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A GENERAL FORM OF NON-FROBENIUS SELF-INJECTIVE ALGEBRAS

BY

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Dedicated to Professor Takeshi Sumioka on the occasion of his sixtieth birthday

Abstract. Applying the classical work of Nakayama [Ann. of Math. 40 (1939)], we exhibit a general form of non-Frobenius self-injective finite-dimensional algebras over a field.

Introduction. Throughout the note, by an *algebra* we mean a finite-dimensional associative K-algebra with an identity over a fixed field K. An algebra A is called *basic* if A_A (equivalently, ${}_AA$) is a direct sum of pairwise non-isomorphic indecomposable modules. Recall that every algebra A is Morita equivalent to a basic algebra A^{b}, uniquely determined up to isomorphism by A (see [1], [4]). Moreover, an algebra A is called *self-injective* if the A-modules ${}_AA$ and A_A are injective.

In 1903 Frobenius studied algebras (over the field of complex numbers) for which the left and right regular representations are equivalent, and gave necessary and sufficient conditions for this equivalence in terms of "parastrophic matrices" depending on the algebra structure constants in a fixed linear basis (see [3, VII.2], [6], [7]). Later, Brauer and Nesbitt [2], [13] pointed out the importance of the algebras studied by Frobenius and named them *Frobenius algebras*. Moreover, they gave a criterion for an algebra to be a Frobenius algebra which was independent of the choice of a basis, and did not involve the parastrophic matrices: an algebra A is a Frobenius algebra if and only if there is a K-linear map $A \to K$ whose kernel contains no non-zero left (equivalently, right) ideals.

In [10]–[12], Nakayama established several important characterizations of Frobenius algebras. In particular, he proved that an algebra A is a Frobe-

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nius algebra if and only if there exists a non-degenerate K-bilinear form $(-,-) : A \times A \to K$ which is associative, in the sense that $(ab, c) = (a, bc)$ for all $a, b, c \in A$. Moreover, Nakayama proved that an algebra A is a Frobenius algebra if and only if the left (equivalently, right) A-modules A and $A^* = \operatorname{Hom}_K(A, K)$ are isomorphic. This immediately implies that all Frobenius algebras are self-injective, and all basic self-injective algebras are Frobenius algebras. Therefore, every self-injective algebra is Morita equivalent to a Frobenius algebra. We also note that the class of self-injective algebras coincides with the class of quasi-Frobenius algebras in the sense of [10], and is closed under Morita equivalences. On the other hand, as we will see later, the class of Frobenius algebras is not closed under Morita equivalences. Finally, recall that classical examples of Frobenius algebras are provided by the group algebras of finite groups, or more generally all finite-dimensional Hopf algebras [9].

In [10, p. 624] Nakayama gave an example of a 9-dimensional self-injective algebra which is not a Frobenius algebra. The aim of this note is to present a general form of all non-Frobenius self-injective algebras.

For basic background on representation theory we refer to [1] and [4], and on self-injective and Frobenius algebras to [5], [8], [14] and [15].

1. A construction. Let A be an algebra and

$$1 = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

be a decomposition of the identity of A into a sum of orthogonal primitive idempotents such that

$$\begin{aligned} &e_{ij}A \cong e_{ij'}A \quad && \text{for all } i \in \{1, \ldots, n_A\} \text{ and } j, j' \in \{1, \ldots, m_A(i)\}, \\ &e_{ij}A \ncong e_{i'j'}A \quad && \text{if } i \neq i'. \end{aligned}$$

Nakayama proved in [10] (see also [15, Theorem 2.2.1]) that A is a self-injective (quasi-Frobenius) algebra if and only if there is a permutation ν of $\{1, \ldots, n_A\}$, presently called the *Nakayama permutation* of A, such that $\operatorname{soc} e_{i1}A \cong \operatorname{top} e_{\nu(i)1}A$ for all $i \in \{1, \ldots, n_A\}$. We recall that A is defined to be *basic* if $m_A(i) = 1$ for all $i \in \{1, \ldots, n_A\}$.

Let Λ be a basic algebra, and

$$\Lambda = P_1 \oplus \cdots \oplus P_n$$

a decomposition of Λ into a direct sum of indecomposable right Λ-modules. Let $m(1), \ldots, m(n)$ be an arbitrary sequence of positive integers. For each $i \in \{1, \ldots, n\}$, let M_i be the direct sum of $m(i)$ copies P_{ij}, $1 \le j \le m(i)$, of P_i. Consider the right Λ-module $M = M_1 \oplus \cdots \oplus M_n$, and the endomorphism algebra

$$\Lambda(m(1),\dots,m(n)) = \mathrm{End}_\Lambda(M). \tag{1.1}$$

For $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m(i)\}$, we denote by e_{ij} the primitive idempotent of $\Lambda(m(1),\dots,m(n))$ given by the composition of the canonical projection $M \to P_{ij}$ and the canonical embedding $P_{ij} \to M$, and set $e_i = \sum_{j=1}^{m(i)} e_{ij}$. Observe that

$$\Lambda \cong e\Lambda(m(1),\dots,m(n))e,$$

where $e = \sum_{i=1}^{n} e_{i1}$. Clearly, for $m(1) = \dots = m(n) = 1$, we have $\Lambda(1,\dots,1) = \Lambda$. We also note that an arbitrary algebra A is isomorphic to an algebra of the form $\Lambda(m(1),\dots,m(n))$, where $\Lambda = A^{\mathrm{b}}$ is the basic algebra of A (see [1], [4]), and $n = n_A$, $m(1) = m_A(1),\dots,m(n) = m_A(n)$, in the notation above.

Lemma 1.2. *Let Λ be a basic self-injective algebra. Then an algebra $\Lambda(m(1),\dots,m(n))$ of the form* (1.1) *is a self-injective algebra and its Nakayama permutation ν coincides with the Nakayama permutation of Λ.*

Proof. This follows from the fact that Λ is self-injective and is the basic algebra of $\Lambda(m(1),\dots,m(n))$. ∎

The following theorem is the main result of this note.

Theorem 1.3. *Let Λ be a basic self-injective algebra. Then a self-injective algebra $\Lambda(m(1),\dots,m(n))$ of the form* (1.1) *is a Frobenius algebra if and only if $m(i) = m(\nu(i))$ for all $i \in \{1,\dots,n\}$.*

For the proof of the theorem, we abbreviate $A = \Lambda(m(1),\dots,m(n))$. We need the following lemma.

Lemma 1.4. *For any $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m(i)\}$, we have $\operatorname{soc} e_{ij}A \cong \operatorname{top} e_{\nu(i)j}A$.*

Proof. Fix $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m(i)\}$. We first claim that there exists an isomorphism

$$\operatorname{soc} e_{ij}A \cong \mathrm{Hom}_\Lambda(M, \operatorname{soc} P_{ij})$$

of right A-modules. Observe that we have isomorphisms of right A-modules $e_{ij}A \cong \mathrm{Hom}_\Lambda(M, P_{ij})$ and $\mathrm{Hom}_\Lambda(M, \operatorname{soc} P_{ij}) \cong \mathrm{Hom}_\Lambda(M, \operatorname{soc} P_{i1})$, because $P_{ij} = P_{i1} = P_i$ as Λ-modules. Obviously, $\mathrm{Hom}_\Lambda(M, \operatorname{soc} P_{ij})$ is a right A-submodule of $\mathrm{Hom}_\Lambda(M, P_{ij})$. Further, we know from Lemma 1.2 that A is a self-injective algebra having the same Nakayama permutation ν as Λ. Hence, the indecomposable projective right A-module $e_{ij}A$ is injective, and consequently $\operatorname{soc} e_{ij}A$ is a simple A-module. Finally, notice that $\mathrm{Hom}_\Lambda(M, \operatorname{soc} P_{i1}) \neq 0$, because $\operatorname{soc} P_{i1} \cong \operatorname{soc} P_i$ and $\mathrm{Hom}_\Lambda(P_{\nu(i)}, \operatorname{soc} P_i) \neq 0$ by the definition of ν. Therefore, in order to prove the claim above, it is enough to show that the right A-module $\mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1})$ is semisimple, or equivalently that $\mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1})(\mathrm{rad}\, A) = 0$. Let $f \in \mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1})$ and $g \in \mathrm{rad}\, A = \mathrm{Hom}_\Lambda(M, \mathrm{rad}\, M)$. Then we have

$$(fg)(M) \subseteq f(\mathrm{rad}\, M) = f(M)(\mathrm{rad}\, \Lambda) \subseteq (\mathrm{soc}\, P_{i1})(\mathrm{rad}\, \Lambda) = 0.$$

This implies $\mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1})(\mathrm{rad}\, A) = 0$, and hence the claim holds.

Therefore, we have an isomorphism of vector spaces

$$(\mathrm{soc}\, e_{ij} A) e_{\nu(i)1} \cong \mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1}) e_{\nu(i)1}$$

and the inclusion

$$0 \neq \mathrm{Hom}_\Lambda(P_{\nu(i)1}, \mathrm{soc}\, P_{i1}) \subseteq \mathrm{Hom}_\Lambda(M, \mathrm{soc}\, P_{i1}) e_{\nu(i)1}.$$

Since $\mathrm{top}\, e_{\nu(i)1} A \cong \mathrm{top}\, e_{\nu(i)j} A$, we get a required isomorphism $\mathrm{soc}\, e_{ij} A \cong \mathrm{top}\, e_{\nu(i)j} A$ of right A-modules. ■

Proof of Theorem 1.3. From the definition of $A = \Lambda(m(1), \ldots, m(n))$ it follows that $\mathrm{top}\, A_A = T_1 \oplus \cdots \oplus T_n$ and $\mathrm{soc}\, A_A = S_1 \oplus \cdots \oplus S_n$, where

$$T_i = \bigoplus_{j=1}^{m(i)} \mathrm{top}\, e_{ij} A, \qquad S_i = \bigoplus_{j=1}^{m(i)} \mathrm{soc}\, e_{ij} A$$

for $i \in \{1, \ldots, n\}$.

Assume now that $m(i) = m(\nu(i))$ for all $i \in \{1, \ldots, n\}$. Applying Lemma 1.4, we conclude that we have isomorphisms of right A-modules

$$\mathrm{soc}\, e_{ij} A \cong \mathrm{top}\, e_{\nu(i)j} A$$

for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m(i)\} = \{1, \ldots, m(\nu(i))\}$. Comparing the numbers of simple summands of S_i and $T_{\nu(i)}$, we then see that the right A-modules S_i and $T_{\nu(i)}$ are isomorphic, for all $i \in \{1, \ldots, n\}$. We also note that, for any primitive idempotent e_{ij}, the right A-module $\mathrm{Hom}_K(Ae_{ij}, K) = e_{ij} \mathrm{Hom}_K(A, K)$ is an indecomposable projective-injective module whose socle is isomorphic to $\mathrm{top}\, e_{ij} A$. Therefore, we conclude that the right A-modules A and $A^* = \mathrm{Hom}_K(A, K)$ are isomorphic, and so A is a Frobenius algebra, by Nakayama's theorem.

Conversely, assume that A is a Frobenius algebra. Then, applying Nakayama's theorem again, we have an isomorphism of right A-modules A and $A^* = \mathrm{Hom}_K(A, K)$, and consequently A-module isomorphisms $S_i \cong T_{\nu(i)}$ for all $i \in \{1, \ldots, n\}$. This obviously implies $m(i) = m(\nu(i))$ for all $i \in \{1, \ldots, n\}$. ■

Recall that a self-injective algebra A is called *weakly symmetric* if $\mathrm{soc}\, P \cong \mathrm{top}\, P$ for any indecomposable projective right (equivalently, left) A-module P. As a direct consequence of Theorem 1.3 one obtains the following fact.

COROLLARY 1.5. *Let A be a weakly symmetric algebra. Then A is a Frobenius algebra.*

2. Self-injective algebras that are not Frobenius. We now exhibit a general form of non-Frobenius self-injective algebras.

THEOREM 2.1. *Let A be a self-injective algebra. Then the following statements are equivalent*:

(i) *A is not Frobenius.*
(ii) *There exist $i, j \in \{1, \dots, n_A\}$ such that $m_A(i) \neq m_A(j)$ and* $\operatorname{soc} e_{i1}A \cong \operatorname{top} e_{j1}A$.
(iii) *$A \cong \Lambda(m(1), \dots, m(n))$ for a basic self-injective algebra Λ with $n = n_A = n_\Lambda$ and a sequence $m(1), \dots, m(n)$ of positive integers with $m(i) \neq m(j)$ and* $\operatorname{soc} e_{i1}\Lambda \cong \operatorname{top} e_{j1}\Lambda$ *for some $i, j \in \{1, \dots, n\}$.*

Proof. We know that $A \cong \Lambda(m(1), \dots, m(n))$, where $\Lambda = A^{\mathrm{b}}$ is the basic algebra of A and $n = n_A$, $m(1) = m_A(1), \dots, m(n) = m_A(n)$, in the notation of Section 1. Moreover, Λ is a self-injective algebra whose Nakayama permutation coincides with the Nakayama permutation of A (see Lemma 1.2). Then the required equivalence of (i), (ii) and (iii) follows from Theorem 1.3. ∎

Theorem 2.1 allows us to construct arbitrarily large connected non-Frobenius self-injective algebras from basic, connected, self-injective but not weakly symmetric algebras. For example, we have the following fact.

COROLLARY 2.2. *Let Λ be a basic, connected, self-injective but not weakly symmetric algebra, and $\Lambda = P_1 \oplus \dots \oplus P_n$ a decomposition of Λ into a direct sum of indecomposable right Λ-modules. Take any $i \in \{1, \dots, n\}$ with* $\operatorname{soc} P_i \not\cong \operatorname{top} P_i$, *an integer $r \geq 2$, and the sequence $m(1), \dots, m(n)$ with $m(j) = 1$ for $j \neq i$ and $m(i) = r$. Then $\Lambda(m(1), \dots, m(n))$ is a non-Frobenius self-injective algebra.*

Proof. Since
$$\operatorname{top} P_{\nu(i)} \cong \operatorname{soc} P_i \not\cong \operatorname{top} P_i,$$
we have $\nu(i) \neq i$. Hence, $m(\nu(i)) = 1 \neq r = m(i)$. Then, applying Lemma 1.2 and Theorem 2.1, we conclude that $\Lambda(m(1), \dots, m(n))$ is a non-Frobenius self-injective algebra. ∎

3. Nakayama's example. We will show that the non-Frobenius self-injective algebra given by Nakayama in [10, p. 624] is isomorphic to an algebra of the form $\Lambda(2,1)$ for a basic, connected, self-injective algebra Λ.

Let Λ be the bound quiver algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and the relations $\alpha\beta = 0$ and $\beta\alpha = 0$ (compare [4, p. 172, Exercise 4]). Then Λ is a basic Nakayama self-injective algebra with $\operatorname{rad}^2 \Lambda = 0$. Moreover,

$$\Lambda = P_1 \oplus P_2$$

where P_1 and P_2 are the indecomposable right projective Λ-modules corresponding to the vertices 1 and 2, respectively. Take

$$M = P_{10} \oplus P_{11} \oplus P_2$$

where $P_{11} = P_1$ and P_{10} is another copy of P_1. Then $m(1) = 2$ and $m(2) = 1$, and $A = \Lambda(2,1)$ is the endomorphism algebra $\mathrm{End}_\Lambda(M)$. Let e_0, e_1, e_2 be the primitive idempotents of A corresponding to the direct summands P_{10}, P_{11}, P_2 of M. Further, denote by ${}_1u_0$ the identity map from P_{10} to P_{11}, and by ${}_0u_1$ the identity map from P_{11} to P_{10}. Finally, let ${}_2\alpha_0 : P_{10} \to P_2$, ${}_2\alpha_1 : P_{11} \to P_2$, and ${}_0\beta_2 : P_2 \to P_{10}$, ${}_1\beta_2 : P_2 \to P_{11}$ be the maps given by the left multiplications by α and β, respectively. Then we have in A the equalities

$$\begin{aligned} e_0 &= {}_0u_1 \cdot {}_1u_0, & e_1 &= {}_1u_0 \cdot {}_0u_1, \\ {}_2\alpha_0 &= e_2 \cdot {}_2\alpha_0 \cdot e_0, & {}_2\alpha_1 &= e_2 \cdot {}_2\alpha_1 \cdot e_1, \\ {}_0\beta_2 &= e_0 \cdot {}_0\beta_2 \cdot e_2, & {}_1\beta_2 &= e_1 \cdot {}_1\beta_2 \cdot e_2. \end{aligned}$$

Then A is isomorphic to the matrix algebra given by the matrices of the form

$$\left[\begin{array}{ccc|ccc} a_0 & {}_0b_1 & {}_0\mu_1 & & & \\ {}_1b_0 & a_1 & {}_1\mu_2 & & 0 & \\ 0 & 0 & a_2 & & & \\ \hline & & & a_2 & {}_2\lambda_0 & {}_2\lambda_1 \\ & 0 & & 0 & a_0 & {}_0b_1 \\ & & & 0 & {}_1b_0 & a_1 \end{array}\right]$$

where $a_0 \in Ke_0$, $a_1 \in Ke_1$, $a_2 \in Ke_2$, ${}_0b_1 \in K{}_0u_1$, ${}_1b_0 \in K{}_1u_0$, ${}_0\mu_2 \in K{}_0\beta_2$, ${}_1\mu_2 \in K{}_1\beta_2$, ${}_2\lambda_0 \in K{}_2\alpha_0$, ${}_2\lambda_1 \in K{}_2\alpha_1$, which is exactly the algebra presented by Nakayama in [10, p. 624].

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