COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 1

DIFFERENTIABLE L^p-FUNCTIONAL CALCULUS FOR CERTAIN SUMS OF NON-COMMUTING OPERATORS

BY

MICHAEL GNEWUCH (Leipzig)

Abstract. We consider a special class of sums of non-commuting positive operators on L^2-spaces and derive a formula for their holomorphic semigroups. The formula enables us to give sufficient conditions for these operators to admit differentiable L^p-functional calculus for $1 \le p \le \infty$. Our results are in particular applicable to certain sub-Laplacians, Schrödinger operators and sums of even powers of vector fields on solvable Lie groups with exponential volume growth.

1. Introduction. Our investigation is stimulated by a multiplier theorem of S. Mustapha stating the following (for a more detailed discussion see Section 3):

Consider a semidirect product G of the real line $\mathbb{R}$ and a stratified nilpotent Lie group N, where $\mathbb{R}$ is acting on N via natural dilations. Then G is a solvable Lie group with exponential volume growth. Let Δ be a (distinguished type of) left invariant sub-Laplacian on G, and let $L^p(G)$ be the L^p-space on G with respect to the right invariant Haar measure. Let $\kappa > 2$, and let $H^\kappa(\mathbb{R})$ be the L^2-Sobolev space of order κ on $\mathbb{R}$. Mustapha showed in [27] that for continuous, compactly supported $f \in H^\kappa(\mathbb{R})$ the restriction of $f(\Delta)$ to $L^2 \cap L^p(G)$ extends to a bounded operator on $L^p(G)$ for all $1 \le p \le \infty$. Roughly speaking: it suffices to control a finite number of derivatives of f to get L^p-boundedness for $f(\Delta)$. (In this situation we say that Δ admits *differentiable L^p-functional calculus*, and f is called an *L^p-multiplier* for Δ.)

This multiplier theorem has many predecessors, starting with the classical results of Mikhlin and Hörmander in the Euclidean setting (cf. [18]). So it is well known that sub-Laplacians on connected Lie groups with polyno-

---

2000 *Mathematics Subject Classification*: Primary 47A60, 47B25; Secondary 22E25, 22E30.

Key words and phrases: selfadjointness, differentiable functional calculus, holomorphic semigroup, spectral multiplier, solvable Lie group, exponential volume growth, sub-Laplacian, Schrödinger operator, sum of even powers of vector fields.

Supported by the European Commission TMR network "Harmonic Analysis" and the Erwin Schrödinger International Institute for Mathematical Physics (ESI).

mial volume growth (including all connected nilpotent Lie groups) always have differentiable L^p-functional calculus [1]. But for a sub-Laplacian on a solvable Lie group with exponential volume growth the validity of such a theorem is *a priori* not clear. In fact there are sub-Laplacians on exponential groups known that do not admit differentiable L^p-functional calculus as defined above—they are of the so-called *holomorphic L^p-type* [5, 17, 23].

A particularly interesting facet of the result of S. Mustapha is the order κ of the Sobolev space that is independent of the "size" of the group, which, e.g., can be expressed by the Euclidean dimension of G or the homogeneous dimension Q of N. Actually, W. Hebisch proved in [16] a multiplier theorem that is identical with the one of Mustapha, except for the order of the Sobolev space, which is $Q/2 + 5/2 + \varepsilon$ instead of $2 + \varepsilon$. And indeed, in the conditions of the multiplier theorems for solvable groups that have been proven so far there regularly appears some quantity describing the size of the underlying group (see, e.g., [1, 2, 4, 6, 7, 11, 14, 18, 24, 25, 26]). In this respect, the result of Mustapha seems remarkable.

One aim of this paper is to present a different proof strategy for Mustapha's multiplier theorem than the one used in [27]: Our approach is purely analytic and relies on bounded functional calculus and Bessel functions instead of on stochastic methods like Brownian motion.

Another aim is to extend the multiplier theorem to a larger class of operators. We define this class in a more abstract way, but it includes several interesting differential operators on the Lie groups mentioned above. These particular operators are not necessarily hypoelliptic, left invariant and of second order like the sub-Laplacians considered by Mustapha.

The article is organized as follows: In Section 2 we consider operators

$$T = -\partial_r^2 \otimes I + e^{2\nu r} \otimes L, \qquad \nu \in \mathbb{R} \setminus \{0\},$$

on $L^2(\mathbb{R} \times X)$, where X is an arbitrary σ-finite measure space, ∂_r the derivative with respect to the parameter $r \in \mathbb{R}$, I the identity operator on $L^2(X)$, $e^{2\nu r}$ the multiplication operator $\varphi(r) \mapsto e^{2\nu r}\varphi(r)$ on $L^2(\mathbb{R})$ and L a positive selfadjoint operator on $L^2(X)$. The operator T can be realized as a selfadjoint operator, and we deduce a representation of the holomorphic semigroup $(e^{-zT})_{\Re(z)>0}$ of T in terms of $(e^{-tL})_{t\geq 0}$. From this it is almost straightforward to derive our central multiplier result, Theorem 2.2.

In Section 3 we present differential operators on Lie groups which satisfy the conditions of our multiplier theorem. There we also discuss the result of S. Mustapha in detail.

The appendix (Section 4) provides a selfadjointness theorem for generalized sub-Laplacians and Schrödinger operators on Lie groups. "Generalized" means here that the operators are not required to satisfy the so-called Hörmander condition, i.e., they are not necessarily hypoelliptic. The result is probably known, but we are not aware of any explicit reference. Since we use the theorem a few times in the main text, we decided to give a proof (whose idea is actually simple).

2. The general multiplier result. Let (X, dx) be a σ-finite measure space and L a positive selfadjoint operator on $L^2(X)$ with domain $\mathrm{D}(L)$. Let dr be the Lebesgue measure on $\mathbb{R}$ and $dr \otimes dx$ the product measure on $\mathbb{R} \times X$. Let $\nu \in \mathbb{R} \setminus \{0\}$ be fixed. We denote the function $r \mapsto e^{2\nu r}$ on $\mathbb{R}$ briefly by $e^{2\nu r}$. For $\kappa \geq 0$ let

$$H^\kappa(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : (1 + |\cdot|)^\kappa \widehat{f} \in L^2(\mathbb{R})\}$$

be the *Sobolev space* of order κ on $\mathbb{R}$, and put

$$\mathscr{A} := H^2(\mathbb{R}) \cap \{f \in L^2(\mathbb{R}) : e^{2\nu r} f \in L^2(\mathbb{R})\}.$$

The positive operator $-\partial_r^2 + e^{2\nu r} L$ is defined on $\mathscr{A} \otimes \mathrm{D}(L) \subseteq L^2(\mathbb{R} \times X)$ and its closure is denoted by T. It is clear that T has a selfadjoint extension, namely its Friedrichs' extension. In fact T itself is already a selfadjoint operator, as we shall prove in Proposition 2.10.

The precise definition of an L^p-multiplier that we use throughout this article is the following: Let $p \in [1, \infty]$. We call a bounded function $f : \mathbb{R} \to \mathbb{C}$ an *L^p-multiplier for T* if there exists a positive constant C such that

$$\|f(T)\varphi\|_{L^p(\mathbb{R}\times X)} \leq C\|\varphi\|_{L^p(\mathbb{R}\times X)} \quad \text{for all } \varphi \in L^2 \cap L^p(\mathbb{R} \times X).$$

Notice that in the case $p = \infty$ this definition does not necessarily imply that $f(T)$ can be extended to a bounded operator on the whole space $L^\infty(\mathbb{R} \times X)$, since, for a non-trivial measure dx, $L^2 \cap L^\infty(\mathbb{R} \times X)$ is not dense in $L^\infty(\mathbb{R} \times X)$. On the other hand, if $f(T)$ is given explicitly, e.g., by integration against a suitable kernel function, then it is often easy to extend $f(T)$ to a bounded operator on $L^\infty(\mathbb{R} \times X)$ in a canonical way.

Before we state the main results of this section, some remarks concerning notation: We use the abbreviations sh, ch and th for the hyperbolic functions sinh, cosh and tanh respectively. $\Re(z)$ stands for the real part, $\Im(z)$ for the imaginary part and z^* for the complex conjugate of the complex number z. For $f \in L^2(\mathbb{R} \times X)$ and $s \in \mathbb{R}$ let f_s denote the function $x \mapsto f(s, x)$ on X.

2.1. *Results of Section 2.* First we give a representation of the holomorphic semigroup of T in terms of the semigroup of L with positive time parameter.

THEOREM 2.1. *Let $z \in \mathbb{C}$ with $\Re(z) > 0$ and $f \in L^2(\mathbb{R} \times X)$. Then*

$$\begin{aligned} (1)\qquad & e^{-zT} f(r, x) \\ &= \bigg(\int_{\mathbb{R}} \int_0^\infty |\nu| \Psi_{\nu^2 z}(\xi) \exp\bigg(-\frac{\mathrm{ch}(\nu(r - s))}{\xi}\bigg) \exp\bigg(-\frac{\xi e^{\nu(r+s)}}{2\nu^2} L\bigg) f_s \, d\xi \, ds\bigg)(x) \end{aligned}$$

for almost all $(r,x)\in\mathbb{R}\times X$, *where the function* $\Psi_z :]0,\infty[\,\to\mathbb{C}$ *is given by*

$$(2)\quad \Psi_z(\xi)=\frac{\xi^{-2}}{\sqrt{4\pi^3 z}}\exp\left(\frac{\pi^2}{4z}\right)\int_0^\infty \mathrm{sh}(\vartheta)\sin\left(\frac{\pi\vartheta}{2z}\right)\exp\left(-\frac{\vartheta^2}{4z}-\frac{\mathrm{ch}(\vartheta)}{\xi}\right)d\vartheta.$$

There exists a $C_z>0$, *only depending on* z, *with* $|\Psi_z(\xi)|\le C_z\xi^{-2}$ *for all* $\xi>0$.

With the help of Theorem 2.1 it is easy to prove estimate (3), which is the key inequality for our main multiplier result:

THEOREM 2.2. *Let* $p\in[1,\infty]$. *If there exists a constant* $C>0$ *such that* $\|e^{-tL}\psi\|_{L^p(X)}\le C\|\psi\|_{L^p(X)}$ *for each* $t>0$ *and every* $\psi\in L^2\cap L^p(X)$, *then*

$$(3)\quad \|e^{-(\varrho+i\sigma)T}\varphi\|_{L^p(\mathbb{R}\times X)} \le C(2+|\nu|\sqrt{\varrho})\exp\left(\frac{\pi^2}{4\nu^2\varrho}\right)\left(1+\frac{|\sigma|}{\varrho}\right)^{3/2}\|\varphi\|_{L^p(\mathbb{R}\times X)}$$

for all $\varrho>0$, $\sigma\in\mathbb{R}$ *and* $\varphi\in L^2\cap L^p(\mathbb{R}\times X)$. *Moreover, for every* $\kappa>2$ *each continuous* $f\in H^\kappa(\mathbb{R})$ *with compact support is an* L^q*-multiplier for* T *if* q *satisfies* $1/q=s+(t-s)/p$ *for some* $s,t\in[0,1]$ *with* $s+t=1$.

The following theorem states our multiplier result in terms of heat or evolution kernels rather than in terms of semigroups:

THEOREM 2.3. *Assume that* L *has a measurable evolution kernel* $(p_t)_{t>0}$, *i.e.,* $e^{-tL}\psi(x)=\int_X p_t(x,y)\psi(y)\,dy$ *for almost all* $x\in X$. *If the operators* $(\Lambda_t)_{t>0}$ *defined by* $\psi\mapsto\int_X|p_t(\cdot,y)|\psi(y)\,dy$ *are bounded on* $L^2(X)$ *with* $\|\Lambda_t\|_{L^2(X)\to L^2(X)}\le\kappa$, κ *independent of* $t>0$, *then* T *has an evolution kernel* $(P_z)_{\Re(z)>0}$, *given by*

$$(4)\quad P_z((r,x),(s,y)) = \int_0^\infty |\nu|\Psi_{\nu^2 z}(\xi)\exp\left(-\frac{\mathrm{ch}(\nu(r-s))}{\xi}\right)p_{\xi e^{\nu(r+s)}/2\nu^2}(x,y)\,d\xi$$

for almost all $((r,x),(s,y))\in(\mathbb{R}\times X)^2$.

If there is in addition a constant $C>0$ *with* $\|p_t(\cdot,x)\|_{L^1(X)}\le C$ *for all* $t>0$ *and* $x\in X$, *then for each* $z\in\mathbb{C}$ *with* $\Re(z)>0$ *and every* $g\in\mathbb{R}\times X$,

$$(5)\quad \|P_z(\cdot,g)\|_{L^1(\mathbb{R}\times X)}\le C(2+|\nu|\sqrt{\Re(z)})\exp\left(\frac{\pi^2}{4\nu^2\Re(z)}\right)\left(1+\frac{|\Im(z)|}{\Re(z)}\right)^{3/2}.$$

In particular, for every $\varepsilon>0$ *and every* $p\in[1,\infty]$, *each compactly supported, continuous* $f\in H^{2+\varepsilon}(\mathbb{R})$ *is an* L^p*-multiplier for* T.

REMARK 2.4. (i) The exponent 3/2 of $1+|\Im(z)|/\Re(z)$ in (3) and (5) is optimal in the sense that there exists an operator $T=-\partial_r^2+e^{2r}L$ with

$$\|e^{-(1+i\sigma)T}\|_{L^1(\mathbb{R}\times X)\to L^1(\mathbb{R}\times X)}\sim(1+|\sigma|)^{3/2}.$$

This was shown in [27] for a sub-Laplacian T on some solvable Lie group G, given by $\mathbb{R}$ acting on $\mathbb{R}^2$ via natural dilations (cf. Section 3). S. Mustapha used a concrete formula from [3] for the heat kernel Φ_z of T to demonstrate that $\|\Phi_{1+i\sigma}\|_{L^1(G)} \sim (1+|\sigma|)^{3/2}$.

(ii) One observes that the term $C\exp(\pi^2/4\nu^2\Re(z))(2+|\nu|\sqrt{\Re(z)})$ in (3) and (5) tends to infinity as $|\nu|$ tends to zero. That reflects the fact that for operators of the form $-\partial_r^2 + L$ in general one cannot achieve estimates like (3) and (5). If, e.g., L is the Laplacian on $\mathbb{R}^n$, then $-\partial_r^2+L$ is the Laplacian on $\mathbb{R}^{n+1}$ and its heat kernel Φ_z satisfies

$$\|\Phi_{1+i\sigma}\|_{L^1(\mathbb{R}^{n+1})} \sim (1+|\sigma|)^{(n+1)/2}.$$

Thus for $n>2$ an estimate like (5) does not hold for $P_z(\cdot,0)=\Phi_z$.

2.2. *Proof of the results.* For $a \geq 0$ we consider the operator $A = A(a) := -\partial_r^2 + ae^{2\nu r}$ defined on the space of test functions $C_c^\infty(\mathbb{R})$. We denote its closure again by $A(a)$. Then $A(a)$ is selfadjoint with domain

$$\mathrm{D}(A(a)) = \begin{cases} H^2(\mathbb{R}) & \text{if } a=0,\\ \mathscr{A} & \text{if } a>0\end{cases}$$

(see, e.g., Theorem 4.1).

Guideline for the proof of Theorems 2.1 and 2.2. The multiplier statement in Theorem 2.2 follows from estimate (3) by utilizing the spectral theorem and the Fourier inversion formula: We have

$$f(T) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{F}(\xi)e^{-(1-i\xi)T}\,d\xi,$$

where $F := f\cdot\exp$. Hence we get for $\varphi \in L^2\cap L^p(\mathbb{R}\times X)$, with $v(\xi) := 1+|\xi|$ and $K := C(2+|\nu|)\exp(\pi^2/4\nu^2)$,

$$\begin{aligned}\|f(T)\varphi\|_{L^p(\mathbb{R}\times X)} &\leq \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{F}(\xi)|\,\|e^{-(1-i\xi)T}\varphi\|_{L^p(\mathbb{R}\times X)}\,d\xi\\ &\leq \frac{K}{2\pi}\int_{\mathbb{R}} |\widehat{F}(\xi)|v(\xi)^{3/2}\,d\xi\,\|\varphi\|_{L^p(\mathbb{R}\times X)}\\ &\leq \frac{K}{2\pi}\,\|v^{-1/2-\varepsilon}\|_{L^2(\mathbb{R})}\|v^{2+\varepsilon}\widehat{F}\|_{L^2(\mathbb{R})}\|\varphi\|_{L^p(\mathbb{R}\times X)}.\end{aligned}$$

To apply a duality argument, we use the following statement: For all $q,q' \in [1,\infty]$ with $1/q+1/q'=1$, and for all $\xi \in L^q(\mathbb{R}\times X)$, we have

$$\|\xi\|_{L^q(\mathbb{R}\times X)} = \sup\{|\langle\xi,\tau\rangle| : \tau \in L^2\cap L^{q'}(\mathbb{R}\times X),\ \|\tau\|_{L^{q'}(\mathbb{R}\times X)}=1\}. \tag{6}$$

Hölder's inequality shows that the right hand side of (6) is bounded from above by the left hand side. For $q \in\,]1,\infty[$ equality follows from the theorem of Hahn–Banach, the fact that $L^{q'}$ is isometrically isomorphic to the dual space of L^q and the fact that $L^2\cap L^{q'}$ is dense in $L^{q'}$. In the remaining cases $q = 1, \infty$ the σ-finiteness of the measure space $\mathbb{R} \times X$ allows us to construct sequences (η_n) in $L^2 \cup L^{q'}(\mathbb{R} \times X)$ satisfying $|\langle \xi, \eta_n \rangle| \to \|\xi\|_{L^q(\mathbb{R}\times X)}$ as $n \to \infty$. (These constructions are straightforward. Nevertheless, the details can be found in [10, §1.1].) From (6) we get, with a simplified notation,

$$\begin{aligned}
&\sup\{\|f(T)\psi\|_{L^{p'}} : \psi \in L^2 \cap L^{p'}, \|\psi\|_{L^{p'}} = 1\} \\
&\quad = \sup\{|\langle f(T)\psi, \tau\rangle| : \psi \in L^2 \cap L^{p'}, \tau \in L^2 \cap L^p, \|\psi\|_{L^{p'}} = 1 = \|\tau\|_{L^p}\} \\
&\quad = \sup\{|\langle \psi, f^*(T)\tau\rangle| : \psi \in L^2 \cap L^{p'}, \tau \in L^2 \cap L^p, \|\psi\|_{L^{p'}} = 1 = \|\tau\|_{L^p}\} \\
&\quad = \sup\{\|f^*(T)\tau\|_{L^p} : \tau \in L^2 \cap L^p, \|\tau\|_{L^p} = 1\} \le \frac{K}{2\pi} \int_{\mathbb{R}} |\widehat{F}(\xi)| v(\xi)^{3/2}\, d\xi,
\end{aligned}$$

since $(f^* \cdot \exp)^\wedge(\xi) = \widehat{F}(-\xi)^*$.

The full multiplier statement follows now from the interpolation theorem of Riesz–Thorin.

Inequality (3) is more or less a direct consequence of identity (1), as we shall show later. Therefore the crucial part of the proof is to verify (1). Actually it is sufficient to establish (1) for $\nu = 1$, since the general case can then be derived using the change of variables $s := \nu r$. We therefore only consider the case $\nu = 1$ to make formulas a little shorter.

The formal idea to establish (1) is the following: Instead of looking directly at $e^{-zT} = e^{-z(-\partial_r^2 + e^{2r}L)}$, we first consider the family of operators $e^{-z(-\partial_r^2 + ae^{2r})} = e^{-zA(a)}$, $a \in [0, \infty[$. These operators have an integration kernel $q_z(a, \cdot, \cdot)$, i.e.,

$$e^{-zA(a)}\varphi = \int_{\mathbb{R}} q_z(a, \cdot, s)\varphi(s)\, ds \quad \text{ for any } \varphi \in L^2(\mathbb{R}),$$

which can be calculated easily. After deriving a suitable representation of q_z, we are able to replace the variable a again by the operator L in the sense of bounded functional calculus, and we get

$$e^{-zT}\varphi \otimes \psi = \int_{\mathbb{R}} (q_z(L, \cdot, s)\psi)\varphi(s)\, ds \quad \text{ for any } \varphi \in L^2(\mathbb{R}),\ \psi \in L^2(X).$$

On the next pages we realize this proof idea in a mathematically rigorous manner. The first step is to calculate the kernel $q_z(a, \cdot, \cdot)$ of $e^{-zA(a)}$, $a \ge 0$.

LEMMA 2.5. *Let $a \in [0, \infty[$ and $z \in \mathbb{C}$ with $\Re(z) > 0$. For $n \in \mathbb{N}$ let the curve $\gamma_n : \mathbb{R} \to \mathbb{C}$ be defined by $\gamma_n(\theta) = (\theta + i2^{-n})^2$.*

(i) *We have*

$$e^{-zA(a)} = -\frac{1}{2\pi i} \int_{\gamma_n} e^{-z\lambda}(\lambda - A(a))^{-1}\, d\lambda; \tag{7}$$

here the parameterized integrand $s \mapsto \gamma_n'(s)e^{-z\gamma_n(s)}(\gamma_n(s) - A(a))^{-1}$ *is an* L^1*-mapping from* $\mathbb{R}$ *into the space of bounded linear operators on* $L^2(\mathbb{R})$.

(ii) *If* $\varphi \in C_c^\infty(\mathbb{R})$, *then for almost all* $r \in \mathbb{R}$,

$$e^{-zA(a)}\varphi(r) = -\frac{1}{2\pi i}\int_{\gamma_n} e^{-z\lambda}((\lambda - A(a))^{-1}\varphi)(r)\,d\lambda. \tag{8}$$

Proof. Formulas like (7) are well known in the literature (see, e.g., [28, §1.7]). From (7) it follows that, for almost all $r \in \mathbb{R}$,

$$e^{-zA(a)}\varphi(r) = -\frac{1}{2\pi i}\Big(\int_{\gamma_n} e^{-z\lambda}(\lambda - A(a))^{-1}\varphi\,d\lambda\Big)(r).$$

Since φ is smooth, one can change the order of integration and point evaluation in r. Thus (8) holds. ∎

Equation (8) indicates that we can calculate q_z if we have a concrete formula for the resolvent of $A(a)$. We found such a formula in [20] (it is also stated in [29, Ex. 4.15]): For $a > 0$ and $\theta \in \mathbb{C}$ we define the function $k_\theta^a : \mathbb{R}^2 \to \mathbb{C}$ by

$$k_\theta^a(r,s) = \begin{cases} -I_{-i\theta}(\sqrt{a}\,e^r)K_{i\theta}(\sqrt{a}\,e^s) & \text{for } s \ge r, \\ -I_{-i\theta}(\sqrt{a}\,e^s)K_{i\theta}(\sqrt{a}\,e^r) & \text{for } s < r, \end{cases}$$

where I and K are the so-called *Bessel functions of imaginary argument.* (Our reference for Bessel functions is [22].) Furthermore, we define $\vartheta(\theta)$ to be the complex number with $\vartheta(\theta)^2 = \theta$ and $\arg(\vartheta(\theta)) \in [0,\pi[$. Then for $\lambda \in \mathbb{C} \setminus [0,\infty[$ and $\varphi \in C_c^\infty(\mathbb{R})$ we have

$$(\lambda - A(a))^{-1}\varphi(r) = \int_{\mathbb{R}} k_{\vartheta(\lambda)}^a(r,s)\varphi(s)\,ds \quad \text{for almost all } r \in \mathbb{R}. \tag{9}$$

Before we calculate q_z, we want to prove another auxiliary lemma:

LEMMA 2.6. *There exists a* $C > 0$ *such that for all* $a > 0$, $r, s \in \mathbb{R}$ *and all* $z \in \mathbb{C}$ *with* $\Im(z) \ge 0$,

$$|k_z^a(r,s)| \le Ca^{-1/4}e^{-(r+s)/4}.$$

Proof. Because of symmetry we can confine ourselves to the case $r \ge s$. If J_0 denotes the Bessel function of first type and order 0, we have

$$K_{iz}(\sqrt{a}\,e^r)I_{-iz}(\sqrt{a}\,e^s) = \frac{1}{2}\int_{r-s}^{\infty} J_0(\sqrt{2ae^{r+s}}\sqrt{\mathrm{ch}(\sigma) - \mathrm{ch}(r-s)})e^{iz\sigma}\,d\sigma.$$

J_0 is bounded on $[0,\infty[$ and there exists a $C > 0$ with $|J_0(x)| \le C/\sqrt{x}$ for all $x \in [0,\infty[$. Therefore

$$|K_{iz}(\sqrt{a}\,e^r)I_{-iz}(\sqrt{a}\,e^s)| \le \frac{C}{2}\,(2a)^{-1/4}e^{-(r+s)/4}\int_{r-s}^{\infty}(\mathrm{ch}(\sigma) - \mathrm{ch}(r-s))^{-1/4}\,d\sigma.$$

With the substitution $\tau := \mathrm{ch}(\sigma)$, $d\sigma = (\tau^2-1)^{-1/2}\,d\tau$ we get

$$\int\limits_{r-s}^{\infty} (\mathrm{ch}(\sigma)-\mathrm{ch}(r-s))^{-1/4}\,d\sigma = \int\limits_{\mathrm{ch}(r-s)}^{\infty} (\tau-\mathrm{ch}(r-s))^{-1/4}\,\frac{d\tau}{\sqrt{\tau^2-1}}$$

$$= \int\limits_0^{\infty} \tau^{-1/4}\,\frac{d\tau}{\sqrt{(\tau+\mathrm{ch}(r-s))^2-1}} \le \int\limits_0^{\infty} \tau^{-3/4}\frac{d\tau}{\sqrt{\tau+2}} < \infty. \ \blacksquare$$

PROPOSITION 2.7. *Let $a>0$, $z\in\mathbb{C}$ with $\Re(z)>0$ and $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$. Then*

$$e^{-zA(a)}\varphi(r) = \int\limits_{\mathbb{R}} q_z(a,r,s)\varphi(s)\,ds \quad \textit{for almost all } r\in\mathbb{R}, \tag{10}$$

where

$$q_z(a,r,s) := -\frac{1}{\pi i}\int\limits_{\mathbb{R}} \theta e^{-z\theta^2} k_\theta^a(r,s)\,d\theta.$$

Proof. Lemma 2.5(ii) and (9) imply, for $\varphi\in C_{\mathrm{c}}^{\infty}(\mathbb{R})$,

$$\begin{aligned} &e^{-zA(a)}\varphi(r)\\ &\quad = -\frac{1}{2\pi i}\int\limits_{\gamma_n} e^{-z\lambda}\int\limits_{\mathbb{R}} k^a_{\vartheta(\lambda)}(r,s)\varphi(s)\,ds\,d\lambda\\ &\quad = -\frac{1}{\pi i}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}(\theta+i2^{-n})\exp(-z(\theta+i2^{-n})^2)k^a_{\theta+i2^{-n}}(r,s)\varphi(s)\,ds\,d\theta \end{aligned}$$

for almost all r and every $n\in\mathbb{N}$. For fixed $r\in\mathbb{R}$ the integrand of the last term converges as $n\to\infty$ pointwise to the function

$$(\theta,s)\mapsto \theta e^{-z\theta^2}k_\theta^a(r,s)\varphi(s).$$

From

$$\begin{aligned} &|(\theta+i2^{-n})\exp(-z(\theta+i2^{-n})^2)k^a_{\theta+i2^{-n}}(r,s)\varphi(s)|\\ &\qquad \le \frac{C}{a^{1/4}}\,(1+|\theta|)\exp(-\Re(z)(\theta^2-1)+2|\Im(z)\theta|)\,e^{-(r+s)/4}|\varphi(s)| \end{aligned}$$

(see Lemma 2.6) and the dominated convergence theorem we get

$$e^{-zA}\varphi(r) = -\frac{1}{\pi i}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\theta e^{-z\theta^2}k_\theta^a(r,s)\varphi(s)\,ds\,d\theta.$$

Hence Fubini's theorem yields (10) for test functions. $\blacksquare$

PROPOSITION 2.8. *Let $z\in\mathbb{C}$ with $\Re(z)>0$ and Ψ_z as in* (2). *For all $a>0$ and all $r,s\in\mathbb{R}$ we have*

$$q_z(a,r,s) = \int\limits_0^{\infty}\Psi_z(\xi)\exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)\exp\Big(-\frac{\xi e^{r+s}}{2}a\Big)\,d\xi. \tag{11}$$

Proof. Let $a > 0$. From

$$K_\nu(\zeta) = \frac{\pi}{2}\,\frac{I_{-\nu}(\zeta) - I_\nu(\zeta)}{\sin(\pi\nu)}, \quad \nu \notin \mathbb{Z}, \quad \text{and} \quad K_{-\nu} = K_\nu$$

we get, for $r \geq s$,

$$q_z(a,r,s) = \frac{2}{\pi^2}\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)e^{-z\theta^2}K_{i\theta}(\sqrt{a}\,e^r)K_{i\theta}(\sqrt{a}\,e^s)\,d\theta.$$

Obviously this result still holds if $r < s$. From the integral representation [22, (5.10.25)] of $K_{i\theta}$ it follows that for $s, r \in \mathbb{R}$,

$$\begin{aligned} q_z(a,r,s) &= \frac{1}{2\pi^2}\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)e^{-z\theta^2+i\theta(r-s)}\left(\int_0^\infty \exp\left(-v-\frac{ae^{2r}}{4v}\right)v^{-i\theta-1}\,dv\right)\\ &\qquad\times\left(\int_0^\infty \exp\left(-w-\frac{ae^{2s}}{4w}\right)w^{i\theta-1}\,dw\right)d\theta\\ &= \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)\exp(-z\theta^2+i\theta(r-s)+i\theta\ln(w/v))\\ &\qquad\times\exp\left(-\frac{a}{4}\left(\frac{e^{2r}}{v}+\frac{e^{2s}}{w}\right)-w-v\right)\frac{dv}{v}\,\frac{dw}{w}\,d\theta. \end{aligned}$$

The identity $q_z(a,r,s) = \frac{1}{2}(q_z(a,r,s)+q_z(a,s,r))$ shows that we can replace the factor $\exp(i\theta(r-s+\ln(w/v)))$ of the integrand by $\cos(\theta(r-s+\ln(w/v)))$. Then, with the substitutions $\nu := 2ve^{-r+s}$, $\mu := 2we^{r-s}$,

$$\begin{aligned} q_z(a,r,s) &= \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)\cos(\theta(s-r+\ln(\mu/\nu)))e^{-z\theta^2}\\ &\qquad\times\exp\left(-\frac{a}{2}e^{r+s}\left(\frac{1}{\nu}+\frac{1}{\mu}\right)-\frac{1}{2}\left(\nu e^{r-s}+\mu e^{s-r}\right)\right)\frac{d\nu}{\nu}\,\frac{d\mu}{\mu}\,d\theta. \end{aligned}$$

Using the change of variables

$$\Phi : \{(\xi,\eta) : \xi > 0, |\eta| < \xi\} \to\,]0,\infty[^2, \qquad (\xi,\eta) \mapsto \left(\frac{2}{\xi+\eta}, \frac{2}{\xi-\eta}\right) =: (\mu,\nu),$$

we get

$$\begin{aligned} q_z(a,r,s) &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_{-\xi}^{\xi} \theta\,\mathrm{sh}(\theta\pi)\cos\left(\theta\left(s-r+\ln\left(\frac{\xi-\eta}{\xi+\eta}\right)\right)\right)e^{-z\theta^2}\\ &\qquad\times\exp\left(-\frac{ae^{r+s}\xi}{2}-\frac{e^{r-s}}{\xi-\eta}-\frac{e^{s-r}}{\xi+\eta}\right)\frac{d\eta\,d\xi}{\xi^2-\eta^2}\,d\theta\\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_{-1}^{1} \theta\,\mathrm{sh}(\theta\pi)\cos\left(\theta\left(s-r+\ln\left(\frac{1-w}{1+w}\right)\right)\right)e^{-z\theta^2}\\ &\qquad\times\exp\left(-\frac{ae^{r+s}\xi}{2}-\frac{1}{\xi}\left(\frac{e^{r-s}}{1-w}+\frac{e^{s-r}}{1+w}\right)\right)\frac{dw\,d\xi}{\xi(1-w^2)}\,d\theta, \end{aligned}$$

where in the last step we employed the change of variables $w = \eta/\xi$. If we define $u := \operatorname{arth}(w)$, then $\ln((1-w)/(1+w)) = -2\operatorname{arth}(w) = -2u$ and $1 - w^2 = 1 - \operatorname{th}(u)^2 = \operatorname{ch}(u)^{-2}$. Furthermore, we get

$$\frac{e^{r-s}}{1-w} + \frac{e^{s-r}}{1+w} = 2\operatorname{ch}(u)\operatorname{ch}(r-s+u).$$

This leads to

$$q_z(a,r,s) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \theta\operatorname{sh}(\theta\pi)\cos(\theta(s-r-2u))e^{-z\theta^2} \times \frac{1}{\xi}\exp\Big(-\frac{ae^{r+s}\xi}{2} - \frac{2}{\xi}\operatorname{ch}(u)\operatorname{ch}(r-s+u)\Big)\,du\,d\xi\,d\theta.$$

With $\vartheta := 2u + r - s$ we have the identity

$$2\operatorname{ch}(u)\operatorname{ch}(r-s+u) = \operatorname{ch}(r-s) + \operatorname{ch}(\vartheta).$$

That implies

$$q_z(a,r,s) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\Big(\int_0^\infty \theta\operatorname{sh}(\theta\pi)\cos(\theta\vartheta)e^{-z\theta^2}\,d\theta\Big) \times \frac{1}{\xi}\exp\Big(-\frac{ae^{r+s}\xi}{2} - \frac{1}{\xi}\left(\operatorname{ch}(r-s) + \operatorname{ch}(\vartheta)\right)\Big)\,d\vartheta\,d\xi.$$

With partial integration and [13, Eq. 4.133] we get for $z \in\,]0,\infty[$,

$$\begin{aligned}\int_0^\infty \theta\operatorname{sh}(\theta\pi)\cos(\theta\vartheta)e^{-z\theta^2}\,d\theta &= \frac{\sqrt{\pi}}{2\sqrt{z}}\exp\left(\frac{\pi^2-\vartheta^2}{4z}\right)\left(\frac{\pi}{2z}\cos\left(\frac{\pi\vartheta}{2z}\right) - \frac{\vartheta}{2z}\sin\left(\frac{\pi\vartheta}{2z}\right)\right)\\ &= \frac{\sqrt{\pi}}{2\sqrt{z}}\exp\left(\frac{\pi^2}{4z}\right)\partial_\vartheta\Big(\exp\left(-\frac{\vartheta^2}{4z}\right)\sin\left(\frac{\pi\vartheta}{2z}\right)\Big).\end{aligned}$$

As all terms in the preceding calculation are holomorphic in z, this identity holds for all $z \in \mathbb{C}$ with $\Re(z) > 0$. After partial integration with respect to the ϑ-variable we finally get formula (11). ∎

Corollary 2.9. *Let $z \in \mathbb{C}$ with $\Re(z) > 0$, and define $q_z(a,r,s)$ for $a = 0$ via* (11). *Then q_z is continuous on $[0,\infty[\, \times \mathbb{R}^2$, and there exists a $C_z > 0$, depending only on z, with*

$$|q_z(a,r,s)| \le C_z\operatorname{ch}(r-s)^{-1} \quad \textit{for all } a \ge 0 \textit{ and } r,s \in \mathbb{R}. \tag{12}$$

Furthermore, identity (10) *holds for all $a \ge 0$ and all $\varphi \in L^2(\mathbb{R})$.*

Proof. It is easy to see that there exists a $C_z > 0$, depending only on z, with $|\Psi_z| \le C_z\xi^{-2}$. This implies

$$|q_z(a,r,s)| \le C_z\int_0^\infty \xi^{-2}\exp\Big(-\frac{\operatorname{ch}(r-s)}{\xi}\Big)\,d\xi = C_z\operatorname{ch}(r-s)^{-1}.$$

The continuity of q_z now follows from the dominated convergence theorem.

Let $a > 0$. Since (12) holds, both sides of (10) define bounded linear operators on $L^2(\mathbb{R})$, which then have to be equal on the whole space $L^2(\mathbb{R})$.

Let now $a = 0$. From (12) and the dominated convergence theorem we have, for $\varphi, \psi \in L^2(\mathbb{R})$ and $a' > 0$,

$$\begin{aligned}\langle e^{-zA(a')}\varphi, \psi\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} q_z(a', r, s)\varphi(s)\psi(r)^*\,ds\,dr\\ &\to \int_{\mathbb{R}}\int_{\mathbb{R}} q_z(0, r, s)\varphi(s)\psi(r)^*\,ds\,dr \qquad \text{for } a' \searrow 0.\end{aligned}$$

As $e^{-zA(a')}\varphi \to e^{-zA(0)}\varphi$ for $a' \searrow 0$ (see, e.g., [31, Thm. 9.17]), (10) holds also for $a = 0$ and arbitrary $\varphi \in L^2(\mathbb{R})$. ∎

Now let $z \in \mathbb{C}$ with $\Re(z) > 0$ and $r, s \in \mathbb{R}$. According to (12) the function $a \mapsto q_z(a, r, s)$ is bounded on $[0, \infty[$, and in the sense of bounded functional calculus we obtain

$$q_z(L, r, s) = \int_0^\infty \Psi_z(\xi)\exp\Big(-\frac{\operatorname{ch}(r-s)}{\xi}\Big)\exp\Big(-\frac{\xi e^{r+s}}{2}L\Big)\,d\xi. \tag{13}$$

The integrand on the right hand side of (13) is an L^1-mapping from $]0, \infty[$ with values in the space of bounded linear operators on $L^2(X)$.

PROPOSITION 2.10. *The positive operator T is selfadjoint and its holomorphic semigroup $(e^{-zT})_{\Re(z)>0}$ is given by*

$$e^{-zT}f(r, x) = \Big(\int_{\mathbb{R}} q_z(L, r, s)f_s\,ds\Big)(x) \tag{14}$$

for all $f \in L^2(\mathbb{R} \times X)$ and almost all $(r, x) \in \mathbb{R} \times X$.

Proof. According to the spectral theorem, L is unitarily equivalent to an operator of multiplication on some $L^2(Y)$, where Y is another σ-finite measure space. It is therefore sufficient to consider the case where $L\psi = m\psi$ for some measurable $m : X \to [0, \infty[$. For $z \in \mathbb{C}$ with $\Re(z) > 0$ let us define the operator $S(z)$ by

$$S(z)f(r, x) = \int_{\mathbb{R}} q_z\big(m(x), r, s\big)f(s, x)\,ds$$

for $f \in L^2(\mathbb{R} \times X)$. It is easy to see from the properties of the holomorphic semigroups $(e^{-zA(a)})_{\Re(z)>0}$, $a \geq 0$, and their kernels $q_z(a, r, s)$ that $(S(z))_{\Re(z)>0}$ is a semigroup of operators on $L^2(\mathbb{R} \times X)$ and that the mapping $z \mapsto \langle S(z)f, g\rangle_{L^2(\mathbb{R}\times X)}$ is holomorphic for all $f, g \in L^2(\mathbb{R} \times X)$. Since the semigroup $(S(t))_{t\geq 0}$ is selfadjoint, the same holds for its infinitesimal generator $-G$. It is straightforward to show that $\mathscr{A} \otimes \mathrm{D}(L) \subseteq \mathrm{D}(G)$ and that $T = G$ on $\mathscr{A} \otimes \mathrm{D}(L)$. Thus $T = G|_{\mathrm{D}(T)}$.

To conclude the proof, we only have to verify that T is a selfadjoint operator, since then $T = G$ and $e^{-zT} = S(z)$. One can surely realize that in several ways; here we want to sketch the following approach: We prove that $\mathrm{D}(T)$ is a core of G (which implies $T = G$). For this it is sufficient to show that $\mathrm{D}(T)$ is invariant under $(S(t))_{t\geq 0}$ [8, Thm. 1.9]. The technical problem here is that our description of $\mathrm{D}(T)$ is somewhat abstract, namely that $\mathrm{D}(T)$ is the completion of $\mathscr{A} \otimes \mathrm{D}(L)$ with respect to the graph norm of T. To handle this, we define some "discrete approximation" of $S(t)$: For $k = 1, \ldots, 2^{2n}$ put $a_{n,k} = n^{-1} + k2^{-n}$ and $V(n,k) = m^{-1}([a_{n,k}, a_{n,k+1}[)$. Furthermore, let $a_{n,0} = 0$ and $V(n,0) = m^{-1}(\{0\})$. We define the operator π_n on $L^2(\mathbb{R}) \otimes L^2(X)$ by

$$\pi_n \eta \otimes \xi = \sum_{k=0}^{2^{2n}} e^{-tA(a_{n,k})} \eta \otimes 1_{V(n,k)} \xi,$$

where $1_{V(n,k)}$ is the characteristic function of $V(n,k)$. It is not hard to see that the range of π_n is contained in $\mathrm{D}(T)$. Moreover, the properties of q_t and the theorem of dominated convergence ensure that $\lim_{n\to\infty} \pi_n \eta \otimes \xi = S(t)\eta \otimes \xi$ for all $\eta \in L^2(\mathbb{R})$, $\xi \in L^2(X)$.

We now prove $\lim_{n\to\infty} T(\pi_n \varphi \otimes \psi) = S(t)(T\varphi \otimes \psi)$ for all $\varphi \in \mathscr{A}$, $\psi \in \mathrm{D}(L)$, which then establishes $S(t)(\mathscr{A} \otimes \mathrm{D}(L)) \subseteq \mathrm{D}(T)$. Since a semigroup and its generator commute on the domain of the generator, we get for $k = 0$:

$$\begin{aligned} T(e^{-tA(0)}\varphi \otimes 1_{V(n,0)}\psi) &= -(e^{-tA(0)}\varphi)'' \otimes 1_{V(n,0)}\psi \\ &= e^{-tA(0)}(-\varphi'') \otimes 1_{V(n,0)}\psi = (e^{-tA(0)} \otimes 1_{V(n,0)})(T\varphi \otimes \psi), \end{aligned}$$

and for $k > 0$:

$$\begin{aligned} T(e^{-tA(a_{n,k})}\varphi \otimes 1_{V(n,k)}\psi) &= (e^{-tA(a_{n,k})} \otimes 1_{V(n,k)})(T\varphi \otimes \psi) \\ &\quad + (e^{-tA(a_{n,k})}\varphi'' - (e^{-tA(a_{n,k})}\varphi)'') \otimes (1 - m/a_{n,k})1_{V(n,k)}\psi. \end{aligned}$$

From this it follows almost directly that

$$\lim_{n\to\infty} T(\pi_n \varphi \otimes \psi) = \lim_{n\to\infty} \pi_n(T\varphi \otimes \psi) = S(t)(T\varphi \otimes \psi).$$

Let us now consider an arbitrary $f \in \mathrm{D}(T)$. Then there exists a sequence (f_n) in $\mathscr{A} \otimes \mathrm{D}(L)$ with $f_n \to f$ and $Tf_n \to Tf$. Since $S(t)$ is bounded, we have $S(t)f_n \to S(t)f$. From what we have shown so far it follows that $TS(t)f_n = S(t)Tf_n \to S(t)Tf$. Thus the sequence $(S(t)f_n)$ in $\mathrm{D}(T)$ converges to $S(t)f$ in the graph norm of T. This finally proves $S(t)(\mathrm{D}(T)) \subseteq \mathrm{D}(T)$ for all $t \geq 0$. ∎

Proof of Theorem 2.1. The result follows from Proposition 2.10 and identity (13).

Proof of Theorem 2.2. Let $\varphi \in L^2 \cap L^p(\mathbb{R} \times X)$. From Proposition 2.10 and (13) it follows that

$$\begin{aligned}
&\|e^{-zT}\varphi\|_{L^p(\mathbb{R}\times X)} \\
&\quad\le \bigg\| \int\limits_{\mathbb{R}} \int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(r-s)}{\xi}\Big)\bigg| \, \|e^{-\xi e^{r+s} L/2}\varphi_s\|_{L^p(X)} \, d\xi \, ds \bigg\|_{L^p(\mathbb{R},\, dr)} \\
&\quad\le C \bigg\| \int\limits_{\mathbb{R}} \int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(s)}{\xi}\Big)\bigg| \, \|\varphi_{s+r}\|_{L^p(X)} \, d\xi \, ds \bigg\|_{L^p(\mathbb{R},\, dr)} \\
&\quad\le C \bigg(\int\limits_{\mathbb{R}} \int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(s)}{\xi}\Big)\bigg| \, d\xi \, ds \bigg) \|\varphi\|_{L^p(\mathbb{R}\times X)}.
\end{aligned}$$

It was shown in [27, Sec. 3] that

$$\begin{aligned}
&\int\limits_{\mathbb{R}} \int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(s)}{\xi}\Big)\bigg| \, d\xi \, ds \\
&\qquad\qquad\le (2+\sqrt{\Re(z)})e^{\pi^2/4\Re(z)}(1+|\Im(z)|/\Re(z))^{3/2}.
\end{aligned}$$

That proves (3). As demonstrated earlier, (3) implies the multiplier statement of Theorem 2.2. Hence the proof of Theorem 2.2 is complete. ∎

Remark 2.11. In a similar manner to the proof of Theorem 2.2 one can try to use representation (1) to obtain (under certain conditions on L) also bounds for $\|e^{-zT}\|_{L^p(\mathbb{R}\times X)\to L^q(\mathbb{R}\times X)}$, $p \neq q$. In [12] we derived, e.g., some sort of ultracontractivity result for e^{-zT}. Depending on L, such a result may be of interest for some operators $T = T(L)$.

Proof of Theorem 2.3. Let the first assumptions on $(p_t)_{t>0}$ and $(\Lambda_t)_{t>0}$ hold. Let $\eta, \tau \in L^2(\mathbb{R}\times X)$. We want to be able to write the integrals appearing in $\langle e^{-zT}\eta, \tau\rangle_{L^2(\mathbb{R}\times X)}$ in any order, so first we show that the theorem of Fubini–Tonelli is applicable: Using the Cauchy–Schwarz inequality we get, with a constant M_z depending only on z,

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_0^\infty\int\limits_X\int\limits_X \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(r-s)}{\xi}\Big) p_{\xi e^{r+s}/2}(x,y)\eta(s,y)\tau(r,x)\bigg| \, dy\, dx\, d\xi\, ds\, dr \\
&\quad\le \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(r-s)}{\xi}\Big)\bigg| |\langle \Lambda_{\xi e^{r+s}/2}|\eta_s|, |\tau_r|\rangle_{L^2(X)}| \, d\xi\, ds\, dr \\
&\quad\le \kappa \int\limits_{\mathbb{R}} \bigg(\int\limits_0^\infty \bigg| \Psi_z(\xi) \exp\Big(-\frac{\operatorname{ch}(s)}{\xi}\Big)\bigg| \, d\xi \bigg) \Big(\int\limits_{\mathbb{R}} \|\eta_{s+r}\|_{L^2(X)}\|\tau_r\|_{L^2(X)}\, dr \Big)\, ds \\
&\quad\le \kappa M_z \|\eta\|_{L^2(\mathbb{R}\times X)}\|\tau\|_{L^2(\mathbb{R}\times X)}.
\end{aligned}$$

The theorem of Fubini–Tonelli ensures now the existence of the integral in (4) for almost all $((r,x),(s,y)) \in (\mathbb{R}\times X)^2$. From (13) and (14) we get

$$\langle e^{-zT}\eta,\tau\rangle_{L^2(\mathbb{R}\times X)} = \int_X\int_{\mathbb{R}}\Big(\int_X\int_{\mathbb{R}} P_z((r,x),(s,y))\eta(s,y)\,ds\,dy\Big)\tau(r,x)^*\,dr\,dx.$$

Thus P_z is the integration kernel of e^{-zT}.

Let now in addition $\|p_t(\cdot,x)\|_{L^1(X)}$ be bounded independently of x and t. Then (5) can be proven in a similar way to (3). As (5) implies (3) for $p=1$, the multiplier statement in Theorem 2.3 follows from Theorem 2.2. ∎

3. Differential operators on solvable Lie groups with exponential volume growth. Let $\mathfrak{n}$ be a real stratified nilpotent Lie algebra, i.e., there exist subspaces $V_1,\ldots,V_q$ of $\mathfrak{n}$ with $\mathfrak{n} = V_1\oplus\cdots\oplus V_q$ and $[V_i,V_j]\subseteq V_{i+j}$ (convention: $V_k=0$ if $k>q$), and V_1 generates the whole Lie algebra $\mathfrak{n}$. (For stratified nilpotent Lie groups and algebras we refer to [9].) Let the derivation D on $\mathfrak{n}$ be defined by $Dv_j = jv_j$ for all $v_j\in V_j$ and the group homomorphism $\theta:\mathbb{R}\to\mathrm{Aut}(\mathfrak{n})$ by $\theta(r)=e^{rD}$. Furthermore, let N be the set $\mathfrak{n}$ endowed with the Campbell–Hausdorff multiplication; thus N is, up to isomorphism, the uniquely determined connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. The exponential mapping $\exp_N$ is the identity $I_{\mathfrak{n}}$ on $\mathfrak{n}$. The Lebesgue measure dn on the Euclidean space $\mathfrak{n}$ is a biinvariant Haar measure on N. If Q denotes the trace of D (the so-called *homogeneous dimension* of N), then

$$\int_N f(e^{rD}n)\,dn = e^{-rQ}\int_N f(n)\,dn \qquad \text{for all } f\in L^1(N).$$

We define the solvable Lie group G by $G := N\rtimes_\theta\mathbb{R}$ ("$\mathbb{R}$ is acting on N via natural dilations"). If dr denotes the Lebesgue measure on $\mathbb{R}$, then $dg := dn\otimes dr$ is a right invariant Haar measure on G. The modular function m on G is given by $m(n,r)=e^{rQ}$, hence G has exponential volume growth (see [30, §IX.1]).

In the section "Improvements and open problems" of [16] W. Hebisch asked whether the evolution kernel P_z of a sum of even powers of vector fields or of a Schrödinger operator T on G satisfies an estimate like $\|P_{1+i\xi}(\cdot,1_G)\|_{L^1(G)} \le C(1+|\xi|^{(Q+4)/2})$. (In [16] Hebisch derived such an inequality for sub-Laplacians on G. This result is obviously not as good as the estimate (5) above, which is independent of the homogeneous dimension of $X:=N$ and was shown by Mustapha in [27] in the case of sub-Laplacians (cf. Subsection 3.1 below). But the methods of Hebisch have the advantage to extend to a reasonably larger class of Lie groups [11].)

For some special cases we can give a positive answer (independent of the homogeneous dimension of N) by employing Theorem 2.2. We use the following notation: For $\mathcal{X}\in\mathfrak{n}$ define vector fields $\mathcal{X}^N$ on N and $\mathcal{X}^G$ on G

by

$$\mathcal{X}^N\psi(x) = \frac{d}{dt}\psi(x\cdot \exp_N(t\mathcal{X}))|_{t=0} \qquad \text{for } \psi \in C^1(N),$$
$$\mathcal{X}^G\varphi(x,r) = \frac{d}{dt}\varphi((x,r)\cdot \exp_G(t\mathcal{X}))|_{t=0} \quad \text{for } \varphi \in C^1(G).$$

Furthermore, put $\mathcal{X}_0^G := \partial_r$. Then $\mathcal{X}_0^G$ is a left invariant vector field on G.

3.1. *Sub-Laplacians.* Let $\mathcal{X}_1, \ldots, \mathcal{X}_m \in V_1$ satisfy *Hörmander's condition*, i.e., generate $\mathfrak{n}$ as a Lie algebra. The operator $-\sum_{j=1}^m (\mathcal{X}_j^N)^2$ is defined on $C_c^\infty(N)$; let L denote its closure. Then L is positive and selfadjoint on $L^2(N)$ (see, e.g., Theorem 4.1), and hypoelliptic [19]. The semigroup of L is given by convolution from the right with a smooth heat kernel ϕ_t, $t>0$, which satisfies $\|\phi_t\|_{L^1(N)} = 1$:

$$e^{-tL}\varphi = \varphi * \phi_t.$$

Thus L induces a semigroup of contractions on $L^1(N)$, i.e.,

$$\|e^{-tL}\varphi\|_{L^1(N)} \le \|\varphi\|_{L^1(N)}\|\phi_t\|_{L^1(N)} = \|\varphi\|_{L^1(N)}.$$

Therefore the sub-Laplacian $T := -\sum_{j=0}^m (\mathcal{X}_j^G)^2$ on G, which is of the form $T = -\partial_r^2 + e^{2r}L$, meets the conditions of Theorem 2.2. Consequently, each compactly supported continuous $f \in H^\kappa(\mathbb{R})$, $\kappa > 2$, is an L^p-multiplier with respect to T for any $p \in [1,\infty]$.

This multiplier result was verified by S. Mustapha in [27]. Mustapha derived the representation (4) for the heat kernel of T by using stochastic methods and a formula from [32].

3.2. *Non-hypoelliptic sums of squares of vector fields.* Here $\mathcal{X}_1, \ldots, \mathcal{X}_m \in V_1$ are not required to satisfy Hörmander's condition. Thus the closure L of $-\sum_{j=1}^m (\mathcal{X}_j^N)^2$ is still positive and selfadjoint (see again Theorem 4.1), but in general not hypoelliptic. Although we cannot expect L to have a smooth heat kernel, it still induces a semigroup of contractions on $L^1(N)$, because

$$e^{-tL}\varphi = \int_N \varphi \circ \varrho_x \, dp_t(x),$$

where ϱ_x is the right translation on N by x and $(p_t)_{t>0}$ is a convolution semigroup of probability measures on N (see [21]). Therefore $T := -\sum_{j=1}^m (\mathcal{X}_j^G)^2$ still meets the conditions of Theorem 2.2. Again each compactly supported continuous $f \in H^\kappa(\mathbb{R})$, $\kappa > 2$, is an L^p-multiplier with respect to T for every $p \in [1,\infty]$.

Theorem 2.2 and this subsection show that the hypoellipticity of the sub-Laplacians plays no crucial role in Mustapha's multiplier result.

3.3. *Schrödinger operators.* Let $\mathcal{X}_1,\dots,\mathcal{X}_m \in V_1$ and $\widetilde{f}_1,\dots,\widetilde{f}_m \in C^1(N,\mathbb{R})$. We consider

$$\widetilde{L} := -\sum_{j=1}^{m}(\mathcal{X}_j^N + i\widetilde{f}_j)^2.$$

Moreover, define $f_0 := 0$ and $f_j := \widetilde{f}_j \otimes e^r$ for $j = 1,\dots,m$. The operator

$$T := -\sum_{j=0}^{m}(\mathcal{X}_j^G + if_j)^2$$

is of the form $T = -\partial_r^2 + e^{2r}\widetilde{L}$. Furthermore, $\widetilde{L}$ and T are essentially selfadjoint (see Theorem 4.1), and we denote their closures again by $\widetilde{L}$ and T. (If $\{\mathcal{X}_1,\dots,\mathcal{X}_m\}$ spans V_1, then $\widetilde{L}$ and T are called *Schrödinger operators.*)

Now let L be the main part of $\widetilde{L}$, i.e., $L = -\sum_{j=1}^m(\mathcal{X}_j^N)^2$. Then

$$\|e^{-t\widetilde{L}}\varphi\|_{L^1(N)} \le \big\|e^{-tL}|\varphi|\big\|_{L^1(N)} \le \|\varphi\|_{L^1(N)}$$

for any $\varphi \in L^1 \cap L^2(N)$ since $|e^{-t\widetilde{L}}\varphi| \le e^{-tL}|\varphi|$ pointwise almost everywhere (see, e.g., [15, Lemma 1.3] and its proof or, for a detailed proof of the whole statement, [10, Lemma 3.21]). Hence Theorem 2.2 implies once again that each $f \in C_{\mathrm{c}} \cap H^\kappa(\mathbb{R})$, $\kappa > 2$, is an L^p-multiplier with respect to T for every $p \in [1,\infty]$.

Notice that, in contrast to sub-Laplacians, the operators T defined in this subsection are not left invariant.

3.4. *Rockland operators on N.* A left invariant differential operator L on N is called *homogeneous of degree $d \in \mathbb{N}$* if $L(\varphi\circ\theta(r)) = e^{dr}(L\varphi)\circ\theta(r)$ for all $\varphi \in C_{\mathrm{c}}^\infty(N)$ and all $r \in \mathbb{R}$. If in addition for every non-trivial irreducible unitary representation π of N the operator $d\pi(L)$ is injective on the space of C^∞-vectors of π (i.e., the set of elements φ of the representation space $\mathscr{H}_\pi$, where $N \ni x \mapsto \pi(x)\varphi$ is a C^∞-function), then L is called a *Rockland operator.* (For Rockland operators see [9].)

We consider here a Rockland operator L which is positive and formally selfadjoint on $C_{\mathrm{c}}^\infty(N)$. Then $L|_{C_{\mathrm{c}}^\infty(N)}$ is essentially selfadjoint and its closure will again be denoted by L. The operator L induces a semigroup on $L^2(N)$ by $e^{-tL}f = f * \phi_t$, $t > 0$, where ϕ_t is in $C^\infty \cap L^1(N)$ and satisfies $\|\phi_t\|_{L^1(N)} = \|\phi_1\|_{L^1(N)} =: C$ for all $t > 0$.

In the notation of Theorem 2.3 we have $p_t(x,y) = \phi_t(y^{-1}x)$, which implies

$$\|p_t(\cdot,m)\|_{L^1(N)} = \int_N |\phi_t(m^{-1}n)|\,dn = \int_N |\phi_t(n)|\,dn = C.$$

By setting

$$(\widetilde{L}f)(n,r) := e^{rd}(Lf(\,\cdot,r))(n)$$

we obtain a left invariant differential operator $\widetilde{L}$ on G. If we define the operator T on G by $T = -\partial_r^2 + \widetilde{L}$, then $T = -\partial_r^2 + e^{rd}L$. From Theorem 2.3 we get, for the convolution kernel $(\Phi_z)_{\Re(z)>0}$ of e^{-zT},

$$\Phi_z(n,r) = P_z((n,r),1_G) = \int_0^\infty \nu\Psi_{\nu^2 z}(\xi)\exp(-\mathrm{ch}(\nu r)/\xi)\phi_{\xi e^{\nu r}/2\nu^2}(n)\,d\xi,$$

where $\nu := d/2$ and 1_G is the neutral element of G. Furthermore,

$$\|\Phi_{1+i\xi}\|_{L^1(G)} \le C_\nu(1+|\xi|)^{3/2},$$

with C_ν independent of ξ, and all compactly supported, continuous $f \in H^{2+\varepsilon}(\mathbb{R})$, $\varepsilon > 0$, are L^p-multipliers of T for any $p \in [1,\infty]$.

The last example is a special case of the preceding class of differential operators:

3.5. *Sums of even powers of vector fields.* Let $\mathcal{X}_1,\ldots,\mathcal{X}_m \in \mathfrak{n}$ generate the Lie algebra $\mathfrak{n}$, and let $k_1,\ldots,k_m \in \mathbb{N}$. Moreover, let the differential operator

$$L := \sum_{j=1}^m (-1)^{k_j}(\mathcal{X}_j^N)^{2k_j}$$

on $C_c^\infty(N)$ be homogeneous of degree 2ν, $\nu = \max\{k_j : 1 \le j \le m\}$. (Hence $\mathcal{X}_j \in V_l$ implies $lk_j = \nu$.) Then L is a positive Rockland operator. If we define the sum of even powers of vector fields T on G by

$$T = -\partial_r^2 + \sum_{j=1}^m (-1)^{k_j}(\mathcal{X}_j^G)^{2k_j},$$

then $T = -\partial_r^2 + e^{2\nu r}L$. We have the results of Subsection 3.4 for $d = 2\nu$.

Subsection 3.5 stresses that a differential operator T does not have to be of second order to satisfy the conditions of our multiplier theorem.

4. Appendix: A selfadjointness theorem. Let G be a real Lie group with a countable number of connected components, $\mathfrak{g}$ its Lie algebra, dg a right invariant Haar measure on G and $L^2 = L^2(G,dg)$. Let $\langle\cdot,\cdot\rangle$ be the scalar product and $\|\cdot\|$ the norm on L^2. We shall identify each $\mathcal{X} \in \mathfrak{g}$ with a left invariant vector field by

$$\mathcal{X}f(g) = \frac{d}{dt}f(g\cdot\exp(t\mathcal{X}))\bigg|_{t=0},$$

where exp denotes the exponential function with respect to G and $\mathfrak{g}$.

Let J be a finite index set, $\mathcal{X}_j$, $j \in J$, left invariant vector fields, and V_j, $j \in J$, real-valued, continuously differentiable functions on G. The operator

$$T := -\sum_{j\in J}(\mathcal{X}_j + iV_j)^2$$

is well defined on $C_c^\infty = C_c^\infty(G)$, the space of test functions on G.

For $f \in L^2$ and for operators Φ, Φ^* defined on C_c^∞ with $\langle \Phi\varphi, \psi\rangle = \langle \varphi, \Phi^*\psi\rangle$ for all $\varphi, \psi \in C_c^\infty$ we shall say that Φf *exists in a weak sense* if there is a function $\widetilde{f} \in L^2$ with $\langle f, \Phi^*\varphi\rangle = \langle \widetilde{f}, \varphi\rangle$ for all $\varphi \in C_c^\infty$. In this case we define $\Phi_{\mathrm{w}} f := \widetilde{f}$, so the domain of the operator Φ_w is given by

$$\mathrm{D}(\Phi_{\mathrm{w}}) = \{f \in L^2 : \Phi f \text{ exists in a weak sense}\}.$$

A helpful tool for the proof of our selfadjointness theorem is the *convolution* of two (suitable) functions φ, ψ on G, defined by

$$\varphi * \psi(x) = \int_G \varphi(xg^{-1})\psi(g)\, dg.$$

We call a sequence (φ_n) in C_c^∞ a *Dirac sequence* if $\varphi_n \geq 0$, $\int \varphi_n\, dg = 1$ for all $n \in \mathbb{N}$, and for each neighborhood U of the unit element of G there exists an $n_0 \in \mathbb{N}$ with $\operatorname{supp}(\varphi_n) \subseteq U$ for every $n \geq n_0$.

We shall also utilize a sequence (ψ_n) in C_c^∞ that approximates the characteristic function of G in the following way:

(i) For any $n \in \mathbb{N}$, $0 \leq \psi_n \leq 1$.
(ii) $(\psi_n^{-1}(\{1\}))$ is an increasing sequence of sets with $\bigcup \psi_n^{-1}(\{1\}) = G$.
(iii) For each left invariant differential operator L there exists a $C > 0$, independent of n, with $|L\psi_n| \leq C$ for all n.

It is easy to see that such a sequence exists. (A construction can be found in [10, Lemma 2.28].)

Furthermore, we will make use of the theory of quadratic forms of selfadjoint operators and the well known theorem about Friedrichs' extension. This can, e.g., be found in [8, Ch. 4] and [31, §5.5].

THEOREM 4.1. *T is essentially selfadjoint on $C_c^\infty \subseteq L^2$. Its selfadjoint closure $\overline{T}$ is given by T_{w}. The domain of the quadratic form of T_{w} is*

$$\mathrm{Q}(T_{\mathrm{w}}) = \bigcap_{j\in J} \mathrm{D}((\mathcal{X}_j + iV_j)_{\mathrm{w}})$$

and the inclusion $\mathrm{D}(T_{\mathrm{w}}) \subseteq \mathrm{Q}(T_{\mathrm{w}})$ holds.

Proof. From the definition of the adjoint operator T^* of T on L^2 it is clear that $T^* = T_{\mathrm{w}}$. As T is positive and symmetric on C_c^∞, there exists Friedrichs' extension S of T with $\mathrm{D}(S) = \mathrm{D}(T_{\mathrm{w}}) \cap \mathrm{Q}(S)$. Here the domain $\mathrm{Q}(S)$ of the quadratic form of S is the set of all $f \in L^2(G)$ for which there exists a sequence (f_n) in C_c^∞ with $L^2\text{-}\lim f_n = f$ such that $((\mathcal{X}_j + iV_j)f_n)$ is an L^2-Cauchy sequence for each $j \in J$.

Obviously we have $\overline{T} \subseteq S = S^* \subseteq T^*$. Our aim is to show that $\overline{T} = S$, because that would imply $\overline{T} = T^* = T_{\mathrm{w}}$. Before we do so, we verify that $\mathrm{Q}(S)$ is equal to $\Lambda := \bigcap_{j\in J} \mathrm{D}((\mathcal{X}_j + iV_j)_{\mathrm{w}})$. Since it is easy to see that $\mathrm{Q}(S) \subseteq \Lambda$, we just have to prove

STATEMENT (a). $\Lambda \subseteq \mathrm{Q}(S)$.

To verify Statement (a), we first consider a compactly supported function $f \in \Lambda$ and a Dirac sequence (φ_n). Then $(\varphi_n * f)$ is a sequence in C_c^∞ with $\|f - \varphi_n * f\| \to 0$ for $n \to \infty$. Let $j \in J$. From $f \in \Lambda$ and $V_j f \in L^2$ it follows that $f \in \mathrm{D}((\mathcal{X}_j)_\mathrm{w})$. Therefore the left invariance of $\mathcal{X}_j$ leads us to $\mathcal{X}_j(\varphi_n * f) = \varphi_n * (\mathcal{X}_j)_\mathrm{w} f$, and

$$\begin{aligned}&\|(\mathcal{X}_j + iV_j)_\mathrm{w} f - (\mathcal{X}_j + iV_j)(\varphi_n * f)\| \\ &\qquad \le \|(\mathcal{X}_j + iV_j)_\mathrm{w} f - \varphi_n * (\mathcal{X}_j + iV_j)_\mathrm{w} f\| + \|\varphi_n * V_j f - V_j(\varphi_n * f)\|.\end{aligned}$$

The first term on the right vanishes as $n \to \infty$. In general for continuous ϑ on G and compactly supported $\widetilde{f} \in L^2$ we have

$$\|\varphi_n * \vartheta \widetilde{f} - \vartheta(\varphi_n * \widetilde{f})\| \le \big\|\varphi_n * |\widetilde{f}|\big\| \sup\{|\vartheta(x^{-1}g) - \vartheta(g)|\}, \tag{15}$$

where the supremum is taken over all $x \in \mathrm{supp}(\varphi_n)$ and $g \in \mathrm{supp}(\varphi_n)\,\mathrm{supp}(\widetilde{f})$. The expression on the right hand side in (15) tends to zero as $n \to \infty$. It follows that f is an element of $\mathrm{Q}(S)$.

Let us now consider a general $f \in \Lambda$, and let (ψ_n) be an approximating sequence for the characteristic function of G as described above. It is trivial that $\psi_n f \in \Lambda$ with

$$(\mathcal{X}_j + iV_j)_\mathrm{w}(\psi_n f) = \psi_n (\mathcal{X}_j + iV_j)_\mathrm{w} f + (\mathcal{X}_j \psi_n) f.$$

Since $\mathrm{supp}(\psi_n f)$ is compact, we have $\psi_n f \in \mathrm{Q}(S)$. Obviously $\psi_n f \to f$ and

$$\|(\mathcal{X}_j + iV_j)_\mathrm{w}(f - \psi_n f)\| \le \|(1 - \psi_n)(\mathcal{X}_j + iV_j)_\mathrm{w} f\| + \|(\mathcal{X}_j \psi_n) f\|.$$

The choice of (ψ_n) implies $\|(\mathcal{X}_j + iV_j)_\mathrm{w}(f - \psi_n f)\| \to 0$ as $n \to \infty$. It follows that $f \in \mathrm{Q}(S)$, i.e., Statement (a) holds.

Statement (b). $\mathrm{D}(S) \subseteq \mathrm{D}(\overline{T})$.

Our strategy is similar to the proof of Statement (a): First let $f \in \mathrm{D}(S)$ be compactly supported. Then $V_j f \in L^2$ and $f \in \mathrm{D}((\mathcal{X}_j)_\mathrm{w})$ for each $j \in J$, because f is contained in $\mathrm{Q}(S) = \Lambda$ (recall that $\mathrm{D}(S) \subseteq \mathrm{Q}(S)$). Moreover, f is contained in $\mathrm{D}(T_\mathrm{w}) \cap \mathrm{D}((V_j \mathcal{X}_j)_\mathrm{w})$, and $V_j^2 f, (\mathcal{X}_j V_j) f \in L^2$ for each $j \in J$, which implies $f \in \mathrm{D}((\sum_{j \in J} X_j^2)_\mathrm{w})$.

Let us again consider a Dirac sequence (φ_n). The sequence (f_n) with $f_n := \varphi_n * f$ is a sequence in C_c^∞ such that $f_n \to f$ in L^2. We obtain

$$\begin{aligned}&\|T_\mathrm{w}(f - f_n)\| \\ &\qquad \le \|T_\mathrm{w} f - \varphi_n * T_\mathrm{w} f\| + \sum_{j \in J} (2\|\varphi_n * (V_j \mathcal{X}_j)_\mathrm{w} f - V_j(\varphi_n * (\mathcal{X}_j)_\mathrm{w} f)\| \\ &\qquad\qquad + \|\varphi_n * (V_j^2 - i(\mathcal{X}_j V_j)) f - (V_j^2 - i(\mathcal{X}_j V_j))\varphi_n * f\|).\end{aligned}$$

With (15) we observe that $T f_n \to T_\mathrm{w} f$ in L^2, which means $f \in \mathrm{D}(\overline{T})$ and $\overline{T} f = T_\mathrm{w} f$.

Now let f be an arbitrary element in $\mathrm{D}(S)$, and let (ψ_n) be as in the proof of Statement (a). Obviously $\psi_n f \in \mathrm{D}(S)$ for each $n \in \mathbb{N}$. As $\psi_n f$ has compact support, we have $\psi_n f \in \mathrm{D}(\overline{T})$. Moreover, $\psi_n f \to f$ in L^2 and

$$\begin{aligned}&\|T_{\mathrm{w}} f - \overline{T}(\psi_n f)\| \\ &\quad \le \|(1-\psi_n)T_{\mathrm{w}} f\| + \sum_{j\in J}(\|(\mathcal{X}_j^2\psi_n)f\| + 2\|(\mathcal{X}_j\psi_n)(\mathcal{X}_j + iV_j)_{\mathrm{w}} f\|).\end{aligned}$$

From our definition of (ψ_n) it follows that $\|T_{\mathrm{w}} f - \overline{T}(\psi_n f)\| \to 0$ as $n \to \infty$. As $\mathrm{D}(\overline{T})$ is closed with respect to the graph norm of T, f is an element of $\mathrm{D}(\overline{T})$. This proves Statement (b), which implies $\overline{T} = S$. ∎

Acknowledgments. I would like to thank Prof. Detlef Müller for introducing me to the subject of this paper and for helpful discussions. I would also like to thank an anonymous referee for useful suggestions.

REFERENCES

[1] G. Alexopoulos, *Spectral multipliers on Lie groups of polynomial growth*, Proc. Amer. Math. Soc. 120 (1994), 973–979.

[2] F. Astengo, *Multipliers for a distinguished Laplacean on solvable extensions of H-type groups*, Monatsh. Math. 120 (1995), 179–188.

[3] P. Bougerol, *Exemples de théorèmes locaux sur les groupes résolubles*, Ann. Inst. H. Poincaré Sect. B 19 (1983), 369–391.

[4] M. Christ, *L^p bounds for spectral multipliers on nilpotent groups*, Trans. Amer. Math. Soc. 328 (1991), 73–81.

[5] M. Christ and D. Müller, *On L^p spectral multipliers for a solvable Lie group*, Geom. Funct. Anal. 6 (1996), 860–876.

[6] M. G. Cowling, S. Giulini, A. Hulanicki and G. Mauceri, *Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth*, Studia Math. 111 (1994), 103–121.

[7] E. David-Guillou, *L^p bounds for spectral multipliers on rank one NA-groups with roots not all positive*, Colloq. Math. 101 (2004), 51–74.

[8] E. B. Davies, *One-Parameter Semigroups*, Academic Press, 1980.

[9] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.

[10] M. Gnewuch, *Zum differenzierbaren L^p-Funktionalkalkül auf Lie-Gruppen mit exponentiellem Volumenwachstum*, Ph.D. thesis, Kiel, 2002.

[11] —, *Spectral multipliers for sub-Laplacians on amenable Lie groups with exponential volume growth*, Math. Z. 246 (2004), 69–83.

[12] —, *On selfadjointness and spectral multipliers for sums of non-commuting operators*, Erwin Schrödinger Internat. Inst. for Mathematical Physics, Vienna, ESI Preprint 1401, 2003.

[13] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals*, *Series and Products*, 5th ed., A. Jeffrey (ed.), Academic Press, 1994.

[14] W. Hebisch, *Multiplier theorem on generalized Heisenberg groups*, Colloq. Math. 65 (1993), 231–239.

[15] —, *Boundedness of L^1 spectral multipliers for an exponential solvable Lie group*, ibid. 73 (1997), 155–164.

[16] —, *Spectral multipliers on exponential growth solvable Lie groups*, Math. Z. 229 (1998), 435–441.

[17] W. Hebisch, J. Ludwig and D. Müller, *Sub-Laplacians of holomorphic L^p-type on exponential solvable groups*, J. London Math. Soc. 72 (2005), 364–390.

[18] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.

[19] —, *Hypoelliptic second order differential equations*, ibid. 119 (1967), 147–171.

[20] A. Hulanicki, *On the spectrum of the Laplacian on the affine group of the real line*, Studia Math. 54 (1976), 199–204.

[21] G. A. Hunt, *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. 81 (1956), 264–293.

[22] N. N. Lebedev, *Special Functions and their Applications*, Prentice-Hall, 1965.

[23] J. Ludwig and D. Müller, *Sub-Laplacians of holomorphic L^p-type on rank one AN-groups and related solvable groups*, J. Funct. Anal. 170 (2000), 366–427.

[24] G. Mauceri and S. Meda, *Vector-valued multipliers on stratified groups*, Rev. Mat. Iberoamericana 6 (1990), 141–154.

[25] D. Müller and E. M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. 73 (1994), 413–440.

[26] S. Mustapha, *Multiplicateurs spectraux sur certains groupes non-unimodulaires*, in: Harmonic Analysis and Number Theory (Montreal, PQ, 1996), CMS Conf. Proc. 21, Amer. Math. Soc., 1997, 11–30.

[27] —, *Multiplicateurs de Mikhlin pour une classe particulière de groupes non-unimodulaires*, Ann. Inst. Fourier (Grenoble) 48 (1998), 957–966.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[29] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, *Part I*, 2nd ed., Oxford Univ. Press, 1962.

[30] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Univ. Press, 1992.

[31] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, 1980.

[32] M. Yor, *On some exponential functionals of Brownian motion*, Adv. in Appl. Probab. 24 (1992), 509–531.

Max Planck Institute for Mathematics in the Sciences
Inselstr. 22
D-04103 Leipzig, Germany
E-mail: gnewuch@mis.mpg.de

Received 10 October 2004;
revised 10 September 2005 (4541)