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MULTIPLIERS OF THE HARDY SPACE H^1 AND POWER BOUNDED OPERATORS

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Abstract. We study the space of functions $\varphi : \mathbb{N} \to \mathbb{C}$ such that there is a Hilbert space H, a power bounded operator T in $B(H)$ and vectors ξ, η in H such that $\varphi(n) = \langle T^n \xi, \eta \rangle$. This implies that the matrix $(\varphi(i+j))_{i,j\ge 0}$ is a Schur multiplier of $B(\ell_2)$ or equivalently is in the space $(\ell_1 \check{\otimes} \ell_1)^*$. We show that the converse does not hold, which answers a question raised by Peller [Pe]. Our approach makes use of a new class of Fourier multipliers of H^1 which we call "shift-bounded". We show that there is a φ which is a "completely bounded" multiplier of H^1, or equivalently for which $(\varphi(i+j))_{i,j\ge 0}$ is a bounded Schur multiplier of $B(\ell_2)$, but which is not shift-bounded on H^1. We also give a characterization of "completely shift-bounded" multipliers on H^1.

0. Introduction. The main motivation of this paper is a question of Peller on power bounded operators [Pe]. To state it, we need some specific notation. For any $c > 1$ and any polynomial $P(z) = \sum a_n z^n$, let

$$|||P|||_c = \sup\Big\{\Big\|\sum a_n T^n\Big\|\Big\}$$

where the supremum runs over all (power bounded) operators T in $B(\ell_2)$ such that $\sup_{n\ge 1} \|T^n\| \le c$. We also let

$$\|P\|_{\mathcal{L}} = \inf\{\|A\|_{\ell_1 \check{\otimes} \ell_1}\}$$

where the infimum runs over all elements $A = \sum A_{ij} e_i \otimes e_j$ in the injective tensor product $\ell_1 \check{\otimes} \ell_1$ such that

$$a_n = \sum_{i+j=n} A_{ij}.$$

As observed by Peller, it follows from Grothendieck's theorem that there is a constant K such that for all $c > 1$ we have

$$|||P|||_c \le K c^2 \|P\|_{\mathcal{L}}.$$

Peller asked whether conversely there is any $c > 1$ for which $|||\ |||_c$ and $\|\ \|_{\mathcal{L}}$ are equivalent. We prove below that it is not so. Unfortunately, Peller's basic

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question whether all the norms $|||\ |||_c$ are equivalent for all $c > 1$ remains open, although we propose a "new" approach for its solution, directly inspired by Peller's ideas in [Pe] but revised in light of the recently developed operator space theory (see especially [BRS, B, BP2]). As we show below, the latter theory clearly suggests that one should replace the iterated injective tensor products $\ell_1 \check{\otimes} \ldots \check{\otimes} \ell_1$ (d times), which Peller uses, by the iterated Haagerup tensor products $\ell_1 \otimes_h \ldots \otimes_h \ell_1$. (Actually, since $\mathbb{N}$ is commutative, we should consider the "symmetrized" iterated Haagerup tensor products of [OP], but this can be left implicit in this note.)

Peller observed that for any two polynomials P, Q we have

$$|||PQ|||_c \le |||P|||_c |||Q|||_c.$$

In other words, $|||\ |||_c$ is a Banach algebra norm. Moreover, its definition clearly shows that it is an "operator algebra" norm, i.e. the resulting Banach algebra can be isometrically embedded into the algebra $B(H)$ of all bounded operators on a Hilbert space H. Thus Peller was led to ask whether $\|\ \|_{\mathcal{L}}$ is equivalent to an operator algebra norm, or merely even to a Banach algebra one. In his review of Peller's paper (Math. Rev. 1983i, 47019), G. Bennett proved that $\|\ \|_{\mathcal{L}}$ is indeed a Banach algebra norm, but we will prove below (see Corollary 2.2) that it is not equivalent to an operator algebra norm. As Peller observed, it suffices to prove that for any $c > 1$ the norms $\|\ \|_{\mathcal{L}}$ and $|||\ |||_c$ are not equivalent.

To prove this, we use a connection with bounded (Fourier) multipliers on the Hardy space H^1, as follows. Given a Banach space B, we denote by $H^1(B)$ the completion the B-valued polynomials $f(z) = \sum x_n z^n$ (here $x_n \in B$) for the norm

$$\|f\|_{H^1(B)} = \int \|f(z)\|_B \, dm(z)$$

where m denotes the normalized Lebesgue measure of the unit circle. When $B = \mathbb{C}$, this is the classical Hardy space, which we simply denote as usual by H^1.

Let $\varphi : \mathbb{N} \to \mathbb{C}$ be a function in $\mathcal{L}^*$, i.e. we assume there is a constant C such that for any polynomial $P(z) = \sum a_n z^n$ (with $a_n \in \mathbb{C}$) we have

$$\Big|\sum \varphi(n) a_n\Big| \le C \Big\|\sum a_n z^n\Big\|_{\mathcal{L}}. \tag{0.1}$$

Then φ defines a multiplier

$$M_\varphi : \sum a_n z^n \to \sum a_n \varphi(n) z^n$$

which is bounded on H^1. Actually, M_φ is *completely bounded* on H^1 (see [P1, §6]), which means that M_φ defines a bounded multiplier on $H^1(S_1)$ where S_1 denotes the Banach space of all trace class operators on ℓ_2 (equipped with the norm $\|x\|_{S_1} = \mathrm{tr}(|x|)$). Conversely, as observed in [P1, Th. 6.2] any

φ such that M_φ is completely bounded on H^1 is in $\mathcal{L}^*$, i.e. satisfies (0.1), and the norms $\|\varphi\|_{\mathcal{L}^*}$ and $\|M_\varphi : H^1(S_1) \to H^1(S_1)\|$ are equivalent (see (1.5) and (1.6) below). The main virtue of this note is the introduction of a "restricted" class of completely bounded (in short c.b.) multipliers on H^1. We will say that M_φ is *shift-bounded* on H^1 if for any $x = \sum_{n\geq 0} x_n z^n$ in H^1 we have

$$\int \sup_{k\geq 0} |M_\varphi(z^k x)|\, dm(z) = \int \sup_{k\geq 0} \Big| \sum_{n\geq 0} x_n \varphi(n+k) z^n \Big|\, dm(z) < \infty.$$

We show below that any φ such that $|||\varphi|||_c^* < \infty$ for some $c > 1$ must define a shift-bounded multiplier on H^1. Thus to show that $\|\ \|_{\mathcal{L}}$ and $|||\ |||_c$ are not equivalent it suffices to produce a multiplier on H^1 which is completely bounded but *not* shift-bounded. This follows from our main result (see Theorem 2.1 below). In §3 we include some remarks on the class of shift-bounded multipliers on H^1 and a characterization of their analogue on S_1-valued H^1, the "completely shift-bounded" ones, which might be of independent interest. Although this uses ideas and techniques from the recently developed "operator space theory" ([BP2, ER1-3]), our formulation (especially in Theorem 3.3) hopefully will be accessible to readers not familiar with it.

1. Notation and background. Let G be a semigroup with unit. Since we mostly concentrate here on the case $G = \mathbb{N}$, we denote the operation of G additively with unit 0. However, most of our notation makes sense for a general (non-commutative) semigroup. In particular we refer to [P5] for a detailed treatment of the case when G is a free group (see also [P2] for related results). After some hesitation, we chose to write separate papers since, although multipliers appear in both of these two cases, the same questions require quite different techniques.

Let $\pi : G \to B(H)$ be a uniformly bounded unital semigroup homomorphism, i.e. we have

$$\pi(s+t) = \pi(s)\pi(t), \qquad \pi(0) = I.$$

We define

$$|\pi| = \sup\{\|\pi(t)\|_{B(H)} \mid t \in G\}.$$

Let $c \geq 1$. We denote by $B_c(G)$ the space of all "matrix coefficients" of the unital semigroup homomorphisms which are uniformly bounded by c. More precisely, $B_c(G)$ is the space of functions $\varphi : G \to \mathbb{C}$ for which there is $\pi : G \to B(H)$ as above with $|\pi| \leq c$ together with vectors $\xi,\ \eta$ in H such that

$$\forall t \in G \quad \varphi(t) = \langle \pi(t)\xi, \eta\rangle. \tag{1.1}$$

Moreover, we define

$$\|\varphi\|_{B_c(G)} = \inf\{\|\xi\| \cdot \|\eta\| \mid \varphi(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle \text{ with } |\pi| \le c\}.$$

Note that when $c = 1$ and G is a group, $B_1(G)$ coincides with the classical space of coefficients of unitary representations of G, usually denoted by $B(G)$, with the same norm. Indeed, it is easy to check in the group case that $|\pi| = 1$ iff π is a unitary representation.

The space $B_c(G)$ is a Banach space (for the above norm). Moreover, for any $c' \ge 1$ we have

$$f \in B_c(G),\ g \in B_{c'}(G) \ \Rightarrow\ f \cdot g \in B_{cc'}(G).$$

Note moreover that if $c \ge c'$ we have a norm one inclusion $B_{c'}(G) \subset B_c(G)$.

In the main case of interest to us here, $G = \mathbb{N}$, we have an isometric identity

$$B_1(\mathbb{N}) = A(D)^*. \tag{1.2}$$

Here $A(D)$ is the disc algebra which can be defined as the completion of the space of (analytic) polynomials P under the sup-norm over the unit disc in $\mathbb{C}$. Indeed, by a well known inequality of von Neumann (see e.g. [P1, §1]) for any such P and for any contraction T in $B(H)$ (meaning $\|T\| \le 1$), we have

$$\|P(T)\| \le \|P\|_{A(D)}.$$

A unital homomorphism $\pi : \mathbb{N} \to B(H)$ is in 1-1 correspondence with a $T \in B(H)$ such that $\pi(n) = T^n$ for all $n \in \mathbb{N}$, thus, for any $\varphi : \mathbb{N} \to \mathbb{C}$, we have $\|\varphi\|_{B_1(\mathbb{N})} \le 1$ iff there is a contraction T and $\xi,\ \eta$ in the unit ball of H such that

$$\varphi(n) = \langle T^n \xi, \eta\rangle, \tag{1.3}$$

and by von Neumann's inequality this holds iff $\|\varphi\|_{A(D)^*} \le 1$. This verifies (1.2). More generallly, for any $c \ge 1$, we have $\|\varphi\|_{B_c(\mathbb{N})} \le 1$ iff there is a power bounded $T \in B(H)$ with $\sup_n \|T^n\| \le c$ and ξ, η in the unit ball of H such that (1.3) holds.

Let $d \ge 1$ be an integer. Let $M_d(G)$ be the space of all functions $\varphi : G \to \mathbb{C}$ such that there are bounded functions $\xi_i : G \to B(H_i, H_{i-1})$ (H_i Hilbert) with $H_0 = \mathbb{C}$, $H_d = \mathbb{C}$ such that

$$\forall t_i \in G \qquad \varphi(t_1 + t_2 + \ldots + t_d) = \xi_1(t_1)\xi_2(t_2)\ldots\xi_d(t_d). \tag{1.4}$$

Here of course we use the identification $B(H_0, H_d) = B(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$. We define

$$\|\varphi\|_{M_d(G)} = \inf\{\sup_{t_1 \in G} \|\xi_1(t_1)\| \ldots \sup_{t_d \in G} \|\xi_d(t_d)\|\}$$

where the infimum runs over all possible ways to write φ as in (1.4). It is quite easy to see that $M_d(G)$ is a Banach algebra for the pointwise product of functions on G.

When $G = \mathbb{N}$, we have $\|\varphi\|_{M_2(\mathbb{N})} \le 1$ iff there are sequences (x_n) and (y_n) in the unit ball of H such that

$$\forall i, j \in \mathbb{N} \quad \varphi(i+j) = \langle x_i, y_j \rangle.$$

Equivalently, if we denote by $u_\varphi : \ell_1 \to \ell_\infty$ the linear operator with (Hankel) matrix $(\varphi(i+j))$, we have $\|\varphi\|_{M_2(\mathbb{N})} = \gamma_2(u_\varphi)$ (here $\gamma_2(\cdot)$ is the norm of factorization through a Hilbert space). A fortiori we have

$$K^{-1}\|\varphi\|_{\mathcal{L}^*} \le \|\varphi\|_{M_2(\mathbb{N})} \le \|\varphi\|_{\mathcal{L}^*} \tag{1.5}$$

where K denotes the Grothendieck constant. Moreover, we have

$$\|\varphi\|_{M_2(\mathbb{N})} = \|M_\varphi : H^1(S_1) \to H^1(S_1)\| \ge \|M_\varphi : H^1 \to H^1\|. \tag{1.6}$$

For example, any bounded φ with support in a lacunary sequence such as $\{2^n \mid n \ge 0\}$ is in $M_2(\mathbb{N})$ (see Lemma 2.4 below for a more general fact). See [Bo] and [P1, §§5–6] for more on all this.

The definition of the spaces $M_d(G)$ (and of the $B(H)$-valued version of these spaces for which we refer to [P5]) is motivated by the work of Christensen–Sinclair on “completely bounded multilinear maps” and the so-called Haagerup tensor product (see [CS]). The connection is explained in detail in [P3, P5], and is important for the results below, but we prefer to skip this in the present, hopefully more accessible, exposition.

Note the following easily checked inclusions, valid when G is any semigroup with unit:

$$\begin{aligned} B(G) = B_1(G) \subset \bigcup_{c>1} B_c(G) \subset M_d(G) \subset M_{d-1}(G) \subset \dots \\ \subset M_2(G) \subset M_1(G) = \ell_\infty(G), \end{aligned}$$

and the estimate

$$\forall m \le d \quad \|f\|_{M_m(G)} \le \|f\|_{M_d(G)}. \tag{1.7}$$

Moreover, we have

$$\forall \varphi \in B_c(G) \quad \|\varphi\|_{M_d(G)} \le c^d \|\varphi\|_{B_c(G)}. \tag{1.8}$$

Indeed, if $\varphi(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle$ with $|\pi| \le c$, then we can write

$$\varphi(t_1 + \ldots + t_d) = \langle \pi(t_1) \ldots \pi(t_d)\xi, \eta \rangle = \xi_1(t_1) \ldots \xi_d(t_d)$$

where $\xi_1(t_1) \in B(H_\pi, \mathbb{C})$, $\xi_d(t_d) \in B(\mathbb{C}, H_\pi)$ and $\xi_i(t_i) \in B(H_\pi, H_\pi)$ $(1 < i < d)$ are defined by $\xi_1(t_1)h = \langle \pi_1(t_1)h, \eta \rangle$ $(h \in H_\pi)$, $\xi_d(t_d)\lambda = \lambda\pi(t_d)\xi$ $(\lambda \in \mathbb{C})$ and $\xi_i(t_i) = \pi(t_i)$ $(1 < i < d)$. Therefore, we have

$$\|\varphi\|_{M_d(G)} \le \sup \|\xi_1\| \ldots \sup \|\xi_d\| \le |\pi|^d \|\xi\| \cdot \|\eta\| \le c^d \|\xi\| \cdot \|\eta\|,$$

which yields the announced inequality (1.8).

Let $UB(G) = \bigcup_{c>1} B_c(G)$. Then $\varphi \in UB(G)$ iff $\sup_{m \ge 1} \|\varphi\|_{M_m(G)}^{1/m} < \infty$. More precisely, let $c(\varphi)$ denote the infimum of the numbers $c \ge 1$ for which

$\varphi \in B_c(G)$. Then (see [P5])

$$c(\varphi) = \limsup_{m\to\infty} \|\varphi\|_{M_m(G)}^{1/m}.$$

Moreover, it follows from [BP2] (see also [P4, §7]) that

$$\|\varphi\|_{B_1(G)} = \sup_m \|\varphi\|_{M_m(G)}.$$

The definition of the spaces $B_c(G)$ and $M_d(G)$ shows that they are dual spaces. There is a natural duality between these spaces and the semigroup algebra $\mathbb{C}[G]$ which we view as the convolution algebra of finitely supported functions on G. Indeed, for any function $f : G \to \mathbb{C}$ and any g in $\mathbb{C}[G]$, we set

$$\langle g, f\rangle = \sum_{t\in G} g(t)f(t),$$

and we define the spaces $X_d(G)$ and $\widetilde{A}_c$ as the completions of $\mathbb{C}[G]$ for the respective norms

$$\|g\|_{X_d(G)} = \sup\{|\langle g, f\rangle| \mid f \in M_d(G),\ \|f\|_{M_d(G)} \le 1\}$$

and

$$|||g|||_c = \sup\{|\langle g, f\rangle| \mid f \in B_c(G),\ \|f\|_{B_c(G)} \le 1\}.$$

Then the following isometric identities are rather easy to check:

$$B_c(G) = (\widetilde{A}_c)^* \quad \text{and} \quad M_d(G) = (X_d(G))^*.$$

Obviously, we can also write (here we can restrict to $H = \ell_2$ if we wish)

$$|||g|||_c = \sup\left\{\left\|\sum g(t)\pi(t)\right\| \,\middle|\, \pi : G \to B(H),\ |\pi| \le c\right\}.$$

Thus, when $G = \mathbb{N}$, $\widetilde{A}_c$ can be identified with the completion of the polynomials (here a polynomial is identified with the sequence of its coefficients) for the norm $|||\ |||_c$ introduced in §0. The last formula shows that $\widetilde{A}_c$ is naturally equipped with an operator algebra structure under convolution: we have $|||g_1 * g_2|||_c \le |||g_1|||_c|||g_2|||_c$.

However, the analogue for the spaces $X_d(G)$ fails in general. (This is the basic idea used by Haagerup to prove that $M_2(\mathbb{F}_\infty) \ne B_c(\mathbb{F}_\infty)$ for any c (see Remark 1.2 below): he proves first in [H] that $X_2(\mathbb{F}_\infty)$ is not a Banach algebra under convolution.)

In sharp contrast, as we already mentioned, $X_2(\mathbb{N})$ is indeed a Banach algebra!

Although $X_d(G)$ is not in general a Banach algebra under convolution, it has the following property: if $g_1 \in X_d(G)$ and $g_2 \in X_k(G)$, then $g_1 * g_2 \in X_{d+k}(G)$ and

$$\|g_1 * g_2\|_{X_{d+k}(G)} \le \|g_1\|_{X_d(G)}\|g_2\|_{X_k(G)}. \tag{1.9}$$

See [P5] for a detailed proof.

To explain the relevance of the spaces $M_d(G)$ for Peller's question, we quote

THEOREM 1.1 ([P4, P5]). *Let G be a semigroup with unit. The following assertions are equivalent*:

(i) *There is a $\theta \geq 1$ such that $B_\theta(G) = B_c(G)$ for all $c > \theta$.*
(i)′ *There is a $\theta \geq 1$ such that $B_\theta(G) = B_c(G)$ for some $c > \theta$.*
(ii) *There are $\theta \geq 1$ and an integer d such that $B_\theta(G) = M_d(G)$.*
(iii) *There is an integer d such that $M_d(G) = M_{2d}(G)$.*
(iv) *There is an integer d such that $X_d(G)$ is (up to isomorphism) a unital operator algebra under convolution.*

Thus to show that the norms $|||\ |||_c$ and $|||\ |||_\theta$ are not equivalent whenever $c \neq \theta$, it suffices to prove the following

CONJECTURE. $M_d(\mathbb{N}) \neq M_{d+1}(\mathbb{N})$ *for all* $d \geq 1$.

REMARK 1.2. When $d = 2$, and G is a group, the space $M_2(G)$ is the classical space of "Herz–Schur multipliers" on G. This space also coincides (see [BoF] or [P2, p. 110]) with the space of all c.b. "Fourier multipliers" on the reduced C^*-algebra $C^*_\lambda(G)$. The question whether $M_2(G) = UB(G)$ remained open for a while but Haagerup [H] showed that it is not the case. More precisely, he showed that if $G = \mathbb{F}_\infty$, we have

$$\forall c > 1 \quad B_c(G) \subsetneq M_2(G).$$

In [P5] we give a different proof of this. More generally we show there that if G is a non-commutative free group, then for any $d \geq 1$, we have

$$M_d(G) \neq M_{d+1}(G),$$

and hence there are elements of $M_d(G)$ which are not coefficients of uniformly bounded representations.

2. Main results. We wish to prove here a special case of the above conjecture, which answers a question of Peller in [Pe].

THEOREM 2.1. $M_2(\mathbb{N}) \neq M_3(\mathbb{N})$.

COROLLARY 2.2. *For any $c > 1$ the norms $\|\ \|_{\mathcal{L}}$ and $|||\ |||_c$ are not equivalent. More generally, $\|\ \|_{\mathcal{L}}$ is not equivalent to any operator algebra norm.*

Proof. Note that by (1.5) we have for all polynomials P,

$$K^{-1}\|P\|_{X_2(\mathbb{N})} \leq \|P\|_{\mathcal{L}} \leq \|P\|_{X_2(\mathbb{N})}.$$

So if $\|\ \|_{\mathcal{L}}$ and $|||\ |||_c$ were equivalent, we would have (by duality) $M_2 = B_c$, hence a fortiori $M_2 = M_3$, which contradicts Theorem 2.1. The second assertion follows from Theorem 1.1. ∎

Remark. This corollary is closely related to the more recent result due to Kalton and Le Merdy [KLM] asserting that, for any $c > 1$, there are power bounded operators which are not similar to operators with powers bounded by c. Indeed, their result implies the operator space (= completely bounded) analogue of the inequivalence of $|||\ |||_c$ and $\|\ \|_{X_d(\mathbb{N})}$ (hence in particular of $|||\ |||_c$ and $\|\ \|_{X_2(\mathbb{N})}$) for all $c > 1$ and $d \geq 1$. It also implies the completely bounded analogue of the above conjecture. Equivalently, this shows that for any d there is a $B(H)$-valued function which is in the operator-valued analogue of $M_d(\mathbb{N})$ but not in the corresponding analogue of $M_{d+1}(\mathbb{N})$.

We will consider the Hardy spaces H^p. We define H^p as the subspace of all functions $x \in L_p(T, m)$ such that the Fourier transform $\widehat{x} : \mathbb{Z} \to \mathbb{C}$ vanishes on the negative integers. We write abusively $x = \sum \widehat{x}(n) z^n$, meaning that x admits $\sum \widehat{x}(n) z^n$ as its formal Fourier series.

Lemma 2.3. *Any φ in $M_3(\mathbb{N})$ defines a shift-bounded multiplier on H^1. More precisely*,

$$\sup_{x \in B_{H^1}} \left\{ \int \sup_{k \geq 0} \Big| \sum_{n \geq 0} \widehat{x}(n) z^n \varphi(n+k) \Big| \, dm \right\} \leq \|\varphi\|_{M_3(\mathbb{N})}. \tag{2.1}$$

Proof. Note that for each fixed integer k the function $n \mapsto \varphi(n+k)$ is in $M_2(\mathbb{N})$, hence defines a bounded multiplier on H^1. Therefore, for any x in H^1, the series $\sum_{n \geq 0} \widehat{x}(n) z^n \varphi(n+k)$ is in H^1, so (2.1) expresses a sort of uniform boundedness of this family of multipliers. Now assume $\|\varphi\|_{M_3(\mathbb{N})} < 1$ so that there are $\xi_i \in \ell_2^*$, $\eta_j \in \ell_2$, $T_k \in B(\ell_2)$ with

$$\max(\|\xi_i\|, \|\eta_j\|, \|T_k\|) < 1 \tag{2.2}$$

such that

$$\varphi(i+k+j) = \langle \xi_i, T_k \eta_j \rangle. \tag{2.3}$$

Let $x \in B_{H^1}$. Then, by a classical result (cf. e.g. [Ga, p. 87] or [Ni]) we can write $x = gh$ with $g, h \in B_{H^2}$. Then

$$\widehat{x}(n) = \sum_{i+j=n} \widehat{g}(i) \widehat{h}(j).$$

Hence

$$\sum_n \widehat{x}(n) z^n \varphi(n+k) = \sum_{i,j} \widehat{g}(i) \widehat{h}(j) z^{i+j} \langle \xi_i, T_k \eta_j \rangle = \langle G(z), T_k H(z) \rangle$$

where $G(z) = \sum_i \widehat{g}(i) z^i \xi_i \in H^2(\ell_2^*)$ and $H(z) = \sum_j \widehat{h}(j) z^j \eta_j \in H^2(\ell_2)$ with

$$\|G\|_{H^2(\ell_2^*)} = \Big(\sum |\widehat{g}(i)|^2 \|\xi_i\|^2 \Big)^{1/2} \leq 1$$

and similarly $\|H\|_{H^2(\ell_2)} \le 1$. Thus we obtain (by Cauchy–Schwarz)

$$\int \sup_k \Big|\sum_n \widehat{x}(n) z^n \varphi(n+k)\Big|\, dm \le \sup_k \|T_k\| \int \|G(z)\|_{\ell_2^*} \|H(z)\|_{\ell_2}\, dm(z)$$
$$\le \|G\|_{H^2(\ell_2^*)} \|H\|_{H^2(\ell_2)} \le 1. \ \blacksquare$$

REMARK. Actually (see Theorem 3.1 below), it is possible to show that $\|\varphi\|_{M_3}$ coincides with the c.b. norm of the "multiplier" defined by φ as above but acting from H^1 to $H^1(\ell_\infty)$.

We will use the following well known lemma (see e.g. [Bo]).

LEMMA 2.4. *For any $\varphi : \mathbb{N} \to \mathbb{C}$ we have*

$$\|\varphi\|_{M_2(\mathbb{N})} \le 4 \sup_{n\ge 0} \Big(\sum_{2^n \le i < 2^{n+1}} |\varphi(i)|^2\Big)^{1/2} + |\varphi(0)|.$$

Proof. Let $(e_i)_{i\ge 0}$ be the canonical basis of ℓ_2. We can write

$$\varphi(i+j) = \langle x_i, e_j\rangle + \langle x_j, e_i\rangle + \langle y_i, e_j\rangle$$

where $x_i = \sum_{i\le k<2i} \varphi(k) e_{k-i}$ and $y_i = \varphi(2i) e_i$. Therefore

$$\|\varphi\|_{M_2(\mathbb{N})} \le 2 \sup_{i\ge 0} \|x_i\| + \sup_i \|y_i\|.$$

Let

$$C = \sup_{n\ge 0} \Big(\sum_{2^n \le i < 2^{n+1}} |\varphi(i)|^2\Big)^{1/2}.$$

We have $x_0 = 0$, $\|x_i\| \le \sqrt{2}\, C$ and $\|y_i\| \le C + |\varphi(0)|$. Hence

$$\|\varphi\|_{M_2(\mathbb{N})} \le (2\sqrt{2} + 1) C + |\varphi(0)|. \ \blacksquare$$

LEMMA 2.5. *Let $F_1, \ldots, F_K$ be analytic trigonometric polynomials with degree $\le 2^{K-1}$. Let $\varphi = \widehat{F}$ with*

$$F = \sum_{K/2<p<K} z^{2^{2p}} F_p.$$

Then

$$\int \sup_{K/2<p<K} |F_p|\, dm \le 3 \|\varphi\|_{M_3(\mathbb{N})}.$$

Proof. Let $x \in H^1$ be the (La Vallée Poussin type) kernel such that: $\widehat{x} \equiv 1$ on the interval $[2^K, 2^{K+1}]$, $\widehat{x}(0) = 0$, $\widehat{x} \equiv 0$ on the interval $[3\cdot 2^K, \infty)$ and $\widehat{x}$ is linear on the remaining intervals $[0, 2^K]$ and $[2^{K+1}, 3\cdot 2^K]$. A well known computation shows that $\|x\|_1 \le 2$. By Lemma 2.3 we have

$$\int \sup_{k\ge 0} \Big|\sum_{n\ge k} \widehat{x}(n-k) \varphi(n) z^n\Big|\, dm \le 2 \|\varphi\|_{M_3(\mathbb{N})}.$$

For each p with $K < 2p < 2K$ we let $k(p) = 2^{2p} - 2^K$. Hence we have

$$\int \sup_{K/2<p<K} \Big| \sum_{n\geq k(p)} \widehat{x}(n-k(p))\varphi(n)z^n \Big| \, dm \leq 2\|\varphi\|_{M_3(\mathbb{N})}.$$

Let $A_p = \{n \mid \widehat{x}(n-k(p)) \neq 0\}$. We have $A_p \subset k(p) + [0, 3\cdot 2^K]$, hence $A_p \subset [2^{2p} - 2^K, 2^{2p} + 2\cdot 2^K]$. Therefore (since φ is supported in the union of the intervals $[2^{2p}, 2^{2p} + 2^{K-1}]$) we find

$$A_p \cap \{n \mid \varphi(n) \neq 0\} \subset [2^{2p}, 2^{2p} + 2^{K-1}].$$

Now since

$$[2^{2p}, 2^{2p} + 2^{K-1}] - k(p) \subset [2^K, 2^{K+1}]$$

we have $\widehat{x}(n-k(p)) = 1$ for all $n \in A_p \cap \{n \mid \varphi(n) \neq 0\}$. Since $A_p \cap \{n \mid \varphi(n) \neq 0\} \subset [2^{2p}, 2^{2p} + 2^{K-1}]$ we must have simply

$$\sum_{n\geq k(p)} \widehat{x}(n-k(p))\varphi(n)z^n = z^{2^{2p}} F_p$$

and we conclude that

$$\int \sup_{K/2<p<K} |F_p| \, dm \leq 2\|\varphi\|_{M_3(\mathbb{N})}. \ \blacksquare$$

Let q be an integer and let $\mathcal{P}(q)$ denote the space of all analytic trigonometric polynomials with degree at most q. We define

$$C(q) = \sup\Big\{\int \sup_{1\leq p\leq q} |F_p| \, dm \,\Big|\, F_p \in \mathcal{P}(q),\ \sup_{1\leq p\leq q} \|F_p\|_2 \leq 1\Big\}.$$

Lemma 2.6. *For each even integer $K > 1$ there is a function $\varphi_K : \mathbb{N} \to \mathbb{C}$ with support in $[0, 2^{2K}]$ such that $\|\varphi_K\|_{M_2(\mathbb{N})} \leq 1$ but*

$$\|\varphi_K\|_{M_3(\mathbb{N})} \geq (1/8)C(K/2-1).$$

Proof. Let $q = K/2-1$. Let $F_1, \dots, F_q \in \mathcal{P}(q)$ be such that $\sup_{p\leq q} \|F_p\|_2 \leq 1$ and

$$\int \sup_{p\leq q} |F_p| \, dm = C(q).$$

We consider the function

$$F = \sum_{K/2<p<K} z^{2^{2p}} F_{p-K/2},$$

and we let $\varphi = \widehat{F}$. Then by Lemma 2.5 (note that $q = K/2 - 1 \leq 2^{K-1}$) we have

$$C(q) \leq 2\|\varphi\|_{M_3(\mathbb{N})},$$

and on the other hand by Lemma 2.4 we have

$$\|\varphi\|_{M_2(\mathbb{N})} \leq 4 \sup_{p\leq q} \|F_p\|_2 \leq 4,$$

whence $\varphi_K = \varphi/4$ has the announced property. $\blacksquare$

The following fact is elementary and well known:

LEMMA 2.7. *There is a number $\delta > 0$ such that for any $q \geq 1$,*

$$\delta\sqrt{q} \leq C(q) \leq \sqrt{q}.$$

Proof. The upper bound is an easy exercise (use $\sup_p |F_p| \leq (\sum |F_p|^2)^{1/2}$). For the lower bound, we use the following well known consequence of S. Bernstein's inequality: There is a finite subset $B_q \subset \mathbb{T}$ such that for any function F in $\mathcal{P}(q)$ we have

$$\|F\|_\infty \leq a \sup_{t \in B_q} |F(t)|$$

and moreover $|B_q| \leq bq$ where $a \geq 1$ and $b \geq 1$ are absolute constants. We then set $S(z) = \sum_{i=0}^{q} z^i$ and for each ξ in B_q we set

$$F_\xi(z) = S(\xi z).$$

We then have for any z in $\mathbb{T}$,

$$\sup_{\xi \in B_q} |F_\xi(z)| \geq (q+1)/a,$$

hence $\int \sup_{\xi \in B_q} |F_\xi(z)|\, dm(z) \geq (q+1)/a$, and on the other hand

$$\sup_{\xi \in B_q} \|F_\xi\|_2 \leq (q+1)^{1/2}.$$

Hence (assuming without loss of generality that b is an integer ≥ 1) this shows that

$$C(bq) \geq (q+1)^{1/2}/a.$$

On the other hand it is easy to check that $C(bq) \leq bC(q)$, hence we finally obtain the announced result with $\delta = (ab)^{-1}$. ∎

Proof of Theorem 2.1. From the preceding two lemmas, it is clear that the norms of $M_2(\mathbb{N})$ and $M_3(\mathbb{N})$ are not equivalent, which proves Theorem 2.1. ∎

To reformulate Theorem 2.1 more precisely, we need more notation: For any integer $d \geq 2$ we define, for any $\varphi : \mathbb{N} \to \mathbb{C}$,

$$\|\varphi\|_{[d]} = \sup\{\|\varphi\psi\|_{M_d(\mathbb{N})}\}$$

where the supremum runs over all ψ in $\ell_\infty(\mathbb{N})$ with $\|\psi\|_\infty \leq 1$. It is well known that $\|\varphi\|_{[2]} < \infty$ iff M_φ maps H^1 boundedly into H^2, which gives (see [Ru, §8.6], see also [Bo]) the following characterization:

$$4^{-1}\|\varphi\|_{[2]} \leq |\varphi(0)| + \sup_{n \geq 0}\Big(\sum_{2^n \leq k < 2^{n+1}} |\varphi(k)|^2\Big)^{1/2} \leq 4\|\varphi\|_{[2]}.$$

(The left side follows from Lemma 2.4, the other one from a routine averaging argument and Khinchin's inequality, which show that if $\|\varphi\|_{[2]} < \infty$ then

M_φ maps H^1 into H^2; then using, say, La Vallée Poussin kernels, one can obtain the right side.)

We do not know how to characterize the functional $\|\varphi\|_{[3]}$ in an analogous fashion. However, the preceding results show:

THEOREM 2.8. *For any* $q \ge 1$, *let*

$$\alpha(q) = \sup\{\|\varphi\|_{[3]}/\|\varphi\|_{[2]}\}$$

where the supremum runs over all non-zero functions $\varphi : \mathbb{N} \to \mathbb{C}$ *with support in* $[0, q]$. *Then*

$$a_1\sqrt{q} \le \alpha(2^q) \le a_2\sqrt{q} \tag{2.4}$$

where a_1, a_2 *are positive absolute constants.*

Proof. By the preceding equivalent description of $\|\varphi\|_{[2]}$, if φ is supported by $[0, 2^q]$, we clearly have $\|\varphi\|_2 \le a_3\|\varphi\|_{[2]}\sqrt{q}$. Recall that by (1.8) and (1.2), $\|\varphi\|_{M_3(\mathbb{N})} \le \|\varphi\|_{B_1(\mathbb{N})} \le \|\varphi\|_2$. Hence $\|\varphi\|_{M_3(\mathbb{N})} \le a_3\sqrt{q}\,\|\varphi\|_{[2]}$. This yields the right side of (2.4). The converse follows from a combination of Lemmas 2.6 and 2.7. ∎

3. $M_3(\mathbb{N})$ **viewed as a space of completely bounded multipliers.** It is known (see [P1, p. 109]) that $\|\varphi\|_{M_2(\mathbb{N})}$ is equal to the c.b. norm of M_φ viewed as a multiplier from H^1 to H^1. Analogously, we now show that $\|\varphi\|_{M_3(\mathbb{N})}$ coincides with the c.b. norm of the "shifted multiplier" defined by φ as above but acting from H^1 to $H^1(\ell_\infty)$. A somewhat similar statement can also be proved for general discrete groups or semigroups, but the next statement uses the commutativity of $\mathbb{N}$.

To state this result, we need some background on operator space theory from [P6]. Following [P6], we call the operator space (o.s. for short) structure on $L_1(\mathbb{T})$ *natural* if the o.s. dual of $L_1(\mathbb{T})$ is completely isometric to $L_\infty(\mathbb{T})$. Since H^1 is a subspace of $L_1(\mathbb{T})$, this also induces a natural o.s. structure on H^1. More generally, for any o.s. E, we equip the space $L_1(\mathbb{T}; E)$ with the o.s. structure defined by the "o.s. projective" tensor product $L_1(\mathbb{T}) \otimes^\wedge E$ introduced by Effros–Ruan [ER1–2] and Blecher–Paulsen [BP1] (see also [P6]). As a Banach space, $L_1(\mathbb{T}; E)$ is the same as the classical projective tensor product $L_1(\mathbb{T}) \hat{\otimes} E$ in Grothendieck's sense, but the o.s. structure encodes additional information that the norm alone does not carry. We define $H^1(E)$ as the closed subspace generated by $H^1 \otimes E$ (or $\mathrm{span}(z^n) \otimes E$) in $L_1(\mathbb{T}; E) = L_1(\mathbb{T}) \otimes^\wedge E$. Again, we will call the o.s. structure induced by the embedding $H^1(E) \subset L_1(\mathbb{T}) \otimes^\wedge E$ *natural.*

For any bounded function $\varphi : \mathbb{N} \to \mathbb{C}$, let

$$\Phi(n) = \sum_{k\ge 0} \varphi(n+k)e_k \in \ell_\infty$$

where $(e_k)_{k\geq 0}$ denotes the canonical basis of ℓ_∞. Then for any polynomial x in H^1 we have

$$\sup_{k\geq 0}\Big|\sum \widehat{x}(n)z^n\varphi(n+k)\Big| = \Big\|\sum \widehat{x}(n)z^n\Phi(n)\Big\|_{\ell_\infty}.$$

Moreover, $z\mapsto \sum \widehat{x}(n)z^n\Phi(n)$ can be viewed as an element of $H^1\otimes\ell_\infty\subset H^1(\ell_\infty)$. Thus, φ defines a mapping $T_\varphi : \mathrm{span}[z^n, n\geq 0]\to H^1(\ell_\infty)$ such that

$$T_\varphi\Big(\sum \widehat{x}(n)z^n\Big) = \sum \widehat{x}(n)z^n\Phi(n). \tag{3.1}$$

Note that φ is shift-bounded on H^1 iff T_φ is bounded from H^1 to $H^1(\ell_\infty)$. As observed in [P1, §6], $\varphi\in M_2(\mathbb{N})$ iff $M_\varphi : H^1\to H^1$ is completely bounded and

$$\|\varphi\|_{M_2(\mathbb{N})} = \|M_\varphi : H^1\to H^1\|_{\mathrm{cb}}.$$

For $M_3(\mathbb{N})$, the analogous result is as follows:

THEOREM 3.1. *Let H^1 and $H^1(\ell_\infty)$ be equipped with their natural o.s. structures as above. Let $\varphi:\mathbb{N}\to\mathbb{C}$ be a bounded function. Then $\varphi\in M_3(\mathbb{N})$ iff $T_\varphi : H^1\to H^1(\ell_\infty)$ is completely bounded. Moreover*

$$\|\varphi\|_{M_3(\mathbb{N})} = \|T_\varphi : H^1\to H^1(\ell_\infty)\|_{\mathrm{cb}}.$$

Proof. Let us write L_∞ instead of $L_\infty(\mathbb{T})$. Let $S=(H^1)^\perp\subset L_\infty$ and let $q: L_\infty\to L_\infty/S$ be the quotient map. As is well known (see [P1, §6]) we have a completely isometric embedding $j: L_\infty/S\to B(\ell_2)$ taking $q(\overline{z}^n)$ to the Hankel operator

$$\gamma_n = \sum_{i+j=n} e_{ij}.$$

Assume $T_\varphi : H^1\to H^1(\ell_\infty)$ is completely bounded. We clearly have a completely contractive (restriction) map $r: L_\infty\otimes_{\min}\ell_1\to (H^1(\ell_\infty))^*$ where ℓ_1 is equipped with its natural (= maximal) o.s. structure. Then let

$$v = j(T_\varphi)^* r : L_\infty\otimes_{\min}\ell_1\to B(\ell_2).$$

We clearly have $\|v\|_{\mathrm{cb}}\leq\|T_\varphi\|_{\mathrm{cb}}$.

We can assume that $\ell_1=\mathrm{span}[U_n\mid n\geq 0]$ with $U_n\in B(H)$, $\|U_n\|=1$. By the fundamental factorization of c.b. maps (see e.g. [Pa1, p. 105] or [P1, p. 57]), we can find suitable operators α,β with $\|\alpha\|\cdot\|\beta\|=\|v\|_{\mathrm{cb}}$ and a representation

$$\pi : L_\infty\otimes_{\min} B(H)\to B(\widehat{H})$$

such that

$$\forall y\in L_\infty\otimes\ell_1\qquad v(y)=\alpha\pi(y)\beta.$$

This implies

$$\forall n,k\geq 0\qquad v(\overline{z}^n\otimes U_k) = \alpha\pi(\overline{z}^n\otimes U_k)\beta.$$

For any polynomial x in H^1, we have

$$\langle r(\overline{z}^n \otimes U_k), T_\varphi(x)\rangle = \widehat{x}(n)\langle U_k, \Phi(n)\rangle = \widehat{x}(n)\varphi(k+n).$$

Hence we find

$$v(\overline{z}^n \otimes U_k) = \gamma_n \varphi(k+n).$$

Letting $U = \pi(\overline{z} \otimes I)$ and $\widehat{U}_k = \pi(I \otimes U_k)$, we obtain

$$\gamma_n \varphi(k+n) = \alpha U^n \widehat{U}_k \beta,$$

hence denoting by S the classical unilateral shift (note U and $\widehat{U}_k$ commute)

$$\begin{aligned}\varphi(i+k+j) &= \langle \alpha U^{i+j}\widehat{U}_k \beta e_j, e_i\rangle = \langle \alpha U^i \widehat{U}_k U^j \beta e_j, e_i\rangle \\ &= \langle S^{i^*}\alpha U^i \widehat{U}_k U^j \beta S^j e_0, e_0\rangle,\end{aligned}$$

which clearly shows that $\|\varphi\|_{M_3(\mathbb{N})} \le \|\alpha\| \cdot \|\beta\|$. So we conclude

$$\|\varphi\|_{M_3(\mathbb{N})} \le \|T_\varphi\|_{\mathrm{cb}}.$$

The converse inequality can be proved by an extension of the argument given above for Lemma 2.3. We simply mention that the c.b. norm of T_φ is the norm of the "same" multiplier acting from $H^1(S_1)$ to $H^1(S_1[\ell_\infty])$. Here $S_1[\ell_\infty] = S_1 \otimes^\wedge \ell_\infty$; moreover, if we are given g, h in the unit ball of S_2 (= the Hilbert–Schmidt class), and operators y_k in the unit ball of $B(\ell_2)$, then the sequence of products $(gy_kh)_{k\ge 0}$ defines an element of the unit ball of $S_1[\ell_\infty]$, and conversely any element of the unit ball is of this form. We leave the details to the reader. See the proof of Theorem 3.3 below for more on this point. ■

Remark 3.2. More generally, let $d \ge 2$. We denote by $\otimes_h$ the Haagerup tensor product for which we refer the reader to either [CS], [BP1] or [P6]. Let

$$L(\infty, d) = (\ell_1 \otimes_h \ldots \otimes_h \ell_1)^* \quad (d \text{ times}).$$

Consider φ in $M_d(\mathbb{N})$. Let $\Phi : \mathbb{N} \to L(\infty, d-2)$ be defined by

$$\Phi(n) = \sum \varphi(n + i_1 + \ldots + i_{d-2}) e_{i_1} \otimes \ldots \otimes e_{i_{d-2}} \in L(\infty, d-2)$$

(note that this series is meant only in the weak-$*$-sense). Now, consider again the multiplier defined in (3.1),

$$T_\varphi : H^1 \to H^1(L(\infty, d-2)).$$

The same argument as above shows that

$$\|\varphi\|_{M_d(\mathbb{N})} = \|T_\varphi\|_{\mathrm{cb}}.$$

Remarks. (i) It is easy to check that $\|\varphi\|_{M_3(\mathbb{N})} \le 1$ iff the mapping (generalized Schur multiplier) taking $e_{ij} \otimes U_k$ to $\varphi(i+k+j)e_{ij}$ extends to a complete contraction from $B(\ell_2) \otimes_{\min} \ell_1$ to $B(\ell_2)$ (or from $K(\ell_2) \otimes_{\min} \ell_1$ to $K(\ell_2)$).

(ii) Since $\|T_\varphi\|_{\mathrm{cb}} \ge \|T_\varphi\|$, Lemma 2.3 is a corollary of Theorem 3.1.

(iii) The multivariable Schur products have been studied in [ER3] mainly with the group case in mind. Our arguments can be viewed as a variation on the same theme.

(iv) The preceding proof is reminiscent of the main point in [J].

One unpleasant feature of Theorem 3.1 is that it is not obvious from it that $M_3(\mathbb{N})$ is a Banach algebra for the pointwise product, although this is clear from the definition of $M_3(\mathbb{N})$. However, this "defect" is corrected in the next statement, for which we use the following notation: Let $x = (x_k)_{k\ge 0}$ be a sequence in $H^1(S_1)$. We write $|||(x_k)||| < c$ if there are factorizations $x_k = g y_k h$ with $g, h \in H^2(S_2)$, $y_k \in B(\ell_2)$ and

$$\|g\|_{H^2(S_2)} \sup_k \|y_k\|_{B(\ell_2)} \|h\|_{H^2(S_2)} < c.$$

Then we set

$$|||(x_k)||| = \inf\{c \mid |||(x_k)||| < c\}.$$

This norm is nothing but an explicit description of the norm in the space $H^1 \otimes^\wedge S_1 \otimes^\wedge \ell_\infty$.

THEOREM 3.3. *For any $\varphi : \mathbb{N} \to \mathbb{C}$ we define*

$$|||\varphi|||_{\mathcal{M}_3} = \sup\{|||(\bar z^k M_\varphi(z^k x_k))_{k\ge 0}|||\}$$

where the supremum runs over all sequences $(x_k)_{k\ge 0}$ in $H^1(S_1)$ with $|||(x_k)||| < 1$. Then

$$\|\varphi\|_{M_3(\mathbb{N})} = |||\varphi|||_{\mathcal{M}_3}.$$

Proof. Assume $\|\varphi\|_{M_3(\mathbb{N})} < 1$ so that (2.3) and (2.2) hold. Let $x_k = g y_k h$ as above. Then

$$\begin{aligned} M_\varphi(z^k x_k) &= \sum_{n\ge 0} z^{k+n} \varphi(k+n)(g y_k h)^\wedge(n) = \sum_{i,j} z^k z^i z^j \langle \xi_i, T_k \eta_j\rangle g_i y_k h_j \\ &= \Big(\sum_i \xi_i \otimes g_i z^i\Big)(z^k T_k \otimes y_k)\Big(\sum_j \eta_j \otimes z^j h_j\Big). \end{aligned}$$

Hence

$$\begin{aligned} &|||(\bar z^k M_\varphi(z^k x_k))||| \\ &\quad \le \Big\|\sum \xi_i \otimes g_i z^i\Big\|_{H^2(\ell_2^*\otimes_2 S_2)} \sup_k \|T_k \otimes y_k\| \Big\|\sum \eta_j \otimes z^j h_j\Big\|_{H^2(\ell_2\otimes_2 S_2)} < 1. \end{aligned}$$

This shows that $\|\varphi\|_{M_3(\mathbb{N})} \ge \|\varphi\|_{\mathcal{M}_3}$. For the converse, note that, if $\|\varphi\|_{\mathcal{M}_3} \le 1$, then a fortiori we clearly have $\|(M_\varphi(z^k x))\|_{L^1\otimes^\wedge S^1 \otimes^\wedge \ell_\infty} \le 1$ for any x in the unit ball of $H^1(S_1)$, and therefore (see [P6, Lemma 1.7, p. 23]) $\|T_\varphi : H^1 \to H^1(\ell_\infty)\|_{\mathrm{cb}} \le 1$. Thus we conclude by Theorem 3.1 that $\|\varphi\|_{M_3(\mathbb{N})} \le \|\varphi\|_{\mathcal{M}_3}$.

REMARK. Theorem 3.3 shows that if we denote by $\mathcal{M}_\varphi : H^1 \otimes^\wedge \ell_\infty \to H^1 \otimes^\wedge \ell_\infty$ the mapping taking $z^n \otimes \alpha$ ($n \ge 0$, $\alpha = (\alpha_k)_{k\ge 0} \in \ell_\infty$) to

$z^n \otimes (\varphi(n+k)\alpha_k)_{k\ge 0}$, then

$$\|\varphi\|_{M_3(\mathbb{N})} = \|\mathcal{M}_\varphi\|_{\mathrm{cb}}.$$

REMARK. At this point, we are unable to prove that $M_3(\mathbb{N}) \neq M_4(\mathbb{N})$. The main difficulty is the lack of a "good" sufficient condition for $\varphi \in M_3(\mathbb{N})$ analogous to Lemma 2.4.

Final remarks on shift-boundedness. The notion of shift-bounded multiplier obviously makes sense also for multipliers from H^p to H^q $(0 < p, q \le \infty)$. More generally, if M_φ is a multiplier which is bounded from $L_p(\mathbb{T})$ to $L_q(\mathbb{T})$, we will say that it is shift-bounded if, for any x in $L_p(\mathbb{T})$, the "maximal function" (two-sided this time)

$$\sup_{k\in\mathbb{Z}} |M_\varphi(z^k x)|$$

is in $L_q(\mathbb{T})$. This definition has an obvious extension to more general function spaces than $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$, for instance it makes sense also if $L_q(\mathbb{T})$ is replaced by the so-called weak-L_q space which we denote by $L_{q,\infty}(\mathbb{T})$.

By the Nikishin–Maurey theorems (see [M]), any shift-bounded multiplier from $L_1(\mathbb{T})$ to $L_{1,\infty}(\mathbb{T})$ is automatically shift-bounded from $L_2(\mathbb{T})$ to $L_{2,\infty}(\mathbb{T})$, and hence by interpolation it is shift-bounded from $L_p(\mathbb{T})$ to $L_p(\mathbb{T})$ for all $1 < p < 2$. A similar result holds for multipliers on the corresponding Hardy spaces.

In passing, it is amusing to observe that Carleson's celebrated theorem on the a.s. convergence of Fourier series in $L_2(\mathbb{T})$ is essentially equivalent to the assertion that the Hilbert transform is shift-bounded from $L_2(\mathbb{T})$ to $L_{2,\infty}(\mathbb{T})$ (and actually it is known [Hu] to be shift-bounded from $L_p(\mathbb{T})$ to $L_p(\mathbb{T})$ for all $1 < p < \infty$).

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