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GROUPS WITH NEARLY MODULAR SUBGROUP LATTICE

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Abstract. A subgroup H of a group G is *nearly normal* if it has finite index in its normal closure H^G. A relevant theorem of B. H. Neumann states that groups in which every subgroup is nearly normal are precisely those with finite commutator subgroup. We shall say that a subgroup H of a group G is *nearly modular* if H has finite index in a modular element of the lattice of subgroups of G. Thus nearly modular subgroups are the natural lattice-theoretic translation of nearly normal subgroups. In this article we study the structure of groups in which all subgroups are nearly modular, proving in particular that a locally graded group with this property has locally finite commutator subgroup.

1. Introduction. A subgroup of a group G is called *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G. It is clear that every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal; thus modularity may be considered as a lattice generalization of normality. Lattices in which all elements are modular are also called *modular*. The structure of groups with modular subgroup lattice has been described by K. Iwasawa [3], [4] and R. Schmidt [9]; in particular, it turns out that non-periodic groups with this property are metabelian. For a detailed account of results concerning modular subgroups of groups, we refer the reader to [10].

A subgroup H of a group G is said to be *nearly normal* if it has finite index in its normal closure H^G. A relevant theorem of B. H. Neumann [7] states that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. If φ is a *projectivity* from a group G onto a group $\overline{G}$ (i.e. an isomorphism from the lattice $\mathfrak{L}(G)$ onto the subgroup lattice of $\overline{G}$), and N is a normal subgroup of G, then the image N^φ of N is a modular element of the lattice $\mathfrak{L}(\overline{G})$. Furthermore, if H and K are subgroups of G such that $H \leq K$ and the index $|K : H|$ is finite, then H^φ has finite index in K^φ (see [10], Theorem 6.1.7). Thus the image of any nearly normal subgroup of G has finite index in a modular subgroup of $\overline{G}$.

We shall say that a subgroup H of a group G is *nearly modular* if it has finite index in a modular subgroup of G. The definition of nearly modular

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element can be given in an arbitrary lattice, and a lattice $\mathfrak{L}$ will be called *nearly modular* if all its elements are nearly modular. Thus every projective image of a group whose subgroups are nearly normal (i.e. of a finite-by-abelian group) is a group with nearly modular subgroup lattice.

The aim of this article is a first study of groups in which all subgroups are nearly modular. It will be proved that the commutator subgroup of a locally graded group with this property is periodic, and that periodic locally graded groups with nearly modular subgroup lattice are locally finite; in particular every torsion-free locally graded group whose subgroups are nearly modular is abelian. It follows that non-periodic locally graded groups with nearly modular subgroup lattice are finite-by-metabelian. Here a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index; it is clear that the class of locally graded groups contains all residually finite and all locally (soluble-by-finite) groups.

In our results the assumption that the group is locally graded cannot be omitted. In fact, there exists a torsion-free group $G = \langle a, b\rangle$ such that $Z(G) = \langle a\rangle \cap \langle b\rangle$ is infinite cyclic and $G/Z(G)$ is a Tarski group (see [1], proof of Theorem 2); then every non-trivial subgroup X of G has finite index in the modular subgroup $XZ(G)$, and hence the subgroup lattice $\mathfrak{L}(G)$ is nearly modular.

Most of our notation is standard and can be found in [8]. Moreover, we shall use the monograph [10] as a general reference for results on subgroup lattices.

2. Statements and proofs. Let $\mathfrak{L}$ be a lattice with least element 0 and greatest element I. Recall that an element x of $\mathfrak{L}$ is *irreducibly covered* by elements $x_1, \dots, x_m$ of the interval $[x/0]$ if, for each element y of $[x/0]$ such that $[y/0]$ is a distributive lattice with the maximal condition, there is $i \leq m$ such that $y \leq x_i$, and the set $\{x_1, \dots, x_m\}$ is minimal with respect to this property. Clearly a subgroup H of a group G is irreducibly covered in the lattice $\mathfrak{L}(G)$ by its subgroups $H_1, \dots, H_m$ if and only if H is the set-theoretic union of $H_1, \dots, H_m$ and none of these subgroups can be omitted from the covering.

An element h of the lattice $\mathfrak{L}$ is said to be *cofinite* if there exists a finite chain in $\mathfrak{L}$,

$$h = h_0 < h_1 < \dots < h_t = I,$$

such that, for every $i = 0, 1, \dots, t-1$, h_i is a maximal element of the lattice $[h_{i+1}/0]$ and satisfies one of the following conditions:

- h_{i+1} is irreducibly covered by finitely many elements $k_1, \dots, k_{n_i}$ of $\mathfrak{L}$ such that $k_1 \wedge \dots \wedge k_{n_i} \leq h_i$,

• for every automorphism φ of the lattice $[h_{i+1}/0]$, the element $h_i \wedge h_i^{\varphi}$ is modular in $[h_{i+1}/0]$ and the lattice $[h_{i+1}/h_i \wedge h_i^{\varphi}]$ is finite.

We shall say that an element a of $\mathfrak{L}$ is *nearly modular* if there exists a modular element h of $\mathfrak{L}$ such that $a \leq h$ and a is a cofinite element of the lattice $[h/0]$. The lattice $\mathfrak{L}$ is called *nearly modular* if all its elements are nearly modular.

A theorem of R. Schmidt yields that a subgroup H of a group G is cofinite in the lattice $\mathfrak{L}(G)$ if and only if H has finite index in G (see [10], Theorem 6.1.10). Therefore, a subgroup X of G is nearly modular if and only if it is a nearly modular element of the lattice $\mathfrak{L}(G)$, and hence the subject of this article is the structure of groups with nearly modular subgroup lattice.

PROPOSITION 1. *Let G be a group whose cyclic subgroups are nearly modular. Then the set of all elements of finite order of G is a subgroup.*

Proof. Assume by contradiction that G contains two elements x and y of finite order such that xy has infinite order. Let X be a modular subgroup of G such that x belongs to X and the index $|X : \langle x\rangle|$ is finite. Then X is normal in the subgroup $\langle X, xy\rangle = \langle X, y\rangle$ (see [11], Corollary 2.2), and hence $\langle X, y\rangle$ is finite. This contradiction shows that the elements of finite order of G form a subgroup. ∎

LEMMA 2. *Let G be a periodic group with nearly modular subgroup lattice. If G is not locally finite, there exists a subgroup H of G satisfying the following conditions*:

(i) *$H = \langle M, X\rangle$, where M and X are modular subgroups of H*;
(ii) *X is finite and M is a maximal and locally finite subgroup of H*;
(iii) *the factor group H/M_H is a Tarski group.*

Proof. Let M be a maximal locally finite subgroup of G. If L is any subgroup of G properly containing M, then the index $|L : M|$ is infinite; since all subgroups of G are nearly modular, it follows that M is a modular subgroup of G. As M is a proper subgroup of G, there exists an element x of $G \setminus M$ such that x^p belongs to M for some prime number p. Then M is a maximal subgroup of $H = \langle M, x\rangle$, and H contains a finite modular subgroup X such that x belongs to X. Therefore $H = \langle M, X\rangle$ and the index $|H : M|$ is infinite, so that the factor group H/M_H is a Tarski group (see [11], Theorem B). ∎

LEMMA 3. *Let $G = \langle y, X\rangle$ be a residually finite p-group, where p is a prime number. If X is a finite modular subgroup of G, then G is finite.*

Proof. Let $\overline{G}$ be any finite homomorphic image of G. Then $\overline{X}$ is a permutable subgroup of $\overline{G}$, and hence $\overline{G} = \langle \overline{y}, \overline{X}\rangle$ has order at most $|\langle y\rangle| \cdot |X|$.

In particular, the group G has finite upper rank, and so it is finite (see [6], Theorem A). ■

LEMMA 4. *Let G be a residually finite p-group, where p is a prime number. If the lattice $\mathfrak{L}(G)$ is nearly modular, then G is locally finite.*

Proof. Assume by contradiction that G is not locally finite, so that by Lemma 2 it contains a subgroup $H = \langle M, X\rangle$, where M is locally finite, X is a finite modular subgroup of H and H/M_H is a Tarski group. Then $H = M_H\langle y, X\rangle$ for some element y of H, and the subgroup $\langle y, X\rangle$ is finite by Lemma 3. This contradiction proves that G is a locally finite group. ■

A normal subgroup N of a group G is said to be *hypercyclically embedded* in G if it has an ascending series with cyclic factors consisting of normal subgroups of G. It has been proved by R. Schmidt that, if H is a modular subgroup of a finite group G, then H^G/H_G is hypercyclically embedded in G/H_G (see [10], Theorem 5.2.5).

THEOREM 5. *Let G be a periodic locally graded group with nearly modular subgroup lattice. Then G is locally finite.*

Proof. Assume by contradiction that G is not locally finite, so that it contains a finitely generated infinite subgroup E. Since G is locally graded, there exists a normal subgroup J of E such that E/J is an infinite residually finite group. Replacing G by E/J, we may suppose without loss of generality that G is residually finite. By Lemma 2 the group G contains a subgroup $H = \langle M, X\rangle$, where M is locally finite, X is a finite modular subgroup of H and H/M_H is a Tarski group. Then there exists an element y of H such that $H = M_H\langle y, X\rangle$. Put $K = \langle y, X\rangle$, and let N be a normal subgroup of K such that K/N is finite and $N \cap X = \{1\}$. Let $\mathcal{L}$ be the residual system of K consisting of all normal subgroups of finite index of K which are contained in N. Consider any element L of $\mathcal{L}$, and write $\overline{K} = K/L$. As $\overline{X}$ is a modular subgroup of the finite group $\overline{K}$, we find that $\overline{X}^{\overline{K}}/\overline{X}_{\overline{K}}$ is hypercyclically embedded in $\overline{K}/\overline{X}_{\overline{K}}$. Thus $\overline{K}/\overline{X}_{\overline{K}}$ is supersoluble, and hence its commutator subgroup $\overline{K}'\overline{X}_{\overline{K}}/\overline{X}_{\overline{K}}$ is nilpotent. If a and b are elements of K' with coprime orders, then $[\overline{a}, \overline{b}]$ belongs to $\overline{X}_{\overline{K}}$, and so $[a, b]$ is an element of XL. Since

$$\bigcap_{L\in\mathcal{L}} XL = X,$$

it follows that X contains the normal subgroup V of K generated by all commutators $[a, b]$, where a and b are elements of K' with coprime orders. Therefore V is contained in X_K, and the residually finite group $K'X_K/X_K$ is residually nilpotent, so that it is the direct product of its Sylow subgroups (see [8], Part 2, Corollary to Theorem 6.14). On the other hand, every Sylow subgroup of $K'X_K/X_K$ is locally finite by Lemma 4, and hence K' is locally finite. Therefore K itself is locally finite, and so even finite. This contradiction proves the theorem. ■

Let G be a non-periodic group whose subgroup lattice is modular, and let T be the subgroup of all elements of finite order of G. It was proved by Iwasawa that all subgroups of T are normal in G, and either G is abelian or the factor group G/T is torsion-free abelian with Prüfer rank 1. We will obtain corresponding results for non-periodic groups with nearly modular subgroup lattice.

LEMMA 6. *Let G be a group whose cyclic subgroups are nearly modular, and let x and y be elements of infinite order of G such that $\langle x\rangle \cap \langle y\rangle = \{1\}$. If $\langle x\rangle$ is a modular subgroup of $\langle x, y\rangle$, then $xy = yx$.*

Proof. Since $\langle x\rangle$ is a modular subgroup of $\langle x, y\rangle$, it is even normal in $\langle x, y\rangle$ (see [11], Theorem 1.3). Moreover, there exists a modular subgroup Y of G such that y belongs to Y and the index $|Y : \langle y\rangle|$ is finite. It follows that $Y \cap \langle x\rangle = \{1\}$, and hence Y is normal in the subgroup $\langle Y, x\rangle$ (see [11], Corollary 2.2). Therefore $\langle y\rangle = Y \cap \langle x, y\rangle$ is a normal subgroup of $\langle x, y\rangle$, and so $xy = yx$. ■

LEMMA 7. *Let $G = \langle x, y\rangle$ be a locally graded group with nearly modular subgroup lattice. If $\langle x\rangle$ is a modular subgroup of G, then G is supersoluble.*

Proof. The set T consisting of all elements of finite order of G is a subgroup by Proposition 1. In particular, if x and y have finite order, G is periodic, and hence even finite by Theorem 5, so that in this case G is supersoluble (see [10], Theorem 5.2.5). Assume now that one of the elements x and y has finite order and the other has infinite order. As $\langle x\rangle$ is modular in G, we have either $T = \langle x\rangle$ or $T = \langle y\rangle$, and hence G is supersoluble. Suppose finally that both x and y have infinite order. If $\langle x\rangle \cap \langle y\rangle = \{1\}$, the group G is abelian by Lemma 6. Thus it can be assumed that $\langle x\rangle \cap \langle y\rangle \neq \{1\}$. Clearly $\langle x\rangle \cap \langle y\rangle$ is contained in $Z(G)$, so that the cosets $xZ(G)$ and $yZ(G)$ have finite order. On the other hand, the factor group $G/Z(G)$ is locally graded (see [5]), and so it follows from the first part of the proof that $G/Z(G)$ is supersoluble. Therefore G is a supersoluble group. ■

LEMMA 8. *Let G be a group with nearly modular subgroup lattice, and let N be a proper normal subgroup of G such that G/N is torsion-free. Then every subgroup of N is nearly normal in G.*

Proof. Let K be any subgroup of N, and let X be a modular subgroup of G containing K such that the index $|X : K|$ is finite. Thus also the index $|XN : N|$ is finite, and so X is contained in N. Let a be any element of the set $G \setminus N$. Then $\langle a\rangle \cap N = \{1\}$, and hence

$$X = \langle X, a\rangle \cap N$$

is a normal subgroup of $\langle X, a\rangle$. It follows that X is normal in G, and so K is nearly normal. ■

COROLLARY 9. *Let G be a non-periodic group with nearly modular subgroup lattice. Then every periodic subgroup of G is nearly normal.*

Proof. The set T of all elements of finite order of G is a proper subgroup of G by Proposition 1, and so the statement follows from Lemma 8. ■

Recall that a group G is said to be an *FC-group* if every element of G has finitely many conjugates; it is well known that torsion-free FC-groups are abelian (see for instance [8], Part 1, Theorem 4.32). Moreover, it is easy to prove that a group is an FC-group if and only if all its cyclic subgroups are nearly normal (see [2], Lemma 2.1).

THEOREM 10. *Let G be a locally graded group with nearly modular subgroup lattice. Then the commutator subgroup G' of G is locally finite. In particular, every torsion-free locally graded group with nearly modular subgroup lattice is abelian.*

Proof. It can obviously be assumed that the group G is not periodic. Let T be the subgroup consisting of all elements of finite order of G, and let x be any element of G of infinite order. As the lattice $\mathfrak{L}(G)$ is nearly modular, there exists a modular subgroup X of G such that x belongs to X and the index $|X : \langle x\rangle|$ is finite. Then $X_0 = X \cap T$ is a finite normal subgroup of X, and X/X_0 is infinite cyclic. Moreover, X_0 is nearly normal in G by Corollary 9, and hence its normal closure $N = X_0^G$ is finite. Therefore the factor group $\overline{G} = G/N$ is locally graded and $\overline{X}$ is an infinite cyclic modular subgroup of $\overline{G}$. Let $\overline{y}$ be any element of infinite order of $\overline{G}$ such that $\overline{X}^{\overline{y}} \neq \overline{X}$. The subgroup $\bar{L} = \langle \overline{X}, \overline{y}\rangle$ is supersoluble by Lemma 7, and $\overline{X} \cap \langle \overline{y}\rangle \neq \{1\}$ (see [11], Theorem 1.3), so that $\bar{L}/\overline{X} \cap \langle \overline{y}\rangle$ is finite. Thus also the commutator subgroup $\bar{L}'$ of $\bar{L}$ is finite (see [8], Part 1, Theorem 4.12), so that $\bar{L}'$ is contained in $\overline{T}$. As $\overline{G}$ is generated by its elements of infinite order, it follows that $\overline{X}\,\overline{T}$ is a normal subgroup of $\overline{G}$, and so XT is normal in G. We have proved that every cyclic subgroup of G/T is nearly normal, so that G/T is a torsion-free FC-group, and hence it is abelian. Therefore the subgroup G' is periodic, and so even locally finite by Theorem 5. ■

COROLLARY 11. *Let G be a non-periodic locally graded group with nearly modular subgroup lattice. Then the subgroup G'' is finite.*

Proof. The commutator subgroup G' of G is periodic by Theorem 10, and so it follows from Lemma 8 that all subgroups of G' are nearly normal in G. In particular, G' is a finite-by-abelian group, and hence G'' is finite. ■

Proposition 12. *Let G be a group whose cyclic subgroups are nearly modular. If G contains two elements a and b of infinite order such that $\langle a\rangle \cap \langle b\rangle = \{1\}$, then G is an FC-group.*

Proof. Let x be any element of G, and let X be a modular subgroup of G such that x belongs to X and the index $|X : \langle x\rangle|$ is finite. Consider now an element y of G of infinite order, and suppose first that $\langle x\rangle \cap \langle y\rangle = \{1\}$. Then $X \cap \langle y\rangle = \{1\}$, and hence $X^y = X$ (see [11], Corollary 2.2). Suppose now that $\langle x\rangle \cap \langle y\rangle \neq \{1\}$, so that in particular x has infinite order and there exists an element z of G of infinite order such that

$$\langle x\rangle \cap \langle z\rangle = \langle y\rangle \cap \langle z\rangle = \{1\}.$$

Thus $X^z = X$. Let Y be a modular subgroup of G such that $y \in Y$ and the index $|Y : \langle y\rangle|$ is finite. Then $Y^z = Y$, and Y is normal in $\langle Y, yz\rangle = \langle Y, z\rangle$, so that yz must have infinite order and $Y \cap \langle yz\rangle = \{1\}$. Clearly $\langle x\rangle \cap \langle yz\rangle = \{1\}$, so that yz normalizes X and $X^y = X$. As G is generated by its elements of infinite order, it follows that X is a normal subgroup of G. Therefore every cyclic subgroup of G is nearly normal, and G is an FC-group. ∎

Corollary 13. *Let G be a non-periodic locally graded group with nearly modular subgroup lattice, and let T be the subgroup consisting of all elements of finite order of G. Then either G is an FC-group or G/T is a torsion-free abelian group with Prüfer rank 1.*

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