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INEQUALITIES FOR TWO SINE POLYNOMIALS

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Abstract. We prove:

(I) For all integers $n \geq 2$ and real numbers $x \in (0, \pi)$ we have

$$\alpha \leq \sum_{j=1}^{n-1} \frac{1}{n^2 - j^2} \sin(jx) \leq \beta,$$

with the best possible constant bounds

$$\alpha = \frac{15 - \sqrt{2073}}{10240} \sqrt{1998 - 10\sqrt{2073}} = -0.1171\ldots, \qquad \beta = \frac{1}{3}.$$

(II) The inequality

$$0 < \sum_{j=1}^{n-1} (n^2 - j^2) \sin(jx)$$

holds for all even integers $n \geq 2$ and $x \in (0, \pi)$, and also for all odd integers $n \geq 3$ and $x \in (0, \pi - \pi/n]$.

1. Introduction. Problems on the infinite divisibility of probability distributions led K. Takano [18]–[24] to the study of several interesting trigonometric sums. In [19] he investigated the sine polynomial

$$T_n(x) = \sum_{j=1}^{n} \frac{1}{(n-j)!(n+j)!} \sin(jx)$$

and proved the identity

$$T_n(x) = \frac{\sin(x)}{(2n)!} \sum_{j=0}^{n-1} \frac{(2(n-j-1))!}{((n-j-1)!)^2} (2\cos(x/2))^{2j}. \tag{1.1}$$

This is a special case of a more general identity for Jacobi polynomials obtained in [13]. See also [25]. From (1.1) we immediately get the inequality

$$0 < T_n(x) \quad (n \in \mathbb{N},\ 0 < x < \pi). \tag{1.2}$$

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Elementary estimates reveal that the following converse of (1.2) is valid:

$$T_n(x) \le \frac{1}{2} \quad (n \in \mathbb{N},\ 0 < x < \pi). \tag{1.3}$$

The bounds given in (1.2) and (1.3) are best possible.

Inequalities for sine and cosine polynomials have attracted the attention of mathematicians since many years. A detailed collection of the most important theorems as well as historical remarks, applications, and numerous references on this subject can be found in the monograph [16, Chapter 4] and the survey paper [9]. Various new results are published in the research articles [1]–[5].

In this paper we study two sine polynomials which are related to T_n. The estimates (1.2) and (1.3) inspired us to ask about sharp constant bounds for

$$S_n(x) = \sum_{j=1}^{n-1} \frac{1}{(n-j)(n+j)} \sin(jx), \qquad S_n^*(x) = \sum_{j=1}^{n-1} (n-j)(n+j) \sin(jx).$$

In what follows, we maintain these notations. The function S_n^* is a companion of Lukács' polynomial

$$L_n(x) = \sum_{j=1}^{n-1} (n-j) \sin(jx) = \frac{n\sin(x) - \sin(nx)}{4(\sin(x/2))^2},$$

which has been studied by several authors. F. Lukács proved that $L_n(x) > 0$ for all $n \ge 2$ and $x \in (0,\pi)$; see [12]. This inequality is important because it represents the positivity of the classical conjugate Fejér kernel; see [27, pp. 91–92]. Variants and generalizations of Lukács' inequality are given in [1], [6], [8]–[11], [15], [16, p. 140].

2. Main results. First, we provide sharp upper and lower bounds for $S_n(x)$.

THEOREM 1. *For all integers $n \ge 2$ and real numbers $x \in (0,\pi)$ we have*

$$\alpha \le \sum_{j=1}^{n-1} \frac{1}{n^2 - j^2} \sin(jx) \le \beta, \tag{2.1}$$

with the best possible constant bounds

$$\alpha = \frac{15 - \sqrt{2073}}{10240} \sqrt{1998 - 10\sqrt{2073}} = -0.1171\ldots, \qquad \beta = \frac{1}{3}.$$

Proof. Let $x \in (0,\pi)$. Then

$$0 < S_2(x) = \frac{1}{3}\sin(x) \le \frac{1}{3} = S_2(\pi/2),$$

and for $n \geq 3$ we obtain

$$\begin{aligned} S_n(x) &\leq \sum_{j=1}^{n-2} \frac{1}{n^2-j^2} + \frac{1}{n^2-(n-1)^2} \\ &\leq \frac{n-2}{n^2-(n-2)^2} + \frac{1}{n^2-(n-1)^2} = \tau_n, \qquad \text{say.} \end{aligned} \tag{2.2}$$

Combining (2.2) and

$$\frac{1}{3} - \tau_n = \frac{(n-2)(2n-5)}{12(n-1)(2n-1)} > 0,$$

we conclude that $S_n(x) < 1/3$.

A short calculation reveals that

$$S_3(x) = \sqrt{1-(\cos(x))^2}\left(\frac{1}{8} + \frac{2}{5}\cos(x)\right)$$

attains its absolute minimum at

$$x_0 = \arccos\left(-\frac{1}{64}\,(5+\sqrt{2073})\right) = 2.4808\dots$$

with

$$S_3(x_0) = \frac{15-\sqrt{2073}}{10240}\sqrt{1998-10\sqrt{2073}} = -0.1171\dots.$$

We denote by U_k the Chebyshev polynomial of the second kind, which is given by

$$U_k(t) = \frac{\sin((k+1)x)}{\sin(x)} \qquad (k = 0, 1, \dots),$$

where $\cos(x) = t$. Then we obtain the representation

$$S_n(x) = \sqrt{1-t^2}\sum_{j=1}^{n-1} \frac{U_{j-1}(t)}{n^2-j^2}.$$

In order to prove the left-hand bound of (2.1) it suffices to show that

$$\sqrt{1-t^2}\sum_{j=1}^{n-1} \frac{U_{j-1}(t)}{n^2-j^2} + 0.117 > 0 \qquad \text{for } -1 < t < 1. \tag{2.3}$$

We define, for $s \in [0,1]$,

$$h(s) = 1 - \frac{1}{2}s - \frac{1}{8}s^2 - \frac{1}{16}s^3 - \sqrt{1-s}.$$

Since $h(0) = 0$ and

$$h'(s) = \frac{(40+15s+9s^2)s^3}{16(8\sqrt{1-s}+(1-s)(8+4s+3s^2))},$$

we conclude that h is positive on $(0,1]$. Thus, for $t \in (-1,1)$ we get

$$\sqrt{1-t^2} \le 1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6.$$

This implies that the validity of

$$\left(1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6\right)\sum_{j=1}^{n-1} \frac{U_{j-1}(t)}{n^2-j^2} + 0.117 > 0 \quad \text{for } -1 < t < 1 \tag{2.4}$$

leads to (2.3). Let P_n be the polynomial on the left-hand side of (2.4). An application of Sturm's theorem (see, for example, [26, p. 248]) shows that for $n = 4, 5, \ldots, 33$ the function P_n has no zero on $[-1,1]$. Since $U_k(1) = k+1$ $(k \ge 0)$, we obtain

$$P_n(1) = \frac{5}{16}\sum_{j=1}^{n-1} \frac{j}{n^2-j^2} + 0.117 > 0.$$

Thus, $P_n(t) > 0$ for $t \in [-1,1]$.

Next, we prove that $S_n(x) > -0.117$ for $n \ge 34$. We set

$$a_j = \frac{\sin(jx)}{n-j}, \quad b_j = \frac{1}{n+j} \quad (j = 1, \ldots, n-1).$$

Applying Abel's lemma (see [17, pp. 32–33]) gives

$$S_n(x) = \sum_{j=1}^{n-1} a_j b_j \ge b_1 \min_{1\le k\le n-1} \sum_{j=1}^{k} a_j = \frac{1}{n+1} \min_{1\le k\le n-1} \sum_{j=1}^{k} \frac{\sin(jx)}{n-j}. \tag{2.5}$$

Let $k \in \{1, \ldots, n-1\}$ and

$$W_{k,n}(x) = \frac{1}{n+1}\sum_{j=1}^{k} \frac{\sin(jx)}{n-j}.$$

We get

$$(n+1)W_{k,n}(x) \ge -\sum_{j=1}^{k} \frac{1}{n-j} = \psi(n-k) - \psi(n),$$

where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivative of Euler's gamma function. Since ψ is strictly increasing on $(0,\infty)$, we obtain

$$(n+1)W_{k,n}(x) \ge \psi(1) - \psi(n) = -\gamma - \psi(n). \tag{2.6}$$

The function

$$Y(x) = 0.117(x+1) - \psi(x) - \gamma$$

is strictly convex on $(0,\infty)$ with

$$Y(34) = 0.0062\ldots, \qquad Y'(34) = 0.0871\ldots.$$

Hence, we have

$$-\gamma - \psi(n) > -0.117(n+1) \qquad \text{for } n \geq 34. \tag{2.7}$$

Combining (2.6) and (2.7) leads to $W_{k,n}(x) > -0.117$. From (2.5) we conclude that $S_n(x) > -0.117$ for $n \geq 34$. This completes the proof of Theorem 1. ∎

In view of Lukács' inequality $L_n(x) > 0$ it is tempting to conjecture that $S_n^*(x)$ is positive for all $n \geq 2$ and $x \in (0, \pi)$. We prove that this is true for even n. If n is odd, then we conclude from $S_n^*(\pi) = 0$ and $S_n^{*\prime}(\pi) = (n^2-1)/4$ that S_n^* is not everywhere positive on $(0, \pi)$. However, as Theorem 2 below indicates, the negative values only appear in a small interval in the vicinity of π.

THEOREM 2. *For all even integers $n \geq 2$ and real numbers $x \in (0, \pi)$ we have*

$$0 < \sum_{j=1}^{n-1} (n^2 - j^2) \sin(jx). \tag{2.8}$$

Moreover, (2.8) *holds for all odd integers $n \geq 3$ and real numbers $x \in (0, \pi - \pi/n]$.*

Proof. We define

$$\sigma_n(x) = \sum_{j=1}^{n-1} \sin(jx) = \frac{\cos(x/2) - \cos((n-1/2)x)}{2\sin(x/2)}.$$

Some elementary calculations lead to

$$\begin{aligned} S_n^*(x) &= n^2 \sigma_n(x) + \sigma_n''(x) \\ &= \frac{\sin(x)(1-\cos(nx)) + n(1-\cos(x))(n\sin(x) - 2\sin(nx))}{2(1-\cos(x))^2}. \end{aligned} \tag{2.9}$$

In order to prove that $S_n^*(x)$ is positive it is sufficient to show that

$$0 < 1 - \cos(nx) + n^2(1-\cos(x))\left(1 - 2\,\frac{\sin(nx)}{n\sin(x)}\right) = F_n(x), \qquad \text{say.}$$

We distinguish three cases.

CASE 1: $0 < x < \pi/n$. Using

$$\frac{\sin(nx)}{n\sin(x)} < 1, \qquad \left(\frac{\sin(nx)}{n\sin(x)}\right)' = \frac{\sin(nx)}{nx\sin(x)}\,(nx\cot(nx) - x\cot(x)) < 0,$$

we get

$$F_n'(x) = n^2\sin(x)\left(1 - \frac{\sin(nx)}{n\sin(x)}\right) - 2n^2(1-\cos(x))\left(\frac{\sin(nx)}{n\sin(x)}\right)' > 0.$$

This yields $F_n(x) > F_n(0) = 0$.

CASE 2: $\pi/n \le x \le \pi - \pi/n$. Applying the inequality

$$\left|\frac{\sin(nx)}{n\sin(x)}\right| \le \frac{1}{3}, \tag{2.10}$$

we conclude that $F_n(x)$ is positive.

CASE 3: $\pi - \pi/n < x < \pi$ and n even. Then we have $\sin(nx) < 0$, which implies that $F_n(x) > 0$. ∎

REMARKS. (1) Equality holds in (2.10) only when $n = 3$ and $x = \pi/2$. Actually (2.10) can be refined to

$$-\frac{1}{3} \le \frac{\sin(nx)}{n\sin(x)} \le \frac{\sqrt{6}}{9}, \quad \frac{\pi}{n} \le x \le \pi - \frac{\pi}{n}, \quad n = 2, 3, \ldots, \tag{2.11}$$

where the constants $-1/3$, $\sqrt{6}/9$ are the best possible. Inequality (2.11) was stated by R. Askey in [7] as a problem, which was solved by A. A. Jagers in [14]. Additional comments on these inequalities as well as references to applications are given by R. Askey in [14]. We note that inequality (2.10) was also used by R. Askey and G. Gasper in [8, p. 727] in the proof of an inequality for Jacobi polynomials.

(2) Since $S_n^*(0) = 0$, we see that the lower bound 0 is sharp. From (2.9) we get the limit relation

$$\lim_{n\to\infty} S_n^*(\pi/n) = \infty,$$

which reveals that there does not exist a constant upper bound for $S_n^*(x)$. And, if n is odd, then we obtain

$$\lim_{n\to\infty} S_n^*(\pi - \pi/(4n)) = -\infty.$$

This also implies that there does not exist a constant lower bound for $S_n^*(x)$ which is valid for all odd n.

(3) Inequality (2.8) is closely related to

$$0 < \sum_{j=1}^{n-1} (n^2 - j^2)\frac{\sin(jx)}{j} \quad (n \ge 2,\ 0 < x < \pi), \tag{2.12}$$

which is given in [11]. Inequalities (2.8) and (2.12) do not imply each other. If (2.8) were true for odd n as well, then (2.12) would follow from (2.8) by summation by parts.

(4) It is natural to ask about sharp inequalities for the cosine polynomials

$$C_n(x) = \sum_{j=1}^{n-1} \frac{1}{n^2 - j^2}\cos(jx), \qquad C_n^*(x) = \sum_{j=1}^{n-1} (n^2 - j^2)\cos(jx).$$

We have $C_2(x) = \cos(x)/3$ and as in (2.2) we get, for $n \geq 3$,

$$|C_n(x)| \leq \tau_n < \frac{1}{3}.$$

Thus,

$$-\frac{1}{3} < C_n(x) < \frac{1}{3} \qquad (n \geq 2,\ 0 < x < \pi),$$

where both bounds are best possible. Since

$$\lim_{n\to\infty} C_n^*(0) = \infty, \qquad \lim_{n\to\infty} C_n^*(\pi) = -\infty,$$

it follows that there do not exist constant bounds for $C_n^*(x)$ which hold for all $n \geq 2$ and $x \in (0, \pi)$.

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