixed point theorems.

Given a continuous function between continua $f : X \to Y$ and a class of mappings $\mathcal{M}$, the problem of determining if one of the following properties implies another has been widely studied:

(a) f belongs to $\mathcal{M}$,
(b) $C(f)$ belongs to $\mathcal{M}$,
(c) 2^f belongs to $\mathcal{M}$.

A discussion on this topic can be found in [2, Section 77].

In this paper, we study this problem for the class $\mathcal{M}$ of (not necessarily continuous) almost continuous functions. Observe that, to define 2^f it is not necessary to require that f is almost continuous. Since the restriction of an almost continuous function to a closed subset of the domain is also almost continuous (see [3, Proposition 2]), if 2^f is almost continuous, then so is $2^f|\{\{p\} \in 2^X : p \in X\}$. This implies that f is almost continuous. Thus (c) implies (a) and, similarly, (b) implies (a).

The first result we obtain is that if a function $f : X \to Y$ and its induced function 2^f are weakly Darboux, then f is continuous. Thus, for the class of weakly Darboux functions, (a) and (c) together imply (b). The second result says that if X is locally connected and the functions f and $C(f)$ are weakly Darboux, then f is continuous, and the third result says that if X is not locally connected, then it is possible to construct a proper almost continuous function $f : X \to [0,1]$ such that $C(f)$ is almost continuous. Thus (a) and (b) together do not imply (c).

Almost continuity of 2^f. Throughout this paper X denotes a continuum with metric d. The symbol $\mathbb{N}$ denotes the set of positive integers. Given $\varepsilon > 0$, $p \in X$ and $A \subset X$, let $B(\varepsilon, p) = \{q \in X : d(p,q) < \varepsilon\}$ and $N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) \subset X : a \in A\}$. An *order arc* in 2^X is a continuous function $\alpha : [0,1] \to 2^X$ such that $\alpha(s) \subset \alpha(t)$ if $0 \le s \le t \le 1$. Conditions for the existence of order arcs are given in Theorem 15.3 of [2]. A *Whitney map* is a continuous function $\mu : 2^X \to [0,1]$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$ for each $p \in X$ and, if $A, B \in 2^X$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$. It is known that every continuum X admits Whitney maps (see [2, Thm. 13.4]).

The following lemma is well known (see [3, Corollary to Proposition 3]). We include it here for completeness.

LEMMA 1. *If $f : X \to Y$ is almost continuous, then f is weakly Darboux.*

Proof. Let $A \in C(X)$. We need to show that $f(A)$ is connected. Suppose to the contrary that $f(A)$ is not connected. Let K, L be nonempty separated subsets of $f(A)$ such that $f(A) = K \cup L$. Since Y is metric, there exist disjoint open subsets V and W such that $K \subset V$ and $L \subset W$. Fix points $a, b \in A$ such that $f(a) \in K$ and $f(b) \in L$. Consider the set

$$\begin{aligned}\mathcal{U} = [(X - A) \times Y] \cup [(X - \{a\}) \times W] \cup [(X - \{b\}) \times V] \\ \cup [(X - \{a, b\}) \times (W \cup V)].\end{aligned}$$

Clearly, $\mathcal{U}$ is an open subset of $X \times Y$ which contains $\Lambda(f)$. Since f is almost continuous, there exists a continuous function $g : X \to Y$ such that $\Lambda(g) \subset \mathcal{U}$.

Given a point $p \in A$, by the definition of $\mathcal{U}$, $g(p) \in W \cup V$. Moreover, $(a, g(a)) \in \mathcal{U}$ implies that $g(a) \in V$. Similarly, $g(b) \in W$. Therefore, $g(A)$ is a connected subset of $W \cup V$ and $g(A) \cap W \neq \emptyset \neq g(A) \cap V$. This is a contradiction. Hence $f(A)$ is connected. ∎

LEMMA 2. *Let $\alpha : [0,1] \to 2^X$ be an order arc. Suppose that $F : 2^X \to 2^Y$ is weakly Darboux and such that $A \subset B$ implies $F(A) \subset F(B)$. Then the function $F \circ \alpha : [0,1] \to 2^Y$ is continuous.*

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ converging $t \in [0,1]$. We need to check that $\lim F(\alpha(t_n)) = F(\alpha(t))$. It is enough to consider the case in which the sequence $\{t_n\}_{n=1}^{\infty}$ is strictly monotone.

CASE 1: $0 < t_1 < t_2 < \cdots$. In this case, $F(\alpha(t_1)) \subset F(\alpha(t_2)) \subset \cdots$. This implies that $\lim F(\alpha(t_n)) = \mathrm{cl}_Y(\bigcup\{F(\alpha(t_n)) : n \in \mathbb{N}\})$. Hence $\lim F(\alpha(t_n)) \subset F(\alpha(t))$ because each $F(\alpha(t_n))$ is contained in $F(\alpha(t))$. If $\lim F(\alpha(t_n)) \neq F(\alpha(t))$, fix a point $p \in F(\alpha(t)) - \lim F(\alpha(t_n))$. Consider the following sets in 2^Y: $\mathcal{K} = \{A \in 2^Y : A \subset \lim F(\alpha(t_n))\}$ and $\mathcal{L} = \{A \in 2^Y : p \in A\}$. It is easy to see that $\mathcal{K}$ and $\mathcal{L}$ are disjoint closed subsets of 2^Y such that $F(\alpha([0,t))) \subset \mathcal{K}$ and $F(\alpha(t)) \in \mathcal{L}$. This contradicts the connectedness of $F(\alpha([0,t]))$ and proves that $\lim F(\alpha(t_n)) = F(\alpha(t))$.

CASE 2: $1 > t_1 > t_2 > \cdots$. In this case, $\lim F(\alpha(t_n)) = \bigcap\{F(\alpha(t_n)) : n \in \mathbb{N}\}$ and $\lim F(\alpha(t_n)) \supset F(\alpha(t))$. If $\lim F(\alpha(t_n)) \neq F(\alpha(t))$, a contradiction can be obtained by considering the sets $\mathcal{K} = \{A \in 2^Y : \lim F(\alpha(t_n)) \subset A\}$ and $\mathcal{L} = \{F(\alpha(t))\}$. Thus $\lim F(\alpha(t_n)) = F(\alpha(t))$.

This completes the proof of the lemma. ∎

Proceeding as in Lemma 2, one can prove the following lemma.

LEMMA 3. *Let $\alpha : [0,1] \to C(X)$ be an order arc. Suppose that $F : C(X) \to C(Y)$ is weakly Darboux and such that $A \subset B$ implies $F(A) \subset F(B)$. Then the function $F \circ \alpha : [0,1] \to C(Y)$ is continuous.*

THEOREM 1. *Suppose that $f : X \to Y$ and $2^f : 2^X \to 2^Y$ are weakly Darboux. Then f is continuous.*

Proof. Let $F = 2^f$. Suppose that f is not continuous. Then there exist points $p \in X$, $q \in Y$ and a sequence $\{p_n\}_{n=1}^{\infty}$ in X such that $\lim p_n = p$, $\lim f(p_n) = q$ and $q \neq f(p)$. Let $\varepsilon = d_Y(f(p), q) > 0$. Notice that $f(X)$ is nondegenerate.

Fix a Whitney map $\mu : 2^Y \to [0,1]$. Since the set $\{B \in 2^Y : \operatorname{diam}(B) \geq \varepsilon/3\}$ is closed in 2^Y and it does not intersect $\mu^{-1}(0)$, there exists $r > 0$ such that $r < \mu(F(X))$ and, if $B \in \mu^{-1}(r)$, then $\operatorname{diam}(B) < \varepsilon/3$. Fix $\eta > 0$ such that $\eta < \varepsilon$ and $\operatorname{diam}(B) > \eta$ for each $B \in \mu^{-1}(r)$. Since $\lim f(p_n) = q$, we may assume that $f(p_n) \in B(\eta/3, q)$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, fix an order arc $\alpha_n : [0,1] \to C(X)$ with $\alpha_n(0) = \{p_n\}$ and $\alpha_n(1) = X$. By Lemma 2, the function $\mu \circ F \circ \alpha_n : [0,1] \to [0,1]$ is continuous. Since $\mu(F(\alpha_n(0))) = \mu(F(\{p_n\})) = \mu(\{f(p_n)\}) = 0$ and $\mu(F(\alpha_n(1))) = \mu(F(X)) > r$, there exists $t_n \in [0,1]$ such that $\mu(F(\alpha_n(t_n))) = r$. Since $r > 0$, $F(\alpha_n(t_n))$ is a nondegenerate subcontinuum of Y. Notice that, by the choice of r, $\operatorname{diam}(F(\alpha_n(t_n))) < \varepsilon/3$. Since $f(p_n) \in f(\alpha_n(t_n)) \subset F(\alpha_n(t_n))$ and $f(p_n) \in B(\varepsilon/3, q)$, it follows that

$$F(\alpha_n(t_n)) \subset B(2\varepsilon/3, q).$$

Since $C(X)$ is compact, we may assume that the sequence $\{\alpha_n(t_n)\}_{n=1}^{\infty}$ converges to an element $A \in C(X)$. Let

$$P = \{p, p_1, p_2, \ldots\} \quad \text{and} \quad E = A \cup \alpha_1(t_1) \cup \alpha_2(t_2) \cup \cdots .$$

Then $P, E \in 2^X$ and $P \subset E$. Let $\beta : [0,1] \to 2^X$ be given by

$$\beta(t) = \begin{cases} P \cup \Big(\bigcup_{i=1}^{n-1} \alpha_i(t_i)\Big) \cup \alpha_n\Big(2^n\Big(t - \Big(1 - \dfrac{1}{2^{n-1}}\Big)\Big)t_n\Big), & \\ \qquad\qquad \text{if } t \in [1 - 1/2^{n-1}, 1 - 1/2^n],\ n \in \mathbb{N}, & \\ E \qquad\qquad \text{if } t = 1. & \end{cases}$$

It is easy to check that β is well defined, continuous in $[0,1)$ and has the property that if $s \leq t$, then $\beta(s) \subset \beta(t)$.

To see that β is continuous at 1, take a sequence $s_1 < s_2 < \cdots$ in $[0,1]$ such that $\lim s_m = 1$. Since $\beta(s_1) \subset \beta(s_2) \subset \cdots$, we have $\lim \beta(s_m) = \operatorname{cl}(\bigcup\{\beta(s_m) : m \in \mathbb{N}\})$. Hence $\lim \beta(s_m) \subset E$ as each $\beta(s_m)$ is a subset of E. Since $\lim \alpha_n(t_n) = A$, we have $A \subset \operatorname{cl}(\bigcup\{\alpha_n(t_n) : n \in \mathbb{N}\})$. Given $n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that $1 - 1/2^n < s_{m_0}$. Thus $\alpha_n(t_n) \subset \beta(1 - 1/2^n) \subset \beta(s_{m_0}) \subset \lim \beta(s_m)$. Therefore, $E \subset \lim \beta(s_m)$. We have shown that $\beta(1) = E = \lim \beta(s_m)$. This completes the proof that β is continuous. Hence β is an order arc in 2^X such that $\beta(0) = P$ and $\beta(1) = E$.

By Lemma 2, the function $F\circ\beta:[0,1]\to 2^Y$ is continuous. Hence $F(\beta(1))=\lim F(\beta(1-1/2^n))$. Given $n\in\mathbb{N}$,

$$\begin{aligned}F(\beta(1-1/2^n)) &= F(P\cup\alpha_1(t_1)\cup\dots\cup\alpha_n(t_n))\\ &\subset \mathrm{cl}(f(\{p,p_1,p_2,\dots\}))\cup F(\alpha_1(t_1))\cup\dots\cup F(\alpha_n(t_n))\\ &\subset f(\{p\})\cup\mathrm{cl}(f(\{p_1,p_2,\dots\}))\cup F(\alpha_1(t_1))\cup\dots\cup F(\alpha_n(t_n))\\ &\subset \{f(p)\}\cup\mathrm{cl}(B(2\varepsilon/3,q)).\end{aligned}$$

Hence, $F(\beta(1))\subset\{f(p)\}\cup\mathrm{cl}(B(2\varepsilon/3,q))$. That is,

$$F(E)\subset\{f(p)\}\cup\mathrm{cl}(B(2\varepsilon/3,q)).$$

Since $A\subset E$, we have $F(A)\subset F(E)\subset\{f(p)\}\cup\mathrm{cl}(B(2\varepsilon/3,q))$. By the choice of ε, $\{f(p)\}$ and $\mathrm{cl}(B(2\varepsilon/3,q))$ are closed in Y and disjoint. By hypothesis, $F(A)$ is connected. Since $p\in A$, $f(p)\in F(A)$. Thus, $F(A)=\{f(p)\}$.

Define $\gamma:[0,1]\to 2^X$ by

$$\gamma(t)=\begin{cases}A\cup P & \text{if } t=0,\\ A\cup P\cup\alpha_n\left(2^n\left(t-\dfrac{1}{2^n}\right)t_n\right)\cup\displaystyle\bigcup_{i=n+1}^{\infty}\alpha_i(t_i) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } t\in[1/2^n,1/2^{n-1}],\ n\in\mathbb{N}.\end{cases}$$

Since $\lim\alpha_n(t_n)=A$, we have $\gamma(t)\in 2^X$ for each $t\in[0,1]$. It is easy to check that γ is well defined, continuous in $(0,1]$ and has the property that if $s\le t$, then $\gamma(s)\subset\gamma(t)$.

To prove that γ is continuous at 0, take a sequence $s_1>s_2>\cdots$ in $[0,1]$ such that $\lim s_m=0$. Given $\delta>0$, let $N\in\mathbb{N}$ be such that $\alpha_n(t_n)\subset N(\delta,A)$ for each $n\ge N$. Fix $M\in\mathbb{N}$ such that $s_M<1/2^N$. Given $m\ge M$, we have $s_m<1/2^N$, thus $\gamma(s_m)\subset\gamma(1/2^N)\subset N(\delta,A\cup P)$. Since $A\cup P\subset\gamma(s_m)\subset N(\delta,\gamma(s_m))$, we conclude that $D(\gamma(s_m),A\cup P)<\delta$. We have shown that $\lim\gamma(s_m)=A\cup P=\gamma(0)$. Hence, γ is continuous at 0.

We have proved that γ is an order arc from $\gamma(0)=A\cup P$ to $\gamma(1)=E$. By Lemma 2, $F\circ\gamma:[0,1]\to 2^Y$ is a continuous function. Thus

$$\begin{aligned}\lim F(\gamma(1/2^n)) = F(\gamma(0)) = F(A\cup P) &= \mathrm{cl}(f(A))\cup\mathrm{cl}(f(P))\\ &\subset\{f(p)\}\cup\mathrm{cl}(B(\eta/3,q)).\end{aligned}$$

Therefore,

$$\lim F(\gamma(1/2^n))\subset\{f(p)\}\cup B(\eta/2,q).$$

On the other hand, $F(\alpha_{n+1}(t_{n+1}))\subset F(\gamma(1/2^n))$, $F(\alpha_{n+1}(t_{n+1}))$ is connected (see [2, Corollary 15.4]) and $\mu(F(\alpha_{n+1}(t_{n+1})))=r$. Notice that, by the choice of η, $\mathrm{diam}(F(\alpha_{n+1}(t_{n+1})))>\eta$. This implies that $F(\alpha_{n+1}(t_{n+1}))\not\subset B(\eta/2,q)$, and moreover, $f(p_{n+1})\in F(\alpha_{n+1}(t_{n+1}))\cap B(\eta/2,q)$. Hence, $\mathrm{bd}(B(\eta/2,q))\cap F(\alpha_{n+1}(t_{n+1}))\neq\emptyset$. This implies that

$$\mathrm{bd}(B(\eta/2,q))\cap\lim F(\alpha_{n+1}(t_{n+1}))\neq\emptyset.$$

Hence, $\mathrm{bd}(B(\eta/2,q)) \cap \lim F(\gamma(1/2^n)) \neq \emptyset$. This contradicts the inclusion $\lim F(\gamma(1/2^n)) \subset \{f(p)\} \cup B(\eta/2,q)$ proved above and completes the proof of the theorem. ∎

Corollary 1. *Suppose that $f : X \to Y$ and $2^f : 2^X \to 2^Y$ are almost continuous functions. Then f is continuous.*

Almost continuity of $C(f)$

Theorem 2. *Suppose that X is locally connected and the functions $f : X \to Y$ and $C(f) : C(X) \to C(Y)$ are weakly Darboux. Then f is continuous.*

Proof. Let $F = C(f)$. In order to prove that f is continuous take a sequence $\{p_n\}_{n=1}^{\infty}$ in X converging to a point $p \in X$.

For each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{A \in C(X) : p, p_n \in A\}$. Since $\mathcal{A}_n$ is a nonempty and compact subset of $C(X)$, there exists $A_n \in \mathcal{A}_n$ such that $D(\{p\}, A_n) = \min\{D(\{p\}, A) : A \in \mathcal{A}_n\}$. Then $p, p_n \in A_n$.

We claim that $\lim A_n = \{p\}$. Let $\varepsilon > 0$. Since X is locally connected, there exists an open and connected subset U of X with $\mathrm{diam}(\mathrm{cl}_X(U)) < \varepsilon$ and $p \in U$. Thus $D(\{p\}, \mathrm{cl}_X(U)) < \varepsilon$. Let $N \in \mathbb{N}$ be such that $p_n \in U$ for each $n \geq N$. Then $D(\{p\}, A_n) \leq D(\{p\}, \mathrm{cl}_X(U)) < \varepsilon$ for each $n \geq N$. Therefore, $\lim A_n = \{p\}$.

For each $n \in \mathbb{N}$, let $B_n = A_n \cup A_{n+1} \cup \cdots$. Clearly, $B_n \in C(X)$, $p, p_n \in B_n$, $\lim B_n = \{p\}$ and $B_1 \supset B_2 \supset \cdots$. Let $\alpha : [0,1] \to C(X)$ be an order arc such that $\alpha(1/n) = B_n$ for each $n \in \mathbb{N}$ and $\alpha(0) = \{p\}$ (such an order arc can be constructed using Theorem 15.3 of [2]).

By Lemma 3, the map $C(f) \circ \alpha : [0,1] \to C(Y)$ is continuous. Thus $\lim C(f)(\alpha(1/n)) = \{f(p)\}$. Since $f(p_n) \in C(f)(\alpha(1/n))$ for each $n \in \mathbb{N}$, $\lim f(p_n) = f(p)$. Therefore, f is continuous. ∎

Corollary 2. *Suppose that X is locally connected and the functions $f : X \to Y$ and $C(f) : C(X) \to C(Y)$ are almost continuous. Then f is continuous.*

Theorem 3. *Let X be a non-locally connected continuum. Then there exists a proper almost continuous function $f : X \to [0,1]$ such that $C(f)$ is almost continuous.*

Proof. Let H be the harmonic fan defined as a subset of the complex plane by

$$H = \{z \in \mathbb{C} : |z| \leq 2 \text{ and } \mathrm{Arg}(z) \in \{0, 1/1, 1/2, 1/3, \ldots\}\}$$

and set

$$H_n = \{z \in H : \mathrm{Arg}(z) \in \{0, 1/1, 1/2, \ldots, 1/n\}\}.$$

By [1], there exists a continuous surjection $g : X \to H$. For each $n \in \mathbb{N}$, define $r_n : H \to H_n$ by

$$r_n(z) = \begin{cases} z & \text{if } z \in H_n, \\ |z| & \text{if } z \in H - H_n. \end{cases}$$

Let $h : H \to [0,1]$ be given by

$$h(z) = \begin{cases} 0 & \text{if } |z| \geq 1 \text{ and } \operatorname{Arg}(z) = 0, \\ \big| |z| - 1 \big| & \text{otherwise.} \end{cases}$$

Finally, put $f = h \circ g$.

The following observations are easy to prove:

(a) r_n is a continuous retraction.
(b) h is not continuous.
(c) $h \circ r_n$ is continuous.
(d) If $M \subset H$ is connected, then $h(M)$ is connected. Hence, f is weakly Darboux.
(e) The function f is not continuous. Indeed, for each $k \in \mathbb{N}$, fix a point $a_k \in g^{-1}(2(\cos(1/k) + i \sin(1/k)))$. By compactness of X there exists a subsequence $\{a_{k_n}\}_{n=1}^{\infty}$ tending to a point $a \in X$. Then, for each $n \in \mathbb{N}$, $f(a_{k_n}) = 1$, and

$$f(a) = h(\lim_{n\to\infty}(2(\cos(1/k_n) + i \sin(1/k_n)))) = h(2) = 0.$$

The following observations are also easy to check.

For a connected subset A of X we have:

(i) If $A \cap g^{-1}(0) \neq \emptyset \neq A \cap g^{-1}(\{z : |z| \geq 1\})$, then $f(A) = [0,1]$ and $(h \circ r_n \circ g)(A) = [0,1]$ for all $n \in \mathbb{N}$.
(ii) If $A \subset g^{-1}(\{z : |z| \leq 1\})$, then $(h \circ r_n \circ g)(A) = f(A)$ for all $n \in \mathbb{N}$.
(iii) If $A \cap g^{-1}(0) = \emptyset$, then $g(A)$ is contained in some convex segment contained in H. Hence, there exists $k \in \mathbb{N}$ such that $(h \circ r_n \circ g)(A) = f(A)$ for all $n \geq k$.

We are ready to show that $C(f)$ is almost continuous. Let U be an open subset of $C(X) \times C([0,1])$ containing the graph of $C(f)$. Note that each function $h \circ r_n \circ g$ is continuous. We claim that the graph of some map $C(h \circ r_n \circ g)$ is contained in U. Suppose to the contrary that, for each $n \in \mathbb{N}$, there exists $A_n \in C(X)$ such that $(A_n, (h \circ r_n \circ g)(A_n)) \notin U$. In particular, $(h \circ r_n \circ g)(A_n) \neq f(A_n)$. By (i)–(iii) we know that $A_n \cap g^{-1}(0) = \emptyset$ and $g(A_n) \subset H - H_n$ since otherwise $f(A_n) = (h \circ r_n \circ g)(A_n)$. By compactness of $C(X)$ we may assume that the sequence $\{A_n\}_{n=1}^{\infty}$ converges to some $A \in C(X)$. By continuity of g, we see that $g(A)$ is contained in the limit convex segment of H. Therefore,

$$(A, (h \circ r_1 \circ g)(A)) = (A, (h \circ g)(A)) = (A, f(A)) \in U.$$

By continuity of $C(h \circ r_1 \circ g)$, it follows that there exists $k \in \mathbb{N}$ such that $(A_n, (h \circ r_1 \circ g)(A_n)) \in U$ for all $n \geq k$. Since $g(A_n) \subset H - H_n$, we have $r_1(g(A_n)) = r_n(g(A_n))$. Hence $(A_n, (h \circ r_n \circ g)(A_n)) \in U$ for all $n \geq k$. This contradicts the choice of the sets A_n and completes the proof that $C(f)$ is almost continuous.

Finally, since $\{\{p\} \in C(X) : p \in X\}$ is closed in $C(X)$, by [3, Proposition 2], it follows that f is almost continuous. ∎

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