COLLOQUIUM MATHEMATICUM

VOL. 105 2006 NO. 1

ISOPARAMETRIC HYPERSURFACES WITH LESS THAN FOUR PRINCIPAL CURVATURES IN A SPHERE

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Dedicated to Professor Hajime Urakawa on his sixtieth birthday

Abstract. We characterize Clifford hypersurfaces and Cartan minimal hypersurfaces in a sphere by some properties of extrinsic shapes of their geodesics.

1. Introduction. In some cases it is possible to determine the shape of a Riemannian submanifold by observing extrinsic shapes of geodesics of the submanifold in an ambient Riemannian manifold. For example, a hypersurface M^n isometrically immersed into a standard sphere S^{n+1} is totally umbilic in S^{n+1} if and only if every geodesic of M is a circle in S^{n+1}. Here, a smooth curve γ parameterized by its arclength on S^{n+1} is called a *circle of curvature* κ (≥ 0) if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma$, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection of S^{n+1}. It is well known that a circle of constant curvature κ on S^{n+1} is a great circle or a small circle according as κ is zero or positive. The differential equation for a circle γ is equivalent to the differential equations $\nabla_{\dot\gamma}\dot\gamma = \kappa Y$, $\nabla_{\dot\gamma}Y = -\kappa\dot\gamma$ with a field of unit vectors Y along γ.

A hypersurface M in S^{n+1} is called *isoparametric* if all of its principal curvatures in S^{n+1} are constant. The isoparametric hypersurfaces are a quite interesting object of study in differential geometry. Totally umbilic hypersurfaces are the simplest examples of isoparametric hypersurfaces. In his papers [C1, C2], É. Cartan extensively studied isoparametric hypersurfaces in a standard sphere, and completely classified them in the case they have less than four principal curvatures. But the classification problem for all isoparametric hypersurfaces in a sphere is still open (see Problem 34 in [Y]).

2000 *Mathematics Subject Classification*: Primary 53C35, 53C20; Secondary 53C22.

Key words and phrases: isoparametric hypersurfaces, Clifford hypersurfaces, Cartan minimal hypersurfaces, spheres, geodesics, circles, principal curvatures.

The first author is partially supported by Grant-in-Aid for Scientific Research (C) (no. 14540075), Ministry of Education, Science, Sports, Culture and Technology.

In this paper, by studying extrinsic shapes of geodesics on hypersurfaces in the ambient space S^{n+1}, we characterize isoparametric hypersurfaces with two principal curvatures and isoparametric *minimal* hypersurfaces with three principal curvatures.

2. Isoparametric hypersurfaces with two or three principal curvatures. We start by studying extrinsic shapes of geodesics on isoparametric hypersurfaces in a standard sphere with less than four principal curvatures. Let M be a hypersurface of a standard sphere $S^{n+1}(c)$ of curvature c through an isometric immersion and $\mathcal{N}$ a unit normal vector field on M. The Riemannian connections $\widetilde{\nabla}$ of $S^{n+1}(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten: For vector fields X and Y tangent to M we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N}, \qquad \widetilde{\nabla}_X \mathcal{N} = -AX,$$

where $\langle\ ,\ \rangle$ denotes the Riemannian metric on M induced from the standard metric $\langle\ ,\ \rangle$ on $S^{n+1}(c)$, and $A : TM \to TM$ is the shape operator of M in $S^{n+1}(c)$. An eigenvector and an eigenvalue of the shape operator A are called a *principal curvature vector* and a *principal curvature*, respectively.

For a curve γ on a hypersurface M we can consider γ as a curve on $S^{n+1}(c)$. In order to distinguish them we call the latter curve the *extrinsic shape* of γ. When γ is a geodesic we see by the Gauss formula that its extrinsic shape satisfies $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N}$. Thus we find the following:

LEMMA.

(1) *The extrinsic shape of a geodesic γ on a hypersurface M is a geodesic if and only if $\langle A\dot\gamma, \dot\gamma\rangle \equiv 0$.*
(2) *The extrinsic shape of a geodesic γ on a hypersurface M is a circle of positive curvature if and only if $\dot\gamma$ is principal and $\langle A\dot\gamma, \dot\gamma\rangle$ is a nonzero constant function.*

When M is an isoparametric hypersurface in $S^{n+1}(c)$, it is well known that each distribution V_λ of eigenspaces is integrable, and each of its leaves is totally geodesic in the hypersurface M and totally umbilic in the ambient space $S^{n+1}(c)$ (see [CR]). Thus every geodesic on such leaves is a geodesic as a curve on M and is a circle as a curve on $S^{n+1}(c)$. Its curvature $\langle A\dot\gamma(0), \dot\gamma(0)\rangle$ is a principal curvature. Thus we have

PROPOSITION 1. *Let γ be a geodesic on an isoparametric hypersurface in a standard sphere. If the initial vector is principal with principal curvature λ, then the extrinsic shape of γ is a circle. Its curvature as a circle is λ.*

Isoparametric hypersurfaces in $S^{n+1}(c)$ with two constant principal curvatures are called *Clifford hypersurfaces*. For a pair (c_1, c_2) of positive numbers satisfying $1/c_1+1/c_2=1/c$ and a positive integer r with $1\le r\le n-1$, we denote by $M_{r,n-r}=M_{r,n-r}(c_1,c_2)$ a naturally embedded hypersurface in $S^{n+1}(c)$ which is isometric to $S^r(c_1)\times S^{n-r}(c_2)$. It has two constant principal curvatures $\lambda_1=c_1/\sqrt{c_1+c_2}$ and $\lambda_2=-c_2/\sqrt{c_1+c_2}$, whose multiplicities are r and $n-r$, respectively. A Clifford hypersurface $M_{r,n-r}(c_1,c_2)$ is minimal in $S^{n+1}(c)$ if and only if $c_1=nc/r$ and $c_2=nc/(n-r)$. Let $TM_{r,n-r}=V_{\lambda_1}\oplus V_{\lambda_2}$ be the decomposition into distributions of eigenspaces corresponding to eigenvalues λ_1,λ_2.

PROPOSITION 2. *Let γ be a geodesic on $M_{r,n-r}(c_1,c_2)$.*

(1) *The extrinsic shape of γ is a geodesic if and only if the initial vector is of the form $\dot\gamma(0)=(\sqrt{c_2}w_1+\sqrt{c_1}w_2)/\sqrt{c_1+c_2}$ with $w_i\in V_{\lambda_i}$ $(i=1,2)$.*
(2) *If the initial vector is neither principal nor of the form in* (1), *then the extrinsic shape is not a circle.*

Proof. Since $M_{r,n-r}$ has parallel shape operator, we find

$$\frac{d}{ds}\langle A\dot\gamma(s),\dot\gamma(s)\rangle=\langle(\nabla_{\dot\gamma}A)\dot\gamma(s),\dot\gamma(s)\rangle=0.$$

Thus we may study geodesics at their initial point. We denote the initial vector by $\dot\gamma(0)=a_1w_1+a_2w_2$ with unit vectors $w_i\in V_{\lambda_i}$ $(i=1,2)$ and nonnegative constants a_1,a_2 satisfying $a_1^2+a_2^2=1$. In this case we have $\langle A\dot\gamma(0),\dot\gamma(0)\rangle=a_1^2\lambda_1+a_2^2\lambda_2$. We hence obtain $\langle A\dot\gamma,\dot\gamma\rangle\equiv 0$ if and only if $a_1=\sqrt{c_2}/\sqrt{c_1+c_2}$ and $a_2=\sqrt{c_1}/\sqrt{c_1+c_2}$, and get the conclusion. ∎

Isoparametric hypersurfaces with three constant principal curvatures are usually called *Cartan hypersurfaces*. If we denote by m_i the multiplicity of a principal curvature λ_i, it is known that these three principal curvatures have the same multiplicity (i.e. $m_1=m_2=m_3$). When a Cartan hypersurface is minimal, it is congruent to one of the following hypersurfaces:

$$\begin{aligned}M^3&=\mathrm{SO}(3)/(\mathbb{Z}_2+\mathbb{Z}_2)\to S^4(c),\\ M^6&=\mathrm{SU}(3)/T^2\to S^7(c),\\ M^{12}&=\mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)\to S^{13}(c),\\ M^{24}&=F_4/\mathrm{Spin}(8)\to S^{25}(c).\end{aligned}$$

Principal curvatures of a Cartan minimal hypersurface are $\sqrt{3c},0,-\sqrt{3c}$.

3. Characterizations of Clifford hypersurfaces and Cartan minimal hypersurfaces. In this section we characterize Clifford hypersurfaces and Cartan minimal hypersurfaces by extrinsic shapes of their geodesics.

THEOREM 1. *A connected hypersurface M^n in $S^{n+1}(c)$ is locally congruent to a Clifford hypersurface $M_{r,n-r}$ with some r if and only if there are a function $d : M \to \mathbb{N}$, a constant α $(0 < \alpha < 1)$ and an orthonormal basis $\{v_1, \dots, v_n\}$ of T_xM at each point $x \in M$ satisfying the following two conditions*:

(i) *All geodesics on M with initial vector v_i $(1 \le i \le n)$ are small circles in $S^{n+1}(c)$.*
(ii) *All geodesics γ_{ij} on M with initial vector $\alpha v_i + \sqrt{1-\alpha^2}\, v_j$ $(1 \le i \le d_x < j \le n)$ are great circles in $S^{n+1}(c)$.*

In this case d is a constant function with $d \equiv r$ and

$$M = M_{r,n-r}(c/\alpha^2, c/(1-\alpha^2)).$$

Proof. $(\Rightarrow)$ For a Clifford hypersurface $M_{r,n-r}$ we decompose its tangent bundle $TM_{r,n-r}$ into subbundles of principal vectors $V_{\lambda_1} \oplus V_{\lambda_2}$. If we take an orthonormal basis $\{v_1, \dots, v_n\}$ of $T_xM_{r,n-r}$ at each point $x \in M_{r,n-r}$ in such a way that $\{v_1, \dots, v_r\}$ is an orthonormal basis of V_{λ_1} and $\{v_{r+1}, \dots, v_n\}$ is an orthonormal basis of V_{λ_2}, we find by Proposition 2 that they satisfy the required conditions.

$(\Leftarrow)$ Consider an open dense subset

$$\mathcal{U} = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{the multiplicity of each principal curvature of } M \text{ in} \\ S^{n+1}(c) \text{ is constant on some neighborhood } U_x \text{ of } x \end{array} \right\}$$

of M. Our discussion below owes much to [KM]. For an orthonormal basis $\{v_1, \dots, v_n\}$ of T_xM at $x \in \mathcal{U}$ which satisfies the conditions, we take geodesics γ_i $(1 \le i \le n)$ on M with initial vector v_i. Since the extrinsic shape of γ_i is a circle of positive curvature, if we denote its curvature by κ_i, then we find by the formulas of Gauss and Weingarten that

$$-\kappa_i^2 \dot\gamma_i = \widetilde\nabla_{\dot\gamma_i} \widetilde\nabla_{\dot\gamma_i} \dot\gamma_i = -\langle A\dot\gamma_i, \dot\gamma_i \rangle A\dot\gamma_i + \langle (\nabla_{\dot\gamma_i} A)\dot\gamma_i, \dot\gamma_i \rangle \mathcal{N}.$$

Comparing the tangential components of the left-hand and right-hand sides of this equality, we obtain $\langle A\dot\gamma_i, \dot\gamma_i \rangle A\dot\gamma_i = \kappa_i^2 \dot\gamma_i$, so that $\langle Av_i, v_i \rangle Av_i = \kappa_i^2 v_i$ at the point x. Hence we have $Av_i = \kappa_i v_i$ or $Av_i = -\kappa_i v_i$ for $1 \le i \le n$, which means that the tangent space T_xM decomposes as

$$\begin{aligned} T_xM = &\{v \in T_xM \mid Av = -k_1 v\} \oplus \{v \in T_xM \mid Av = k_1 v\} \\ &\oplus \cdots \oplus \{v \in T_xM \mid Av = -k_g v\} \oplus \{v \in T_xM \mid Av = k_g v\}, \end{aligned}$$

where $0 < k_1 < \ldots < k_g$ and g is the number of distinct positive κ_i $(i = 1, \dots, n)$. We decompose T_xM in that way at each point $x \in \mathcal{U}$. Then each k_j turns out to be a smooth function on U_x for each $x \in \mathcal{U}$.

We shall show k_j is locally constant. We consider an arbitrary point $y \in U_x$. Let $\{v_1, \dots, v_n\}$ be the orthonormal basis of T_yM satisfying (i). If k_j is the curvature of the extrinsic shape of geodesic with initial vector v_{i_j}, we find by (i) that $v_{i_j} k_j = 0$. In order to study $v_l k_j$ for other v_l, we extend $\{v_1, \ldots, v_n\}$ to principal curvature unit vector fields $\{V_1, \ldots, V_n\}$ on some neighborhood W_y ($\subseteq U_x$) satisfying $\nabla_{V_{i_j}} V_{i_j}(y) = 0$ and $(V_{i_j})_y = v_{i_j}$ (for details, see p. 76 in [KM]). For simplicity, we only treat the case $Av_{i_j} = k_j v_{i_j}$. Thanks to the Codazzi equation $\langle (\nabla_X A)Y, Z\rangle = \langle (\nabla_Y A)X, Z\rangle$, we find

$$\begin{aligned}
\langle (\nabla_{v_{i_j}} A)v_l, v_{i_j}\rangle &= \langle (\nabla_{v_l} A)v_{i_j}, v_{i_j}\rangle = \langle (\nabla_{V_l} A)V_{i_j}, V_{i_j}\rangle(y) \\
&= \langle \nabla_{V_l}(k_j V_{i_j}) - A\nabla_{V_l} V_{i_j}, V_{i_j}\rangle(y) \\
&= \langle (V_l k_j)V_{i_j} + (k_j I - A)\nabla_{V_l} V_{i_j}, V_{i_j}(y)\rangle = v_l k_j, \\
\langle v_l, (\nabla_{v_{i_j}} A)v_{i_j}\rangle &= \langle V_l, (\nabla_{V_{i_j}} A)V_{i_j}\rangle(y) \\
&= \langle V_l, \nabla_{V_{i_j}}(k_j V_{i_j}) - A\nabla_{V_{i_j}} V_{i_j}\rangle(y) \\
&= \langle v_l, (v_{i_j} k_j)v_{i_j}\rangle = 0.
\end{aligned}$$

Since A is symmetric, we see that $\langle (\nabla_{v_{i_j}} A)v_l, v_{i_j}\rangle = \langle v_l, (\nabla_{v_{i_j}} A)v_{i_j}\rangle$, hence $v_l k_j = 0$. Thus the differential of k_j vanishes at y and k_j is constant on U_x. Hence every principal curvature of M is locally constant on the open dense subset $\mathcal{U}$ of the connected hypersurface M. This, together with the fact that all principal curvatures are continuous functions on M, shows that every hypersurface satisfying (i) is isoparametric in the ambient space $S^{n+1}(c)$.

Consider a fixed point x_0. The above argument shows that every v_i is principal. If we denote its principal curvature by λ_i, then by (ii) we have $\alpha\lambda_i + \sqrt{1-\alpha^2}\,\lambda_j = 0$ for $1 \le i \le d_{x_0} < j \le n$. Hence M has just two distinct principal curvatures, and we obtain our result. ∎

REMARK. In Theorem 1, we only need the second condition at some point $x \in M$.

THEOREM 2. *Let M^n be a connected hypersurface of $S^{n+1}(c)$. Suppose that at each point x in M there exists an orthonormal basis $\{v_1, \ldots, v_m\}$ of the orthogonal complement of* $\ker A$ *in* T_xM ($m = \operatorname{rank} A$) *such that*

(i) *all geodesics with initial vector v_i ($1 \le i \le m$) are small circles in $S^{n+1}(c)$,*
(ii) *they have the same curvature κ_x.*

Then M^n is locally congruent either to a totally umbilic hypersphere, a Clifford hypersurface $M_{r,n-r}(2c, 2c)$, $1 \le r \le n-1$, or a Cartan minimal hypersurface.

Proof. A totally geodesic hypersphere satisfies the conditions trivially. By the discussion in the proof of Theorem 1, a hypersurface M^n satisfying the hypothesis has at most three distinct constant principal curvatures $\kappa, -\kappa, 0$. This yields the result. ∎

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Received 13 July 2004;
revised 30 September 2005 (4474)