WEAK* CONVERGENCE OF ITERATES OF LASOTA–MACKEy–TYRCHA OPERATORS

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. We show that Lasota–Mackey–Tyrcha stochastic operators, which are used in mathematical modeling of cell cycles, have weak* convergent iterates.

1. Introduction. Let \((X, d)\) be a separable metric space such that all finite closed balls \(K(x_0, r) = \{x \in X : d(x, x_0) \leq r\}\) are compact. Given a \(\sigma\)-finite measure \(\mu\) on the Borel \(\sigma\)-algebra \(\mathcal{B}\) of subsets of \((X, d)\) we denote by \((L^1(\mu), \| \cdot \|)\) the Banach lattice of \(\mu\)-integrable functions on \(X\). Functions from \(L^1(\mu)\) which are equal \(\mu\)-almost everywhere are identified. Then instead of \(\mathcal{B}\) we will rather think of its \(\mu\)-completion \(\tilde{\mathcal{B}}\). If not stated otherwise also all inequalities are in the \(\mu\)-a.e. sense. The convex set \(\{f \in L^1(\mu) : f \geq 0, \int_X f \, d\mu = 1\}\) of all densities is denoted by \(\mathcal{D}_\mu\).

A linear operator \(P : L^1(\mu) \to L^1(\mu)\) which preserves \(\mathcal{D}_\mu\) (i.e. \(P(\mathcal{D}_\mu) \subseteq \mathcal{D}_\mu\)) is called markovian (or a Markov operator). If there exists a Borel measurable function \(k : X \times X \to \mathbb{R}_+\) such that \(Pf(x) = \int_X k(x, y)f(y) \, d\mu(y)\), then the Markov operator \(P\) is called a kernel operator. Clearly, in this case, for every \(y \in X\) we have \(\int k(x, y) \, d\mu(x) = 1\). Here we require \(k(\cdot, y) \in \mathcal{D}_\mu\) for all \(y\), instead of almost all, to extend \(P\) to all finite measures on \(X\). Otherwise, this extension would be defined only on a set of full \(\mu\)-measure. Now \(P\) is extended to a positive contraction (also denoted by \(P\)), acting on the Banach lattice \((\mathcal{M}(X), \| \cdot \|)\) of all bounded signed measures \(\nu\) on \((X, \mathcal{B})\), by

\[
P\nu(A) = \int \int k(x, y)1_A(x) \, d\mu(x) \, d\nu(y).
\]

Obviously \(P\nu \in L^1(\mu)\).

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The paper is devoted to Markov operators on $L^1([0, \infty))$, the Banach lattice of Lebesgue integrable functions on $[0, \infty)$, with kernels

$$k(x, y) = \begin{cases} -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) & \text{if } 0 \leq y \leq \lambda(x), \\ 0 & \text{otherwise.} \end{cases}$$

The functions $H, Q, \lambda : [0, \infty) \to [0, \infty)$ are absolutely continuous and satisfy:

(H) $H(0) = 1$, $\lim_{x \to \infty} H(x) = 0$, $H$ is nonincreasing,

(Q$\lambda$) $Q(0) = \lambda(0) = 0$, $\lim_{x \to \infty} Q(x) = \lim_{x \to \infty} \lambda(x) = \infty$, and $Q, \lambda$ are nondecreasing.

The class of Markov operators with kernels (1) such that $H, Q, \lambda$ satisfy conditions (H), (Q$\lambda$) is denoted by LMT (after Lasota, Mackey and Tyrcha whose contribution to mathematical modeling of cell cycles is crucial; see [GL], [LM1], [LM2], [LM3], [LMT], [KT], [R2] and [T] for more details).

A kernel stochastic operator on $L^1(\mu)$ is called strong Feller in the strict sense if the mapping

$$(X, d) \ni y \mapsto k(\cdot, y) \in (D_\mu, \| \cdot \|)$$

is continuous. We note that for the adjoint operator $P^*$ the condition (SFS) implies that, whenever $h \in L^\infty(\mu)$, then the image $P^*h$ is continuous. This easily follows from $P^*h(y) = \int_X k(x, y)h(x)\,d\mu(x)$.

In this paper Markov operators $P$ are (SFS) with $P^*$ preserving $C_0(X)$, the Banach lattice of continuous functions $h$ vanishing at infinity (i.e. $h$ is continuous and for every $\varepsilon > 0$ there exists a compact set $E_\varepsilon \subseteq X$ satisfying $|h(x)| \leq \varepsilon$ for all $x \notin E_\varepsilon$). As usual $C_0(X)$ is endowed with the sup-norm $\| \cdot \|_{\sup}$. If there exists $x_0$ such that for every $\varepsilon > 0$ there is a function $r_\varepsilon : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} (t - r_\varepsilon(t)) = \infty$ and

$$\inf_{y \in X} \left\{ \int_{K(y, r_\varepsilon(d(x_0, y)))} k(x, y)\,d\mu(x) > 1 - \varepsilon, \right\}$$

and the kernel $k(x, y)$ has property (SFS), then $P^*C_0(X) \subseteq C_0(X)$. It has been noticed in [B] that LMT operators satisfy (SFS) and $P^*$ preserves $C_0([0, \infty))$.

We recall that the set of all subprobabilistic positive measures on $X$ is compact for the vague topology. We say that a variation norm bounded sequence of measures $\nu_n$ is vaguely convergent to $\nu$ if $\lim_{n \to \infty} \int_X h\,d\nu_n = \int_X h\,d\nu$ for all $h \in C_0(X)$. Since $\mathcal{M}(X)$ may be identified with the adjoint space $C_0(X)^*$, this is simply weak* convergence.

A Markov operator $P$ on $L^1(\mu)$ is said to be asymptotically stable if there exists a (unique) $f_* \in D_\mu$ such that $\lim_{n \to \infty} \|P^n f - (\int f\,d\mu)f_*\| = 0$ for all $f \in L^1(\mu)$. Obviously $f_*$ is $P$-invariant, i.e. $Pf_* = f_*$. The Banach sublattice of all $P$-invariant functions is denoted by $L^1_*(\mu)$. 
The opposite concept to stability is sweeping. Given a family $A_*$ of subsets of $B$ we say that $P$ is $A_*$-sweeping if $\lim_{n \to \infty} \int_A P^n f \, d\mu = 0$ for all $A \in A_*$ and $f \in D_\mu$. Here $A_*$ satisfies the natural conditions that $0 < \mu(A) < \infty$ for all $A \in A_*$, $A_*$ is closed under finite unions, and $\bigcup_{j=1}^{\infty} A_j = X$ for some sequence $A_j \in A_*$. In our case $A_*$ is the family of all compact subsets (balls) of $(X,d)$. It has recently been proved in [M] (see also [KM], [KT], [ŁR], [R1] and [LM2]) that an LMT operator either has an invariant density $f_*$ or it is sweeping with respect to compact sets (thus the Foguel alternative holds).

In this paper we obtain a stronger result. Namely, we show that the iterates $P^n f$ of an LMT operator $P$ are weak* convergent to $Q f$, where $Q$ is a submarkovian projection onto $L^1_*(\mu)$ (i.e. $Q^2 = Q \geq 0$ and $\int_X Q f \, d\mu \leq 1$ for every $f \in D_\mu$). Moreover, if $F \in B$ denotes the minimal (modulo sets of measure zero) set which carries supports of all $P$-invariant densities, then $Q_F = Q|_{L^1(F,\mu)}$ is a markovian projection onto $P$-invariant functions, and $\|P^n f - Q_F f\| \to 0$ for all $f \in L^1(F,\mu)$. We note that $L^1(F,\mu)$ is $P$-invariant and that $F$ is well defined, because $L^1(\mu)$ is separable. If $F_j = \text{supp}(f_{*j})$, where $f_{*j}$ is an ergodic invariant density, then $P_j = P|_{L^1(F_j,\mu)}$ is simply asymptotically stable on its domain. We note that for each $f \in D_\mu$ we have $\int_{F_j} Q f \, d\mu = \lim_{n \to \infty} \int_{F_j} P^n f \, d\mu = \Lambda_j(f)$, where the last sequence does converge as it is bounded and nondecreasing ($F_j$ is invariant). This implies $Q f = \Lambda_j(f) f_{*j}$ on $F_j$. If there are no invariant densities at all, then obviously $Q \equiv 0$, and $\lim_{n \to \infty} \int_0^\infty P^n f h \, d\mu = 0$ for every $h \in C_0[0,\infty)$. In particular, $P$ is sweeping with respect to the family of compact sets.

The rest of our notation is consistent with [F1]. We also use some of its results. Let us briefly recall them. Given a stochastic operator $P$ on $L^1(\mu)$, the space $X$ may be divided into two disjoint parts $C \cup D = X$. The conservative part $C$ is characterized by $C = \{x \in X : f \geq 0 \Rightarrow \sum_{n=0}^{\infty} P^n f(x) \text{ is either } 0 \text{ or } \infty\}$. Because $P^*1_C \geq 1_C$, the markovian operator $P_C f = P(1_C f)$ is well defined. $D$ is called the dissipative part. Obviously $P^*1_D \leq 1_D$ and $\sum_{n=0}^{\infty} P^n f(x) < \infty$ for all $f \in L^1(\mu)$ and $x \in D$. In particular, $\lim_{n \to \infty} P^n f(x) = 0$ for $x \in D$. If $C = X$ then we say that the operator $P$ is conservative. We denote by $\Sigma_i(P)$ the $\sigma$-algebra of all invariant sets $A$, i.e. such that $P^*1_A = 1_A$. If $\Sigma_i(P) = \{\emptyset, X\}$ then $P$ is called ergodic. The deterministic $\sigma$-algebra $\Sigma_d(P)$ is defined as $\{B \in B : P^*n 1_B = 1_{B_n} \text{ for every natural } n\}$. We say that $B_0, B_1, \ldots , B_{d-1}$ (from $\Sigma_d(P)$) form a cycle if $P^*_d 1_{B_0} = 1_{B_0}$ and $P^*1_{B_j} = 1_{B_{j+1}}$ for all $0 \leq j \leq d-2$. Clearly $\Sigma_i(P) \subseteq \Sigma_d(P)$, but in general these two $\sigma$-algebras may differ. If it happens that $\Sigma_i(P) = \Sigma_d(P)$ then we say that the Markov operator $P$ does not allow cycles.
2. Main result. In this section we describe the asymptotic properties of the iterates of (SFS) Markov operators acting on abstract $L^1(\mu)$ spaces. We start with

**Lemma 1.** Let $P$ be a (SFS) kernel Markov operator on $L^1(X, \mathcal{B}, \mu)$ such that $P^*$ preserves $C_0(X)$. If $P_C$ does not allow cycles then

$$\lim_{n \to \infty} \int_{K \cap (C \setminus F)} P^n f \, d\mu = 0$$

for every compact set $K \subseteq X$ and arbitrary $f \in L^1(X, \mathcal{B}, \mu)$.

**Proof.** Obviously $F \subseteq C$. Since $P^n_C 1_F \geq 1_F$, we have $P^n_C 1_{C \setminus F} \leq 1_{C \setminus F}$. It is well known that $P_C$, considered as a Markov operator on $L^1(C, \mathcal{B} \cap C, \mu|_C)$, is conservative. In particular, $P^n_C 1_{C \setminus F} = 1_{C \setminus F}$ and $P^n_C 1_F = 1_F$. Clearly $P_C$ is a kernel operator. Hence $P_C$ is Harris. This implies that $\Sigma_\delta(P_C)$ is atomic (see [F1] for all details). Let $B \subseteq C$ be an atom of $\Sigma_\delta(P_C)$. Consider the sequence $P^n_C 1_B = 1_{B_n}$. Clearly $B_n \in \Sigma_\delta(P_C)$ are atoms as well. Since $P_C$ is conservative, it follows that $\sum_{n=0}^{\infty} P^n_C 1_B = \infty$ on $B$. In particular, for some $n > 0$ we have $\mu(B_n \cap B) > 0$. Hence $B_n \cap B = \emptyset$. This implies that $P^n_C 1_B \geq 1_B$. By conservativity $P^n_C 1_B = 1_B$. Since $P_C$ does not allow cycles, we have $P_C 1_B = 1_B$. Now let $P_B$ denote the restriction of $P$ to $L^1(B, \mathcal{B}_B, \mu|_B)$. Again $P_B$ is conservative.

We show that $P_B$ is totally ergodic (i.e. for each natural $n$ the operator $P^n_B$ is ergodic). In fact, if $P^n_B 1_A = 1_A$ for some $0 < \mu(A) < \mu(B)$ then $P^n_B 1_A = 1_{A_j}$, as $P_B$ is nondisappearing (see Lemma 0 of [KL]). Hence $A \in \Sigma_\delta(P_C)$, contradicting the assumption that $B$ is an atom. A conservative and totally ergodic Markov operator $P_B$ satisfies the assumptions of the 0-2 law, which says that either $\|P^n_B - P^{n+1}_B\| \to 0$ in the operator norm as $n \to \infty$, or for a fixed $y \in B$ the densities $k_n(\cdot, y)$, corresponding to $P^n \delta_y$, are pairwise orthogonal (see [OS] and [F2] for all details). The latter is excluded as $P_B$ is conservative, and $P^n \delta_y \in \mathcal{D}_\mu$ for all $n \geq 1$. In particular

$$\sum_{n=1}^{\infty} P^n_B \delta_y(\cdot) = \infty \quad \text{on } B.$$

Hence, for all atoms $B \in \Sigma_\delta(P_C)$,

$$\lim_{n \to \infty} \sup_{f \in \mathcal{D}_\mu} \|P^n_B f - P^{n+1}_B f\| = 0.$$

Now assume $B \subseteq C \setminus F$, and let $K$ be an arbitrary compact subset of $B$. Suppose that

$$\lim_{n \to \infty} \int_K P^n f \, d\mu > 0 \quad \text{for some } f \in \mathcal{D}_\mu.$$

Since $B$ is invariant we may assume that $f$ is concentrated on $B$. Let
$P^{n_j} f \to \nu$ as $j \to \infty$ for the vague topology, where $\nu$ is a nonzero positive measure. By (3) we have $\|P \nu - \nu\| = 0$. In particular, $d\nu / d\mu = f_\nu \in L^1(\mu)$ and $\int_B f_\nu \, d\mu \geq \int_K d\nu > 0$, contradicting the inclusion $B \subseteq C \setminus F$. We conclude that

$$\lim_{n \to \infty} \int_K P^n f \, d\mu = 0$$

for every compact set $K$ included in an atom $B \subseteq C \setminus F$. This convergence may be extended to (2) in the following way. Suppose that for some $K$, $f \in D_\mu$ and $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \int_{K \cap C \setminus F} P^n f \, d\mu \geq \varepsilon.$$ 

Let $B_1, B_2, \ldots \in \Sigma_i(P_C)$ be atoms included in $C \setminus F$. Because they are $P$-invariant and disjoint, it follows that there exists a natural $m$ such that

$$\lim_{n \to \infty} \sum_{j=m+1}^{\infty} \int_{B_j} P^n f \, d\mu < \frac{\varepsilon}{2}.$$ 

Without loss of generality we may assume that

$$\lim_{n \to \infty} \int_{K \cap B_1} P^n f \, d\mu > \frac{\varepsilon}{2m}.$$ 

Using again the fact that $B_1$ is invariant, we may assume that the density $f$ is concentrated on $B_1$. Now we choose a subsequence $n_j \not\to \infty$ such that $\int_{K \cap B_1} P^{n_j} f \, d\mu > \varepsilon/(2m)$, and $P^{n_j} f \to \nu$ vaguely. We have already noticed that $\nu$ is absolutely continuous and its density $f_\nu$ is $P$-invariant. Hence $\nu(B_1) = 0$. Now let $\text{supp}_{\text{top}}(\nu)$ be the topological support of $\nu$, and let $y_0 \in \text{supp}_{\text{top}}(\nu)$. Using the (SFS) condition we may find $\delta > 0$ small enough that

$$k(\cdot, y') \wedge k(\cdot, y'') \neq 0$$

for all $y', y'' \in K(y_0, \delta)$. Choosing $j$ large enough we get

$$\int_{K(y_0, \delta) \cap K \cap B_1} P^{n_j} f \, d\mu > 0.$$ 

Hence

$$(P^{n_j+1} f) \wedge k(\cdot, y'') \geq \int_{K(y_0, \delta) \cap K \cap B_1} k(\cdot, y') P^{n_j} f(y') \, d\mu(y') \wedge k(\cdot, y'') \neq 0$$

for all $y'' \in K(y_0, \delta)$. Now, if $f_1$ is an arbitrary density concentrated on $K(y_0, \delta) \cap \text{supp} f_\nu \subseteq F$, then

$$P^{n_j+1} f \wedge P f_1 = (P^{n_j+1} f) \wedge \int k(\cdot, y'') f_1(y'') \, d\mu(y'') \neq 0,$$ 

contradicting the fact that $B_1$ and $F$ are both invariant. ■
The following lemma is an easy consequence of a result from [BB].

**Lemma 2.** Let $P$ be a kernel Markov operator. If $\Sigma_d(P) \cap F = \Sigma_i(P) \cap F$ then $P_F$ has strong operator convergent iterates.

**Proof.** It is enough to show that each $P_B$ is asymptotically stable on $L^1(B, \mathcal{B} \cap B, \mu)$, where $B \in \Sigma_d(P) \cap F$ is an atom. For this we notice that $P_B$ satisfies the “0” alternative of the “0-2” law, and clearly it is mean ergodic (as $P_B$ has an invariant density). Now by Corollary on page 22 of [BB] we conclude that for every density $f$ concentrated on $B$, we have $P^n_B f \to f_{B^*}$ in the $L^1$ norm, where $f_{B^*}$ is a unique $P$-invariant density concentrated on $B$. Finally we get $P^n_B f \to \sum_j (\delta_{B_j} f) d\mu f_{B_j^*}$ in the $L^1$ norm. ■

The next lemma describes the dynamics of the process $P^n f$ on the dissipative part.

**Lemma 3.** Let $P$ be a (SFS) Markov operator acting on $L^1(X, \mathcal{B}, \mu)$ such that $P^*$ preserves $C_0(X)$. Then for every compact set $K \subseteq X$ and $f \in D_\mu$ we have

$$\lim_{n \to \infty} \int_{K \cap D} P^n f d\mu = 0.$$  

**Proof.** By [F1] there exists an increasing sequence of measurable sets $B_m \subseteq D$ such that $\bigcup_{m=1}^{\infty} B_m = D$ and $P^{*n} \mathbf{1}_{B_m} \to 0$ pointwise as $n \to \infty$. In particular, $1_{B_m \cap K(x)} \not\to 1_{D \cap K(x)}$ for $\mu$-almost all $x$. Since $P$ is a kernel operator, it follows that

$$P^* \mathbf{1}_{B_m \cap K}(y) = \int_{X} k(x, y) \mathbf{1}_{B_m \cap K}(x) d\mu(x)$$

$$\not\to \int_{X} k(x, y) \mathbf{1}_{B \cap K}(y) d\mu(x) = P^* \mathbf{1}_{D \cap K}(y)$$

as $m \to \infty$, for all $y \in X$. We notice that $P^* \mathbf{1}_{B_m \cap K} \leq P^* \mathbf{1}_{D \cap K}$ and both functions belong to $C_0(X)$, as $K$ is compact and the kernel $k$ satisfies the (SFS) condition. By the Dini theorem and the assumption that all closed balls in $(X,d)$ are compact we get the uniform convergence $P^* \mathbf{1}_{B_m \cap K} \Rightarrow P^* \mathbf{1}_{D \cap K}$. Suppose that $\int_{D \cap K} P^n f d\mu \geq \varepsilon$ for some $f \in D_\mu$, where $n_j \not\to \infty$ and $\varepsilon > 0$. Choosing $m$ large enough we obtain $\|P^* \mathbf{1}_{B_m \cap K} - P^* \mathbf{1}_{D \cap K}\|_{sup} \leq \varepsilon/2$. Hence

$$\int_{B_m \cap K} P^{n_j} f d\mu = \left( \int_{B_m \cap K} - \int_{D \cap K} + \int_{D \cap K} \right) P^{n_j} f d\mu$$

$$\geq \varepsilon - \left| \int P^{n_j-1} f \cdot (P^* \mathbf{1}_{B_m \cap K} - P^* \mathbf{1}_{D \cap K}) d\mu \right|$$

$$\geq \varepsilon - \int P^{n_j-1} f \cdot \|P^* \mathbf{1}_{B_m \cap K} - P^* \mathbf{1}_{D \cap K}\|_{sup} d\mu$$

$$\geq \varepsilon/2 \quad \text{for all} \; j = 1, 2, \ldots \;$$
On the other hand, \( f \cdot P^{*n_j} 1_{B_m \cap K} \to 0 \) as \( j \to \infty \) for \( \mu \) almost all \( x \). By the Lebesgue dominated convergence theorem we get
\[
\lim_{j \to \infty} \int_{B_m \cap K} P^{n_j} f \, d\mu = 0,
\]
a contradiction. ■

**Theorem 1.** Let \((X, d)\) be a metric space such that all closed balls are compact. If a kernel Markov operator \( P \) on \( L^1(X, \mathcal{B}, \mu) \) satisfies \((SFS)\), \( P^* \) preserves \( C_0(X) \), and \( \Sigma_i(P_C) = \Sigma_d(P_C) \), then for every compact set \( K \subseteq X \) and every \( f \in L^1(X, \mathcal{B}, \mu) \) we have
\[
\lim_{n \to \infty} \int_K P^n f \, d\mu = \int_K Sf \, d\mu,
\]
where \( S \) is a substochastic projection onto the sublattice of \( P \)-invariant functions. Moreover, on \( L^1(F, \mathcal{B}_F, \mu|_F) \) the above convergence holds for the \( L^1 \) norm.

**Proof.** Given a density \( f \in \mathcal{D}_\mu \) and an atom \( B \) in \( \Sigma_d(P_F) = \Sigma_i(P_F) \), we define
\[
S_B f = \left( \lim_{n \to \infty} \int_B P^n f \, d\mu \right) \cdot f_{B^*},
\]
where \( f_{B^*} \) is a (unique) \( P \)-invariant density concentrated on \( B \). The limit \( \lim_{n \to \infty} \int_B P^n f \, d\mu \) is well defined, as the sequence \( \int_B P^n f \, d\mu \) is nondecreasing and bounded by 1. Finally we set
\[
Sf = \sum_{B \in \Sigma_i(P_F)} S_B f.
\]
On each atom \( B \in \Sigma_i(P_F) \) the operator \( P_B \) is asymptotically stable. We obtain \( \lim_{n \to \infty} \|P^n f - S_B f \| = 0 \). This convergence can be extended to the whole of \( F \) in an obvious way. Finally by Lemmas 1 and 3 we get
\[
\lim_{n \to \infty} \int_K P^n f \, d\mu = \lim_{n \to \infty} \left( \int_{K \cap F} + \int_{K \cap (C \backslash F)} + \int_{K \cap D} \right) P^n f \, d\mu
\]
\[
= \lim_{n \to \infty} \int_{K \cap F} P^n f \, d\mu = \int_{K \cap F} Sf \, d\mu = \int_K Sf \, d\mu.
\]
Since each \( f \in L^1(\mu) \) is a linear combination of densities the theorem is proved. ■

The following corollary is an immediate consequence of \((SFS)\) and \( P^*C_0(X) \subseteq C_0(X) \). Such operators are actually defined on \( \mathcal{M}(X) \). We have

**Corollary 1.** Let \( P \) be a \((SFS)\) kernel Markov operator defined on \( L^1(\mu) \) such that \( P^*C_0(X) \subseteq C_0(X) \). If \( P_C \) does not allow cycles, then there
exists a substochastic projection \( S^{**} = S : \mathcal{M}(X) \to L^1(X)(\mu) \) onto the sublattice of \( P \)-invariant functions such that \( P^n\mu \approx P^{**n}\mu \to \mu \) as \( n \to \infty \) for the vague (weak\(^*\)) topology.

3. LMT operators. The last section of the paper is devoted to LMT operators. The result we present has been proved in [B] with some restrictions on \( P \). This also generalizes [KM], [M] as it is obvious that in the absence of invariant densities, the projection \( S \) is zero. In particular, for every compact \( K \) we have the convergence \( \lim_{n \to \infty} \int_K P^n f \, d\mu = 0 \) whenever \( L^1(K)(\mu) \) is trivial. In other words, an LMT operator \( P \) satisfies the Foguel alternative, i.e. either \( P \) has an invariant density, or it is sweeping with respect to the family of compact sets. We have

\[
\text{THEOREM 2. Let } P \text{ be an LMT operator on } L^1([0, \infty)), \text{ where } H, Q, \lambda \text{ satisfy conditions (H) and (QL)}. \text{ Then for every compact set } K \subseteq [0, \infty) \text{ we have}
\]

\[
\lim_{n \to \infty} \int_K P^n f \, d\mu = \int_K S f \, d\mu,
\]

where \( S \) is a substochastic projection onto the Banach sublattice of \( P \)-invariant functions. Moreover \( \|(P^n f - S f)1_F\| \to 0 \) as \( n \to \infty \).

\textbf{Proof.} By Theorem 1 it is sufficient to show that \( P_C \) does not allow cycles. Since \( P_C \) is conservative it follows that given an atom \( B \in \Sigma_a(P_C) \), there exists \( n \) such that \( P_C^n1_B = 1_B \). Set \( 1_{B_j} = P_C^j1_B \) for \( j = 0, 1, \ldots, n-1 \) \((B_n = B_0 = B)\). To show that the cycle \( B_j \) is trivial we use essentially the same arguments as in the proof of Theorem 2 of [B]. Namely, it follows directly from the formula (1) of LMT kernels that

\[
P^*1_{[c,d]}(y) = \begin{cases} 
H(Q(\lambda(c)) - Q(y)) - H(Q(\lambda(d)) - Q(y)) & \text{if } 0 \leq y < \lambda(c), \\
1 - H(Q(\lambda(d)) - Q(y)) & \text{if } \lambda(c) \leq y < \lambda(d), \\
0 & \text{if } \lambda(d) \leq y.
\end{cases}
\]

Substituting \( d = \infty \) we get

\[
P^*1_{[c,\infty]}(y) = \begin{cases} 
H(Q(\lambda(c)) - Q(y)) & \text{if } 0 \leq y < \lambda(c), \\
1 & \text{otherwise}.
\end{cases}
\]

Let \( c_j = \text{ess inf } B_j, j = 0, 1, \ldots, n - 1 \). Then we have (by continuity of \( P^*1_{B_j} \))

\[
P^*1_{B_j}(y) = \begin{cases} 
1 & \text{if } y \in \overline{B}_{j+1}, \\
0 & \text{if } y \in \overline{B}_s \text{ for all } 0 \leq s \leq n - 1 \text{ and } s \neq j + 1.
\end{cases}
\]

In particular, all \( c_j \) must be different. Without loss of generality we assume that \( c_0 = \max\{c_0, c_1, \ldots, c_{n-1}\} \). Note that \( c_0 < \lambda(c_0) \). Otherwise we would have \( P^*1_{[c_0, \infty]} \geq 1_{[c_0, \infty]} \). On the conservative part we get \( P_C^*1_{[c_0, \infty]} \cap C = \ldots \).
1_{[c_0,\infty)} \cap C$. This implies $0 \leq P_C^*1_{[0,1]} = P_C^*1_{(c_0,\infty)} \cap B_0 \leq 1_{(c_0,\infty)} \cap C$. Hence $P_C^*1_{B_0}(y) \equiv 0$ for all $0 \leq y < c_0$, contradicting $P_C^*1_{B_0} = 1_{B_1}$ and $c_1 < c_0$. We also have $1 \geq H(Q(\lambda(c_0)) - Q(c_1)) = P_C^*1_{(c_0,\infty)}(c_1) \geq P_C^*1_{B_0}(c_1) = 1$, as $P_C^*1_{B_0}$ is continuous, $P_C^*1_{B_0} = 1_{B_1}$ and $c_1 \in 0_1$. Combining these facts we conclude that $H(Q(\lambda(c_0)) - Q(y)) = 1$ if $c_1 \leq y \leq \lambda(c_0)$, as $H$ is nonincreasing, and $Q$ and $\lambda$ are nondecreasing. In particular,

$$P_C^*1_{[c_0,\infty)} \geq 1_{[c_1,\infty)} \geq 1_{[c_0,\infty)};$$

which means that $P_C^*1_{[c_0,\infty)} \cap C = 1_{[c_0,\infty)} \cap C$ on the conservative part. Therefore, $P_C^*1_{B_0} \leq 1_{[c_0,\infty)}$, contradicting $P_C^*1_{B_0} = 1_{B_1}$ if $n > 1$ and $c_1 < c_0$. The resulting contradiction forces $n = 1$ and the triviality of the cycle $B_0, \ldots, B_{n-1}$.  

We instantly get

**Corollary 2.** Given an LMT Markov operator $P$ on $L^1([0,\infty))$, there exists a substochastic projection $S : M([0,\infty)) \to L^1([0,\infty))$ onto $L^1_*(\mu)$, the sublattice of all $P$-invariant functions, such that

$$\lim_{n \to \infty} P^n \mu(K) = S\mu(K)$$

for every compact set $K \subseteq [0,\infty)$. In particular, if $P$ has no invariant density (i.e. $L^1_*(\mu)$ is trivial), then

$$\lim_{n \to \infty} P^n \mu(K) = 0$$

for every compact set $K \subseteq [0,\infty)$; in particular, $P$ is sweeping.

**Remark.** If there exists a compact set $K( \subseteq F)$ such that for every density $f \in \mathcal{D}_\mu$ we have

$$\lim_{n \to \infty} \int_K P^n f \, d\mu > 0$$

(compare [B] and [BL]) then $S$ is finite-dimensional, being a compact projection. For this we note that $S = S \circ P = P \circ S$. Now the operator $S_K^* : L^\infty(\mu) \to C(K)$ defined by $S_K^*f = (S^*f)|_K$ is compact. Therefore $\Sigma_1(P) \cap K$ must be finite. Clearly dim$(S^*) = \dim(S_K^*)$ by the assumption (5).

**References**


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