COLLOQUIUM MATHEMATICUM

VOL. 88 2001 NO. 1

SELFINJECTIVE ALGEBRAS OF EUCLIDEAN TYPE WITH ALMOST REGULAR NONPERIODIC AUSLANDER–REITEN COMPONENTS

BY

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Abstract. We give a complete description of finite-dimensional selfinjective algebras of Euclidean tilted type over an algebraically closed field whose all nonperiodic Auslander–Reiten components are almost regular. In particular, we describe the tame selfinjective finite-dimensional algebras whose all nonperiodic Auslander–Reiten components are almost regular and generalized standard.

Introduction. Throughout the paper K denotes a fixed algebraically closed field. By an algebra we mean a finite-dimensional K-algebra (associative, with an identity). For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\mathrm{Hom}_K(-, K)$, by Γ_A the Auslander–Reiten quiver of A, by Γ_A^{s} the stable Auslander–Reiten quiver of A, obtained from Γ_A by removing the nonstable vertices and arrows attached to them, and by τ_A and τ_A^- the Auslander–Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively. We shall identify an indecomposable module from $\operatorname{mod} A$ with the corresponding vertex of Γ_A. A connected component $\mathcal{C}$ of Γ_A which coincides with its stable part $\mathcal{C}^{\mathrm{s}}$ is said to be *regular*. An algebra A is called *tame* if the indecomposable A-modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. If there is a common bound for the numbers of one-parameter families then the algebra A is said to be *domestic*. Moreover, if n is the smallest common bound for the numbers of one-parameter families then the algebra is called *n-parametric*. Finally, by a *Dynkin* (respectively, *Euclidean*) *quiver* we mean a quiver Δ without oriented cycles whose underlying graph $\overline{\Delta}$ is a Dynkin graph $\mathbb{A}_m$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$ (respectively, a Euclidean graph $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$) (for the definition of Dynkin and Euclidean graphs see [22]).

An algebra A is called *selfinjective* if all projective modules in $\operatorname{mod} A$ are injective. The classical examples of selfinjective algebras are blocks of group algebras KG of finite groups G, and more generally Hopf algebras.

2000 *Mathematics Subject Classification*: 16D50, 16G20, 16G60, 16G70.

An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra [16] (locally finite-dimensional, without identity)

$$\widehat{B} = \begin{bmatrix} \ddots & \ddots & & & & 0 & \\ & Q_{i-1} & B_{i-1} & & & & \\ & & Q_i & B_i & & & \\ & & & Q_{i+1} & B_{i+1} & & \\ & 0 & & & \ddots & \ddots & \end{bmatrix}$$

of an algebra B, where $B_i = B$ and $Q_i = {}_BD(B)_B$ for all $i \in \mathbb{Z}$, the algebras B_i are placed on the main diagonal of $\widehat{B}$, all the remaining entries are zero, the matrices in $\widehat{B}$ have only finitely many nonzero elements, addition is the usual addition of matrices, multiplication is induced from the B-bimodule structure of $D(B)$ and the zero map $D(B) \otimes_B D(B) \to D(B)$, and G is an admissible group of K-linear automorphisms of $\widehat{B}$ (the induced action of G on the isomorphism classes of indecomposable projective $\widehat{B}$-modules is free and has a finite number of orbits). Denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$ shifting B_i to B_{i+1} and Q_i to Q_{i+1} for all $i \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \ltimes D(B)$ of B by $D(B)$. We note that if B is of finite global dimension then the stable module category $\underline{\mathrm{mod}}\,\widehat{B}$ of $\mathrm{mod}\,\widehat{B}$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\,B)$ of bounded complexes over $\mathrm{mod}\,B$ [14].

Let Δ be a finite connected quiver without oriented cycles, H the path algebra $K\Delta$ of Δ, T a tilting H-module and $B = \mathrm{End}_H(T)$ the associated tilted algebra of type Δ. An algebra A of the form $\widehat{B}/G$, where G is an admissible group of K-linear automorphisms of $\widehat{B}$, is called a *selfinjective algebra* of tilted type Δ. It is known that $A = \widehat{B}/G$ is representation-finite (respectively, representation-infinite tame) if and only if Δ is a Dynkin quiver [16] (respectively, Δ is a Euclidean quiver [1], [24]). Moreover, if A is a representation-infinite selfinjective algebra of tilted type Δ then Γ_A^{s} admits at least one component of the form $\mathbb{Z}\Delta$ [1], [10]. A connected component $\mathcal{C}$ of Γ_A with $\mathcal{C}^{\mathrm{s}} = \mathbb{Z}\Delta$ for a Euclidean quiver Δ is said to be a *component of Euclidean type*. It has been proved in [11, Section 4] that if G is a finite group and Λ is a block of KG, then $\Gamma_\Lambda^{\mathrm{s}}$ admits a component of the form $\mathbb{Z}\Delta$ (for a finite quiver Δ without oriented cycles) if and only if Γ_Λ admits a component of Euclidean type, and if and only if K is of characteristic 2 and the defect groups of Λ are Klein 4-groups. Moreover, if this is the case then Λ is Morita equivalent to a selfinjective algebra of Euclidean tilted type $\widetilde{\mathbb{A}}_1$ or $\widetilde{\mathbb{A}}_5$.The following characterization of selfinjective algebras of Euclidean tilted type has been established in [24]: *A basic connected selfinjective algebra A is of Euclidean tilted type if and only if A is representation-infinite domestic and admits a simply connected Galois covering*. It has also been conjectured by the second named author that every basic connected selfinjective algebra Λ whose Auslander–Reiten quiver admits a component of Euclidean type is a deformation (in the sense of algebraic geometry) of a selfinjective algebra of Euclidean tilted type. We refer to [1], [19], [24] for more details on selfinjective algebras of Euclidean tilted type.

In the paper we are interested in the following problem. It is known (see [3], [22]) that if A is a selfinjective algebra then Γ_A (respectively, Γ_A^{s}) does not contain a component of the form $\mathbb{Z}\Delta$ with Δ a Euclidean (respectively, Dynkin) quiver. We call a connected component $\mathcal{C}$ of the Auslander–Reiten quiver Γ_A of an algebra A *almost regular* if its stable part $\mathcal{C}^{\mathrm{s}}$ is obtained from $\mathcal{C}$ by removing exactly one projective-injective module. The main aim of the paper is to describe the selfinjective algebras of Euclidean tilted type whose nonperiodic components are all almost regular. Moreover, applying results established by the second named author and K. Yamagata [26], [27], we obtain a complete description of all tame selfinjective algebras which admit nonperiodic Auslander–Reiten components, and whose nonperiodic Auslander–Reiten components are all almost regular and generalized standard (in the sense of [25]).

The paper is organized as follows. In Section 1 we formulate our main results. Section 2 is devoted to the description of some two-parametric tilted algebras of extended Euclidean type playing a fundamental role in our investigations. In Section 3 we complete the proofs of our main results. Finally, in Section 4 we present several examples illustrating our considerations.

For basic background on the topics considered here we refer to [2], [9], [12], [22] [23]. For terminology and notation used in the paper and not introduced here we refer to [22].

The paper has been written during the authors' stay at the University of Bielefeld, supported by SFB 343 (first named author) and the Alexander von Humboldt Foundation (second named author). The research of both authors has also been supported by the Polish Scientific Grant KBN No. 2PO3A 012 14.

1. The main results. Let B be a tilted algebra of Euclidean type Δ. Then the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ is of the form

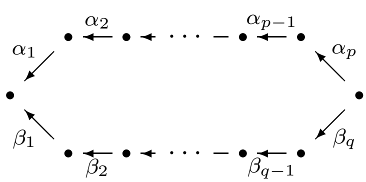
$$\Gamma_{\widehat{B}} = \bigvee_{p\in\mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_p),$$

where, for each $p \in \mathbb{Z}$, $\mathcal{X}_p$ is a component with the stable part $\mathbb{Z}\Delta$, $\mathcal{R}_p$ is a $\mathbb{P}_1(K)$-family of components whose stable parts are stable tubes, and

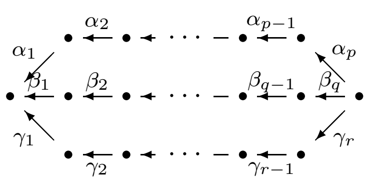
$\nu_{\widehat{B}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$ and $\nu_{\widehat{B}}(\mathcal{R}_p) = \mathcal{R}_{p+2}$ (see [1]). Then a K-linear automorphism ψ of $\widehat{B}$ is said to be *positive* (respectively, *strictly positive*) if $\psi(\mathcal{X}_0) = \mathcal{X}_q$ for some $q \geq 0$ (respectively, $q > 0$). Let G be an admissible group of K-linear automorphisms of $\widehat{B}$. It is shown in [24, (2.9)] that G is infinite cyclic and generated by a positive automorphism ψ of $\widehat{B}$. Moreover, there exists a positive integer m such that $\psi(\mathcal{X}_p) = \mathcal{X}_{p+m}$ and $\psi(\mathcal{R}_p) = \mathcal{R}_{p+m}$ for any $p \in \mathbb{Z}$. Consider the Galois covering $F^B : \widehat{B} \to \widehat{B}/G$ (in the sense of [13]) and the associated push-down functor $F_\lambda^B : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G$. Then it follows from [1] that $\widehat{B}$ is locally support-finite, and consequently the functor F_λ^B is dense and preserves Auslander–Reiten sequences (see [8, (2.5)]). Thus $\Gamma_{\widehat{B}/G}$ is obtained from $\Gamma_{\widehat{B}}$ by identifying (via F_λ^B) $\mathcal{X}_p$ with $\mathcal{X}_{p+m}$ and $\mathcal{R}_p$ with $\mathcal{R}_{p+m}$ for all $p \in \mathbb{Z}$, that is

$$\Gamma_{\widehat{B}/G} = F_\lambda^B(\mathcal{X}_0) \vee F_\lambda^B(\mathcal{R}_0) \vee \ldots \vee F_\lambda^B(\mathcal{X}_{m-1}) \vee F_\lambda^B(\mathcal{R}_{m-1}).$$

We note that there exists a representation-infinite tilted algebra B' of the same Euclidean type Δ such that $\widehat{B} \simeq \widehat{B}'$ [1, Sections 2 and 3]. Moreover, the class of representation-infinite tilted algebras of Euclidean type is formed by all domestic tubular extensions and domestic tubular coextensions of tame concealed algebras (see [22, (4.9)]). A special class of tame concealed algebras is formed by the following canonical algebras of Euclidean type: the path algebras $\Lambda(p,q)$, $1 \leq p \leq q$, of the quiver

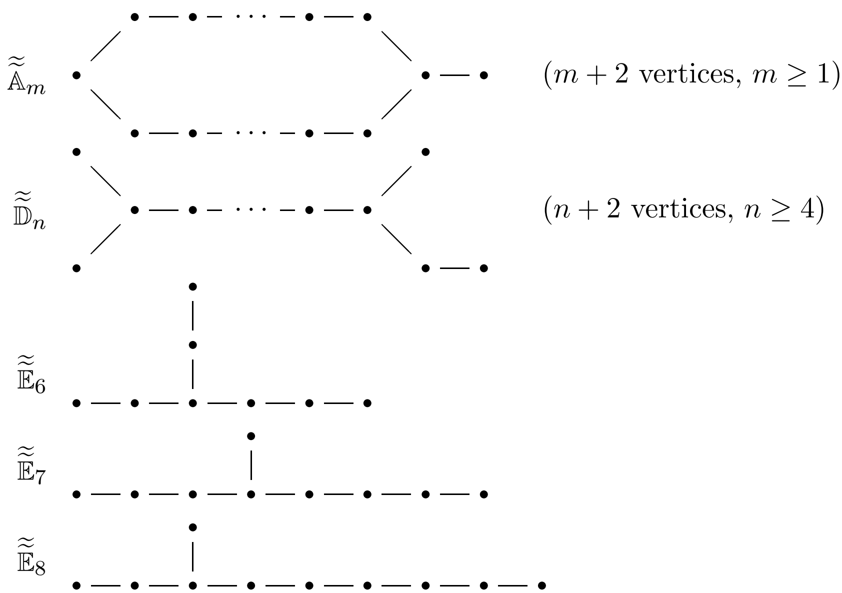


and the bound quiver algebras $\Lambda(p,q,r) = K\Delta(p,q,r)/I(p,q,r)$, $(p,q,r) = (2,3,3), (2,3,4), (2,3,5), (2,2,n-2)$ with $n \geq 4$, where $\Delta(p,q,r)$ is the quiver



and $I(p,q,r)$ is the ideal in $K\Delta(p,q,r)$ generated by $\alpha_p \ldots \alpha_1 + \beta_q \ldots \beta_1 + \gamma_r \ldots \gamma_1$. The algebras $\Lambda(p,q)$, $\Lambda(2,2,n-2)$, $\Lambda(2,3,3)$, $\Lambda(2,3,4)$, $\Lambda(2,3,5)$ are tame concealed algebras of type $\widetilde{\mathbb{A}}_{p+q}$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, respectively. Finally, by an extended Euclidean quiver we mean an arbitrary quiver Δ without oriented cycles whose underlying graph $\bar{\Delta}$ is one of the extended

Euclidean graphs



The main aim of the paper is to prove the following facts.

THEOREM 1. *Let A be a basic connected selfinjective algebra. The following conditions are equivalent*:

(i) *A is of Euclidean tilted type and every nonperiodic component of Γ_A is almost regular.*

(ii) *A is of Euclidean tilted type and every nonperiodic component of Γ_A has a section of extended Euclidean type.*

(iii) *A is of tilted type and every nonperiodic component of Γ_A is almost regular and admits a section of extended Euclidean type.*

(iv) *$A \simeq \widehat{B}/G$, where B is a domestic tubular extension of a canonical tame concealed algebra Λ and G is an admissible infinite cyclic group of K-linear automorphisms of $\widehat{B}$.*

(v) *$A \simeq \widehat{B}/G$, where B is a domestic tubular coextension of a canonical tame concealed algebra Λ and G is an admissible infinite cyclic group of K-linear automorphisms of $\widehat{B}$.*

Here, by a *section* in a component $\mathcal{C}$ of Γ_A we mean a full convex subquiver Σ without oriented cycles of $\mathcal{C}$ and intersecting each τ_A-orbit of $\mathcal{C}$ exactly once. Moreover, following [25] a translation subquiver $\mathcal{C}$ of Γ_A is called *generalized standard* if for any modules X and Y in $\mathcal{C}$ the infinite radical $\operatorname{rad}^\infty(X, Y)$ is zero. As a consequence of the above theorem and [27, Theorem 5.5] we get the following result.

THEOREM 2. *Let A be a basic connected selfinjective algebra. The following conditions are equivalent*:

(i) *A is of Euclidean tilted type, Γ_A has at least two nonperiodic components, and every nonperiodic component of Γ_A is almost regular.*

(ii) *Every nonperiodic component of Γ_A is almost regular and Γ_A admits a generalized standard left stable full translation subquiver of Euclidean type which is closed under predecessors in Γ_A.*

(iii) *Every nonperiodic component of Γ_A is almost regular and Γ_A admits a generalized standard right stable full translation subquiver of Euclidean type which is closed under successors in Γ_A.*

(iv) *$A \simeq \widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a domestic tubular extension* (*respectively, coextension*) *of a canonical tame concealed algebra Λ and φ is a positive automorphism of $\widehat{B}$.*

Recall that the *annihilator* $\operatorname{ann}\mathcal{C}$ of a component $\mathcal{C}$ in Γ_A is the intersection of the annihilators of all modules in $\mathcal{C}$. In the next section we shall define families $\mathbf{A}(u,v,w,t)$, $\mathbf{D}'(u,w)$, $\mathbf{D}''(u,w)$, $\mathbf{E}'(u,w,t)$, $\mathbf{E}''(u,v,w,t)$ of two-parametric tilted algebras of extended Euclidean type. Then we have the following consequence of the above theorem, [27, Corollary 5.6], and of the facts proved in the next section.

THEOREM 3. *Let A be a basic connected selfinjective algebra. The following conditions are equivalent*:

(i) *A is of Euclidean tilted type, Γ_A has at least three nonperiodic components, and every nonperiodic component of Γ_A is almost regular.*

(ii) *A is tame, Γ_A has at least one nonperiodic indecomposable nonprojective module, and every nonperiodic component of Γ_A is almost regular and generalized standard.*

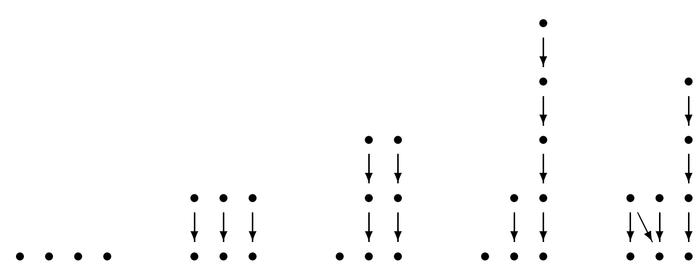
(iii) *Γ_A contains a nonperiodic component $\mathcal{C}$ such that $D = A/\operatorname{ann}\mathcal{C}$ is one of the tilted algebras $\mathbf{A}(u,v,w,t)$, $u,v,w,t \geq 1$, $\mathbf{D}'(u,w)$, $u,w \geq 1$, $u+w \geq 3$, $\mathbf{D}''(u,w)$, $u \geq 2$, $w \geq 1$, $\mathbf{E}'(u,w,t)$, $u,w \geq 1$, $(u+w,t) = (4,3),(5,3),(6,3),(4,4),(4,5)$ or $\mathbf{E}''(u,v,w,t)$, $u,v \geq 2$, $w,t \geq 1$, $(u+w, v+t) = (4,4),(4,5),(4,6)$ and $\mathcal{C}$ is the unique connecting component of Γ_D.*

(iv) *$A = \widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a domestic tubular extension* (*respectively, coextension*) *of a canonical tame concealed algebra Λ and φ is a strictly positive automorphism of $\widehat{B}$.*

2. Tilted algebras of extended Euclidean type. The aim of this section is to describe a class of two-parametric tilted algebras with almost regular connecting component, playing the crucial role in our investigations.

We shall assume (without loss of generality) that all algebras considered are basic. For such an algebra B, there exists an isomorphism $B \simeq KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_B$ of B and I is an admissible ideal in KQ. Equivalently, $B = KQ/I$ may be considered as a K-category whose object class is the set Q_0 of vertices of Q, and the set of morphisms $B(x, y)$ from x to y is the quotient of the K-space $KQ(x, y)$ formed by the linear combinations of paths in Q from x to y, by the subspace $I(x, y) = KQ(x, y) \cap I$. An algebra B with Q_B having no oriented cycles is said to be *triangular*. A full subcategory C of B is said to be *convex* if any path in Q_B with source and target in Q_C lies entirely in Q_C. Finally, for a vertex x of Q_B, we shall denote by $P(x) = P_B(x)$ and $I(x) = I_B(x)$ the indecomposable projective B-module and the indecomposable injective B-module corresponding to x, respectively.

In order to obtain conditions for one-point extensions and coextensions of algebras to be representation-infinite we shall use vector space category methods (see [21], [22], [23]). Recall that a *vector space category* $\mathbb{K}$ is an additive K-category together with a faithful functor $|-| : \mathbb{K} \to \operatorname{mod} K$. Given a vector space category $\mathbb{K}$, the subspace category $\mathcal{U}(\mathbb{K})$ is defined as follows: its objects are triples (V, X, φ), where V is an object of $\operatorname{mod} K$, X is an object of $\mathbb{K}$, and $\varphi : V \to |X|$ is a K-linear morphism. A morphism from (V, X, φ) to (V', X', φ) in $\mathcal{U}(\mathbb{K})$ is given by a pair (α, β), where $\alpha : V \to V'$ is a K-linear map, $\beta : X \to X'$ is a morphism in $\mathbb{K}$, and $|\beta|\varphi = \varphi'\alpha$. An example of a vector space category is provided by the additive category $\operatorname{add} KS$ of the incidence category KS of a partially ordered set S (see [21, p. 202]). For a finite partially ordered set S we have the following well-known result due to M. Kleiner [18] (see also [21, p. 204]). *The subspace category* $\mathcal{U}(\operatorname{add} KS)$ *is of finite representation-type if and only if* S *contains no full subcategory whose Hasse diagram has one of the following forms*:



$(1,1,1,1)$ $(2,2,2)$ $(1,3,3)$ $(1,2,5)$ $(N,4)$

We recall from [21] that a *one-point extension* of an algebra B by a B-module M is the algebra

$$B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $B[M]$ contains the quiver of B as a convex subquiver and there is an additional (extension) vertex which is a source. We shall denote by $\operatorname{Hom}(M, \operatorname{mod} B)$ the vector space category whose objects are the spaces of B-homomorphisms $\operatorname{Hom}_B(M, Y)$, where Y are modules from $\operatorname{mod} B$ without indecomposable direct summand Y' with $\operatorname{Hom}_B(M, Y') = 0$, the morphisms $\operatorname{Hom}_B(M, Y) \to \operatorname{Hom}_B(M, Z)$ are of the form $\operatorname{Hom}_B(M, f)$, where $f : Y \to Z$ is a morphism in $\operatorname{mod} B$, and $|\operatorname{Hom}_B(M, Y)|$ is just the underlying K-space of $\operatorname{Hom}_B(M, Y)$. Then there is a bijective correspondence between the isoclasses of indecomposable objects in $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} B))$ and the isoclasses of indecomposable $B[M]$-module which are not B-modules which assigns to the isoclass of an indecomposable object $(V, \operatorname{Hom}_B(M, Y), \varphi)$ in $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} B))$ the isoclass of an indecomposable $B[M]$-module defined on the vector space $V \oplus Y$ with the multiplication given by

$$[\, v \quad y \,] \begin{bmatrix} \lambda & m \\ 0 & b \end{bmatrix} = [\, \lambda v \quad \varphi(v)(m) + yb \,]$$

for $v \in V$, $y \in Y$, $\lambda \in K$, $m \in M$ and $b \in B$ (see [23, Chapter 17] for details). Note that if we have $\dim_K \operatorname{Hom}_B(M, N) \leq 1$ for any indecomposable module N from $\operatorname{mod} B$ then $\operatorname{Hom}(M, \operatorname{mod} B)$ is the additive category $\operatorname{add} KS$ of the incidence category KS of the partially ordered set (poset) S whose vertices are the isoclasses $[\operatorname{Hom}_B(M, Y)]$ of indecomposable objects $\operatorname{Hom}_B(M, Y)$ of $\operatorname{Hom}(M, \operatorname{mod} B)$, and $[\operatorname{Hom}_B(M, Y)] \leq [\operatorname{Hom}_B(M, Z)]$ if and only if there is a nonzero morphism $\operatorname{Hom}_B(M, Y) \to \operatorname{Hom}_B(M, Z)$ in $\operatorname{Hom}(M, \operatorname{mod} B)$. If $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} B))$ is of finite representation type and $\operatorname{End}_B(Y) \simeq K$ for any indecomposable B-module Y with $\operatorname{Hom}_B(M, Y) \neq 0$ then $\operatorname{Hom}(M, \operatorname{mod} B)$ is of the poset type described above. Moreover, if $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} B))$ is tame and $\dim_K \operatorname{Hom}_B(M, Y) \geq 2$ for an indecomposable module Y in $\operatorname{mod} B$ with $\operatorname{End}_B(Y) \simeq K$, then $\dim_K \operatorname{Hom}_B(M, Y) = 2$ and for any indecomposable object $\operatorname{Hom}_B(M, Z)$ of $\operatorname{Hom}(M, \operatorname{mod} B)$ we have a nonzero morphism in $\operatorname{Hom}(M, \operatorname{mod} B)$ from $\operatorname{Hom}_B(M, Y)$ to $\operatorname{Hom}_B(M, Z)$ or from $\operatorname{Hom}_B(M, Z)$ to $\operatorname{Hom}_B(M, Y)$ (see [21, (2.4)]).

In the paper, we will denote indecomposable objects $\operatorname{Hom}_B(M, Y)$ with $\dim_K \operatorname{Hom}_B(M, Y) = 1$ by $\bullet$, and with $\dim_K \operatorname{Hom}_B(M, Y) = 2$ by $\blacksquare$, respectively. Let B be representation-finite. We say that a family of B-modules $Y_1, \dots, Y_t$ *makes the one-point extension* $B[M]$ *representation-infinite* provided $\operatorname{Hom}_B(M, Y_1), \dots, \operatorname{Hom}_B(M, Y_t)$ are indecomposable objects of $\operatorname{Hom}(M, \operatorname{mod} B)$ and the full subcategory of $\operatorname{Hom}(M, \operatorname{mod} B)$ given by these objects is a minimal representation-infinite full subcategory of $\operatorname{Hom}(M, \operatorname{mod} B)$. Dually, one defines the one-point coextension

$$[M]B = \begin{bmatrix} B & D(M) \\ 0 & K \end{bmatrix}$$

of B by M and the corresponding vector space category $\operatorname{Hom}(\operatorname{mod} B, M) \simeq \operatorname{Hom}(D(M), \operatorname{mod} B^{\mathrm{op}})$ whose subspace category describes (as above) the indecomposable modules in $\operatorname{mod}[M]B$ which are not B-modules.

Let B be a tilted algebra of Dynkin type, and M be a B-module. The extension $B[M]$ will be called *canonical* if $B[M]$ is a tilted algebra of Euclidean type corresponding to the type of B, that is, if B is tilted of type $\mathbb{A}_m$, $m \geq 1$, then $B[M]$ is tilted of type $\widetilde{\mathbb{A}}_m$, if B is of type $\mathbb{D}_n$, $n \geq 4$, then $B[M]$ is of type $\widetilde{\mathbb{D}}_n$, and, if B is of type $\mathbb{E}_p$, $p = 6, 7, 8$, then $B[M]$ is of type $\widetilde{\mathbb{E}}_p$. Dually, we define canonical coextensions of tilted algebras of Dynkin type.

Let B be a tilted algebra of Dynkin type and X be an indecomposable B-module. The function $f : \Gamma' \to \mathbb{Z}$, where Γ' is a full translation subquiver of Γ_B formed by all successors of X, given by the formula $f(Y) = \dim_K \operatorname{Hom}_B(X, Y)$ for each successor Y of X, is an additive function on Γ' called the *additive function starting at* X. Similarly, an additive function g on the full translation subquiver Γ'' of Γ_B given by the set of all predecessors of X, defined by $g(Y) = \dim_K \operatorname{Hom}_B(Y, X)$ for each predecessor Y of X, will be called the *additive function stopping at* X. Note that both the additive function starting at X and the additive function stopping at X cannot take negative values. The additive functions stopping and starting at a given vertex of Γ_B have been discussed in [12, (6.5)].

The path $X_0 \to X_1 \to \ldots \to X_{k-1} \to X_k$ in the Auslander–Reiten quiver Γ_B of an algebra B is called *sectional* if $\tau_B X_i \not\simeq X_{i-2}$, $i = 2, \ldots, k$. For a B-module M we denote by $\Sigma^+(M)$ the full subquiver of Γ_B given by the indecomposable modules X such that there exists a path in Γ_B from an indecomposable direct summand of M to X and each path in Γ_B for an indecomposable direct summand of M to X is sectional. Similarly using the path ending at indecomposable direct summands of M we define the quiver $\Sigma^-_B(M)$. If $\Sigma^+_B(M)$ and $\Sigma^-_B(M)$ are sections in a component of Γ_B then $\Sigma^-_B(M)$ is the opposite quiver of $\Sigma^+_B(M)$, which is obtained from $\Sigma^+_B(M)$ by reversing the directions of arrows.

Let $B_1, \ldots, B_k$ be connected algebras of finite-representation type and for each $i = 1, \ldots, k$ let M_i be a B_i-module. Put $B = \prod_{i=1}^k B_i$ and $M = \bigoplus_{i=1}^k M_i$. Note that $\Sigma^-_B(M)$ is the disjoint union of $\Sigma^-_{B_i}(M_i)$, $i = 1, \ldots, k$. We say the $\Sigma^-_B(M)$ form a *family of sections* in Γ_B if $\Sigma^-_{B_i}(M_i)$ is a section in Γ_{B_i} for each $i = 1, \ldots, k$. The analogous definition may be formulated for $\Sigma^+_B(M)$.

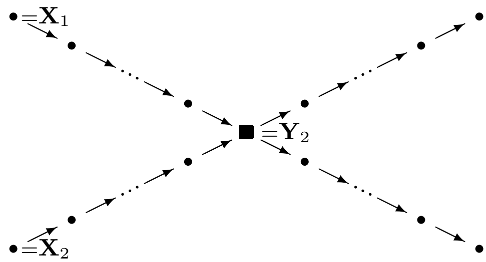
Proposition 2.1. *Let B be a product of tilted algebras of Dynkin type and M be a B-module. Assume that $\Sigma^+_B(M)$ and $\Sigma^-_B(M)$ form a family* *of sections in Γ_B and $B[M]$ and $[M]B$ are representation-infinite tilted algebras of Euclidean type. Then there exist vertices $x_1, \ldots, x_t$ in Q_B such that the modules $I(x_1), \ldots, I(x_t)$ form a unique (up to isomorphism) family of indecomposable modules making $B[M]$ representation-infinite and $P(x_1), \ldots, P(x_t)$ form a unique family of indecomposable modules making $[M]B$ representation-infinite. Moreover, $t = 1$ if and only if B is connected and both the extension $B[M]$ and the coextension $[M]B$ are canonical.*

Proof. Since $\Sigma_B^-(M)$ form a family of sections in Γ_B, $\Sigma_{B[M]}^-(P)$ is a section in the preprojective component of $\Gamma_{B[M]}$, where P is the unique indecomposable projective $B[M]$-module which is not a B-module. Note that $\Sigma_{B[M]}^-(P)$ is obtained from $\Sigma_B^-(M)$ by adding the vertex P and arrows connecting indecomposable direct summands of M with P. The number of arrows connecting a given indecomposable direct summand X of M with P is equal to the multiplicity of X in M. Similarly, $\Sigma_{[M]B}^+(I)$ is a section in the preinjective component of $\Gamma_{[M]B}$ for the unique indecomposable injective $[M]B$-module which is not a B-module and is obtained form $\Sigma_B^+(M)$ by adding I and arrows connecting I with indecomposable direct summands of M. Since $\Sigma_B^-(M)$ is the opposite quiver of $\Sigma_B^+(M)$, the quiver $\Sigma_{B[M]}^-(P)$ is the opposite quiver of $\Sigma_{[M]B}^+(I)$. In particular, if $B[M]$ is a tilted algebra of type Δ_1 and $[M]B$ is a tilted algebra of type Δ_2 then Δ_1 and Δ_2 are Euclidean quivers of the same type. Moreover the extension $B[M]$ is canonical if and only if the coextension $[M]B$ is canonical.

It follows that we have to consider all possible quivers Δ of Euclidean type with the unique source x, where Δ is the type of $[M]B$. Then B is a product of tilted algebras of types $\Sigma_1, \ldots, \Sigma_k$, where $\Sigma_1, \ldots, \Sigma_k$ are connected components of the quiver obtained from Δ by removing the vertex x. The indecomposable direct summands of M correspond to arrows starting at x and two distinct indecomposable direct summands of M are isomorphic if and only if the corresponding arrows have the same end point.

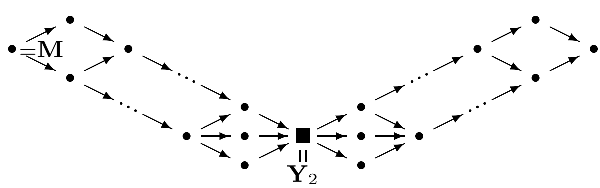
Assume first that Δ is a quiver of type $\widetilde{\mathbb{A}}_m$, $m \geq 1$. Then B has to be tilted of type $\mathbb{A}_m$ and M is a direct sum of two indecomposable modules X_1 and X_2. If $m = 1$, then $B = K$ and $X_1 \simeq X_2 \simeq K$. The vector space category $\operatorname{Hom}(M, \operatorname{mod} B)$ has the unique indecomposable object $\mathbf{X}_1 \simeq \mathbf{X}_2$, where we use the convention that for each X in $\operatorname{mod} B$ we denote by $\mathbf{X}$ the corresponding object $\operatorname{Hom}_B(M, X)$ (respectively, $\operatorname{Hom}_B(X, M)$) in $\operatorname{Hom}(M, \operatorname{mod} B)$ (respectively, in $\operatorname{Hom}(\operatorname{mod} B, M)$). Note that $\dim_K \mathbf{X}_1 = 2$, $X_1 = I(x_1) = P(x_1)$ for the unique vertex x_1 of Q_B, and the extension $B[M]$ is canonical.

Let $m > 1$. Then $X_1 \not\simeq X_2$ and $\operatorname{ind} \operatorname{Hom}(M, \operatorname{mod} B)$ is a full subcategory of



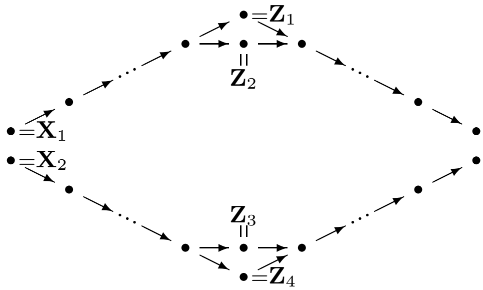
In particular, it must contain $\mathbf{Y}_2$. Similarly, the dual vector space category $\operatorname{Hom}(\operatorname{mod} B, M)$ must contain an indecomposable object $\mathbf{Y}_1$ with $\dim_K \operatorname{Hom}_B(Y_1, M) = 2$. If we consider the additive function starting at Y_1, then we conclude that Y_2 must be injective. Similarly using the additive function stopping at Y_2, we infer that Y_1 is projective. Hence there exists a vertex x_1 of Q_B such that $Y_1 = P(x_1)$. Since $\dim_K \operatorname{Hom}_B(P(x_1), M) = \dim_K \operatorname{Hom}_B(M, I(x_1))$, we get $\dim_K \operatorname{Hom}_B(M, I(x_1)) = 2$. Thus $Y_2 = I(x_1)$, because Y_2 is a unique indecomposable B-module Z with the property $\dim_K \operatorname{Hom}_B(M, Z) = 2$. It is obvious that the extension $B[M]$ is canonical.

Assume Δ is of type $\widetilde{\mathbb{D}}_n$, $n \geq 4$. Suppose first that the extension $B[M]$ is canonical, so B is of type $\mathbb{D}_n$, M is indecomposable and the category $\operatorname{ind}\operatorname{Hom}(M, \operatorname{mod} B)$ is a full subcategory of



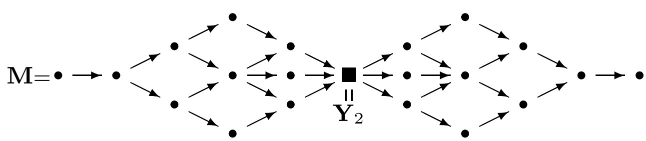
Then $\operatorname{Hom}(M, \operatorname{mod} B)$ must contain $\mathbf{Y}_2 = \operatorname{Hom}_B(M, Y_2)$, and dually the vector space category $\operatorname{Hom}(\operatorname{mod} B, M)$ must contain an indecomposable two-dimensional object $\mathbf{Y}_1$. A simple analysis of the additive functions starting at Y_1 and stopping at Y_2 shows that Y_1 is projective and Y_2 is injective. It follows again that $Y_1 = P(x_1)$ and $Y_2 = I(x_1)$ for a vertex x_1 of Q_B.

Assume now that $B[M]$ is not canonical. Then we have to consider some cases: B can be a product of four algebras of type $\mathbb{A}_1$, for $n = 4$; a product of two algebras of type $\mathbb{A}_1$ and an algebra of type $\mathbb{A}_3$, for $n = 5$; a product of two algebras of type $\mathbb{A}_1$ and one algebra of type $\mathbb{D}_{n-2}$, for $n \geq 6$; a product of two algebras of type $\mathbb{A}_3$, for $n = 6$; a product of an algebra of type $\mathbb{A}_3$ and an algebra of type $\mathbb{D}_{n-3}$, for $n \geq 7$; and finally, a product of an algebra of type $\mathbb{D}_p$ and an algebra of type $\mathbb{D}_q$, where $p, q \geq 4$, and $p + q = n$, for $n \geq 8$. We only deal with the last case, since the arguments in the remaining cases are similar. In this situation, M is the direct sum of two indecomposable modules X_1 and X_2 and the category $\operatorname{ind}\operatorname{Hom}(M, \operatorname{mod} B)$ is a full subcategory of



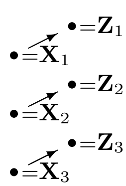
Thus the category $\operatorname{Hom}(M, \operatorname{mod} B)$ must contain the objects $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$. Similarly, the vector space category $\operatorname{Hom}(\operatorname{mod} B, M)$ has to contain four pairwise incomparable one-dimensional objects $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ and $\mathbf{Y}_4$. Considering the additive functions starting at Y_1, Y_2, Y_3 and Y_4, we see that the modules Z_1, Z_2, Z_3 and Z_4 are injective. Similarly, we prove that Y_1, Y_2, Y_3 and Y_4 are projective, using the additive functions stopping at Z_1, Z_2, Z_3, Z_4. Hence, there exist vertices x_1, x_2, x_3 and x_4 of Q_B such that $Y_i = P(x_i)$, $i = 1, \ldots, 4$. It follows that the objects $\operatorname{Hom}_B(M, I(x_i))$, $i = 1, \ldots, 4$, are one-dimensional and pairwise incomparable, and hence, by the uniqueness of the family making $B[M]$ representation-infinite, there exists a permutation σ of $\{1, 2, 3, 4\}$ with $Z_i \simeq I(x_{\sigma(i)})$, for $i = 1, \ldots, 4$.

Now suppose that Δ is of type $\widetilde{\mathbb{E}}_6$. We again start with the canonical case. This means that B is tilted of type $\mathbb{E}_6$, M is indecomposable, and $\operatorname{ind} \operatorname{Hom}(M, \operatorname{mod} B)$ is a full subcategory of



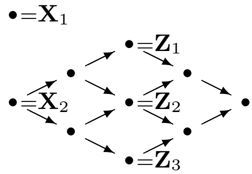
Hence the category $\operatorname{Hom}(M, \operatorname{mod} B)$ must contain the two-dimensional object $\mathbf{Y}_2$. Dually, the vector space category $\operatorname{Hom}(\operatorname{mod} B, M)$ contains an indecomposable two-dimensional object $\mathbf{Y}_1$. Similarly, it follows that $Y_1 = P(x_1)$ and $Y_2 = I(x_1)$ for a vertex x_1 of Q_B.

There are two possibilities when $B[M]$ is not canonical. Namely, the algebra B can be a product of three tilted algebras of type $\mathbb{A}_2$, or a product of an algebra of type $\mathbb{A}_1$ and an algebra of type $\mathbb{A}_5$. In the first situation, M is the direct sum of three pairwise nonisomorphic indecomposable modules X_1, X_2 and X_3, and $\operatorname{ind} \operatorname{Hom}(M, \operatorname{mod} B)$ is a full subcategory of



Thus the vector space category $\mathrm{Hom}(M, \mathrm{mod}\, B)$ has to contain the objects $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$. Similarly, the category $\mathrm{Hom}(\mathrm{mod}\, B, M)$ must contain the objects $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, where $Y_i = \tau_B Z_i$, $i = 1, 2, 3$. Then it follows that Y_1, Y_2, Y_3 are projective, X_1, X_2, X_3 are projective-injective, and Z_1, Z_2, Z_3 are injective. In particular, these are unique families of indecomposable B-modules making $B[M]$ and $[M]B$ representation-infinite. Obviously $Y_1 = P(x_1)$, $Y_2 = P(x_2)$, $Y_3 = P(x_3)$, $X_1 = P(x_4) = I(x_1)$, $X_2 = P(x_5) = I(x_2)$, $X_3 = P(x_6) = I(x_3)$, $Z_1 = I(x_4)$, $Z_2 = I(x_5)$, $Z_3 = I(x_6)$, where x_1, x_2, x_3, x_4, x_5, x_6 constitute the set of all vertices of Q_B.

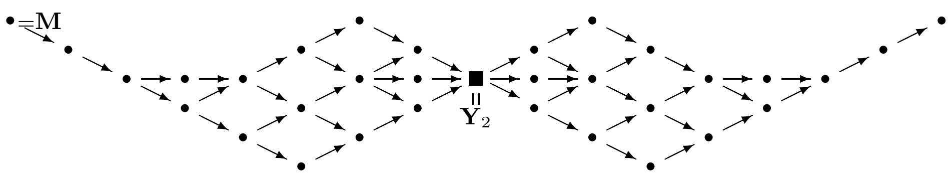
In the second case, M is the direct sum of two indecomposable modules X_1 and X_2, and $\mathrm{ind}\,\mathrm{Hom}(M, \mathrm{mod}\, B)$ is a full subcategory of



so has to contain the objects $\mathbf{X}_1$, $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$. Dually, $\mathrm{Hom}(\mathrm{mod}\, B, M)$ has to contain four pairwise incomparable one-dimensional objects $\mathbf{X}_1$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$. It is now straightforward that X_1, Z_1, Z_2, Z_3 have to be injective, X_1, Y_1, Y_2, Y_3 projective, and there exist vertices x_1, x_2, x_3 and x_4 of Q_B, and a permutation σ of $\{2, 3, 4\}$, such that $X_1 = P(x_1) = I(x_1)$, $Y_i = P(x_{i+1})$ and $Z_i = I(x_{\sigma(i+1)})$, for $i = 1, 2, 3$.

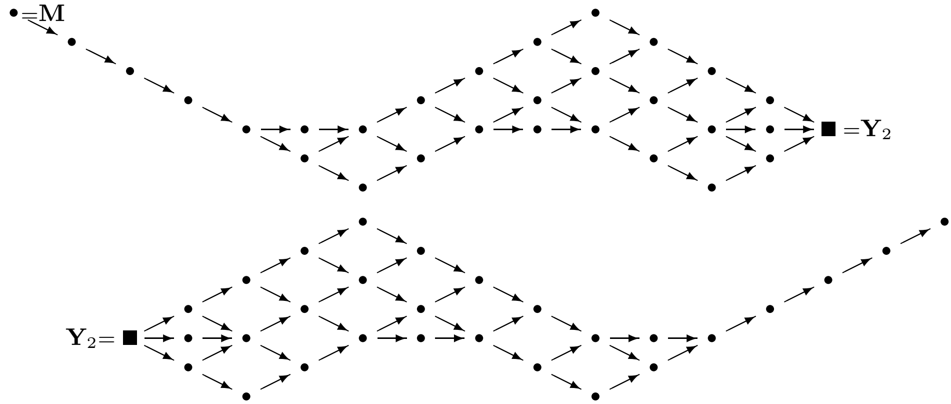
It remains to consider fourteen cases relating to Δ of type $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$. We do not present all of them here, since the considerations are very similar to the ones above. For the sake of our future proofs we only exhibit the canonical cases.

Assume that Δ is of type $\widetilde{\mathbb{E}}_7$ and the extension $B[M]$ is canonical. Then B is of type $\mathbb{E}_7$, M is indecomposable and $\mathrm{ind}\,\mathrm{Hom}(M, \mathrm{mod}\, B)$ is a full subcategory of



Thus $\mathrm{ind}\,\mathrm{Hom}(M, \mathrm{mod}\, B)$ has to contain the object $\mathbf{Y}_2$, and the dual category $\mathrm{Hom}(\mathrm{mod}\, B, M)$ must contain an indecomposable two-dimensional object $\mathbf{Y}_1$. It follows that $Y_1 = P(x_1)$ and $Y_2 = I(x_1)$ for a vertex x_1 of Q_B.

Finally, if Δ is of type $\widetilde{\mathbb{E}}_8$ and the extension $B[M]$ is canonical, then B is of type $\mathbb{E}_8$, M is indecomposable, and $\mathrm{ind}\,\mathrm{Hom}(M, \mathrm{mod}\, B)$ is a full subcategory of



Then $\operatorname{Hom}(M, \operatorname{mod} B)$ must contain the object $\mathbf{Y}_2$, and the $\operatorname{Hom}(\operatorname{mod} B, M)$ must contain an indecomposable two-dimensional object $\mathbf{Y}_1$. As usual, we conclude that Y_1 is projective, Y_2 is injective, and there exists a vertex x_1 of Q_B such that $Y_1 = P(x_1)$ and $Y_2 = I(x_1)$. ∎

We have the following useful corollary.

Corollary 2.2. *Let B be a product of tilted algebras of Dynkin type, and M be a B-module such that $B[M]$ and $[M]B$ are representation-infinite tilted algebras of Euclidean type. In addition, let a and b denote the corresponding extension and coextension vertices, and $x_1, \dots, x_t$ be vertices of Q_B such that the modules $I(x_1), \dots, I(x_t)$ form the unique family of indecomposable B-modules making $B[M]$ representation-infinite, and the modules $P(x_1), \dots, P(x_t)$ constitute the unique family of indecomposable B-modules making $[M]B$ representation-infinite. Then the smallest convex subcategory of $B[M]$ containing the vertices a, $x_1, \dots, x_t$ is a unique tame concealed convex subcategory of $B[M]$, while the smallest convex subcategory of $[M]B$ containing the vertices b, $x_1, \dots, x_t$ is a unique tame concealed convex subcategory of $[M]B$.*

Proof. Let C be the smallest convex subcategory of $B[M]$ containing the vertices a, $x_1, \dots, x_t$, and B' be the algebra obtained from C by deleting the vertex a. It is clear that $C = B'[M']$, where $M' = M|_{B'}$ is the restriction of M to B'. Since B' is a convex subcategory of B, the modules $I(x_1), \dots, I(x_t)$ remain injective when restricted to B'. Hence, the full subcategory of $\operatorname{Hom}(M', \operatorname{mod} B')$ formed by $\operatorname{Hom}_{B'}(M', I(x_1)|_{B'}), \dots, \operatorname{Hom}_{B'}(M', I(x_t)|_{B'})$ is isomorphic to the full subcategory of $\operatorname{Hom}(M, \operatorname{mod} B)$ formed by the objects $\operatorname{Hom}_B(M, I(x_1)), \dots, \operatorname{Hom}_B(M, I(x_t))$, thus C is representation-infinite. On the other hand, since $I(x_1), \dots, I(x_t)$ form the unique family of indecomposable modules making $B[M]$ representation-infinite, it follows that each proper convex subcategory of C is representation-finite, and hence C is tame concealed. Similarly, we prove the second assertion. ■

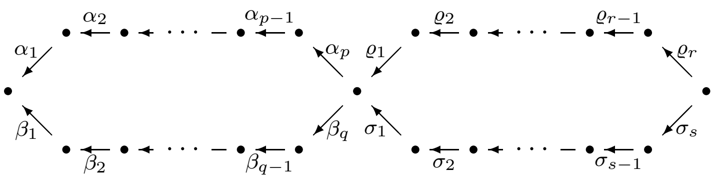
Let D be a two-parametric tilted algebra with almost regular connecting component $\mathcal{C}$. Denote by V the unique indecomposable projective-injective D-module. There exist vertices a and b of Q_D such that $V = P_D(a) = I_D(b)$. Since the component $\mathcal{C}$ is almost regular it follows that $\Sigma_D^+(\operatorname{rad} V)$ and $\Sigma_D^-(V/\operatorname{soc} V)$ are sections in $\mathcal{C}$. Let D_1 and D_2 denote the algebras obtained from D by deleting the vertices a and b, respectively. We call them *left end* and *right end algebras* of D, respectively. Then it follows that D_1 and D_2 are representation-infinite tilted algebras of Euclidean type. Moreover, $\operatorname{rad} V$ is a D_1-module, $D = D_1[\operatorname{rad} V]$ and $\Sigma_{D_1}^+(\operatorname{rad} V)$ is a section in the preinjective component of Γ_{D_2}. Similarly $V/\operatorname{soc} V$ is a D_2-module, $D = [V/\operatorname{soc} V]D_2$ and $\Sigma_{D_2}^-(V/\operatorname{soc} V)$ is a section in Γ_{D_2}. Let B denotes the algebra obtained from D by deleting both vertices a and b. Then B is a product of tilted algebras of Dynkin type, and $D_1 = [M]B$ and $D_2 = B[M]$, where $M = \operatorname{rad} V/\operatorname{soc} V$. Since $\Sigma_B^+(M)$ is obtained from $\Sigma_{D_1}^+(\operatorname{rad} V)$ by deleting the vertex $\operatorname{rad} V$, it follows that $\Sigma_B^+(M)$ form a family of sections in Γ_B. Similarly $\Sigma_B^-(M)$ form a family of sections in Γ_B. If both the extension $B[M]$ and the coextension $[M]B$ are canonical, the algebra D is said to be a *canonical two-parametric tilted algebra*. Note that each canonical two-parametric tilted algebra is of extended Euclidean type, but in Section 4 we exhibit an example of a two-parametric tilted algebra of extended Euclidean type with almost regular connecting component which is not canonical.

The following lemma allows us to reduce our considerations to the sincere case.

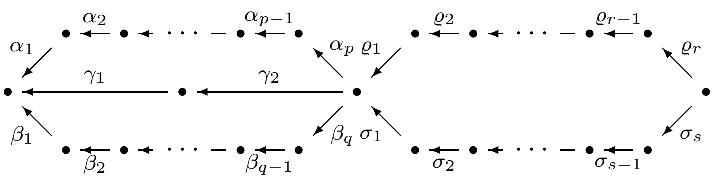
LEMMA 2.3. *Let D be a two-parametric tilted algebra with almost regular connecting component and V be the unique indecomposable projective-injective D-module. Then* $\operatorname{supp} V$ *is a two-parametric tilted algebra with almost regular connecting component. Moreover*, $\operatorname{supp} V$ *is canonical if and only if D is canonical.*

Proof. It is known (see [22, p. 375]) that $\operatorname{supp} V$ is a tilted algebra. Hence, it is enough to show that $\operatorname{supp} V$ is two-parametric. Recall also that $\operatorname{supp} V$ is a convex subcategory of D (see [6]). Let a, b, B and M be as above. It follows from Proposition 2.1 that there exist vertices $x_1, \ldots, x_t$ such that $I_B(x_1), \ldots, I_B(x_t)$ is a unique family of indecomposable modules making $B[M]$ representation-infinite, while $P_B(x_1), \ldots, P_B(x_t)$ is a unique family of indecomposable modules making $[M]B$ representation-infinite. In particular, the vertices $x_1, \ldots, x_t$ belong to $\operatorname{supp} V$. Of course, a and b also belong to $\operatorname{supp} V$, hence it follows from Corollary 2.2 that both the unique tame concealed subcategory of $B[M]$ and the unique tame concealed subcategory of $[M]B$ are contained in $\operatorname{supp} V$. Hence $\operatorname{supp} V$ is two-parametric, and clearly its connecting component is almost regular. The last assertion also follows from Proposition 2.1, since D (respectively, $\operatorname{supp} V$) is canonical if and only if $t = 1$. ∎

Now we introduce some families of algebras. First, for each quadruple of positive integers p, q, r, s, we denote by $\Delta_1(p, q, r, s)$ the quiver



and for $p, q \geq 2$, $r, s \geq 1$, by $\Delta_2(p, q, r, s)$ the quiver



We define $A(p, q, r, s)$, $p, q, r, s \geq 1$, as the bound quiver algebra

$$A(p, q, r, s) = K\Delta_1(p, q, r, s)/I(p, q, r, s),$$

where $I(p, q, r, s)$ is the ideal generated by $\varrho_1\alpha_p$, $\sigma_1\beta_q$, $\sigma_s \ldots \sigma_1\alpha_p \ldots \alpha_1 - \varrho_r \ldots \varrho_1\beta_q \ldots \beta_1$. For $p, r \geq 1$, $p + r \geq 3$, we consider the bound quiver algebra

$$D'(p, r) = K\Delta_1(p, 2, r, 2)/I'(p, q),$$

where the ideal $I'(p, q)$ is generated by the elements $\varrho_1\alpha_p$, $\sigma_1\alpha_p \ldots \alpha_1 - \sigma_1\beta_2\beta_1$, $\varrho_r \ldots \varrho_1\beta_2 - \sigma_2\sigma_1\beta_2$. For $p \geq 2$, $r \geq 1$, we consider the bound quiver algebra of the form

$$D''(p, r) = K\Delta_2(p, 2, r, 1)/I''(p, r),$$

where the generators of $I''(p, r)$ are the elements $\alpha_p \ldots \alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1$, $\varrho_1\alpha_p$, $\varrho_r \ldots \varrho_1\gamma_2 - \sigma_1\gamma_2$, $\sigma_1\beta_2$. For $p, r \geq 1$ and s such that $(p + r, s) = (4, 3), (5, 3), (6, 3), (4, 4), (4, 5)$, we define

$$E'(p, r, s) = K\Delta_1(p, 2, r, s)/J'(p, r, s),$$

where $J'(p, r, s)$ is generated by $\varrho_1\alpha_p$, $\varrho_r \ldots \varrho_1\beta_2 - \sigma_s \ldots \sigma_1\beta_2$, $\sigma_1\alpha_p \ldots \alpha_1 - \sigma_1\beta_2\beta_1$. Finally, for $p, q \geq 2$, and $r, s \geq 1$ such that $(p + r, q + s) = (4, 4), (4, 5), (4, 6)$, we have the algebra

$$E''(p,q,r,s) = K\Delta_2(p,q,r,s)/J''(p,q,r,s),$$

with the generators of the ideal $J''(p,q,r,s)$ being the elements $\alpha_p \dots \alpha_1 + \beta_q \dots \beta_1 + \gamma_2\gamma_1$, $\varrho_1\alpha_p$, $\sigma_1\beta_q$, $\varrho_r \dots \varrho_1\gamma_2 - \sigma_s \dots \sigma_1\gamma_2$.

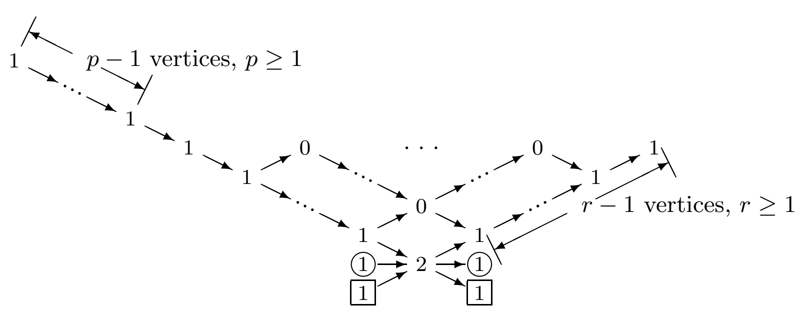
We are now able to prove the following fact.

Proposition 2.4. *Let D be a canonical two-parametric tilted algebra with a sincere projective-injective indecomposable module. Then D or D^{op} belongs to one of the families $A(p,q,r,s)$, $p,q,r,s \geq 1$, $D'(p,r)$, $p,r \geq 1$, $p+r \geq 3$, $D''(p,r)$, $p \geq 2$, $r \geq 1$, $E'(p,r,s)$, $p,r \geq 1$, $(p+r,s) = (4,3),(5,3),(6,3),(4,4),(4,5)$, $E''(p,q,r,s)$, $p,q \geq 2$, $r,s \geq 1$, $(p+r,q+s) = (4,4),(4,5),(4,6)$.*

Proof. Let a and b be the vertices of Q_D such that $V = P(a) = I(b)$ is the unique indecomposable projective-injective D-module. If we denote by B the algebra obtained from D by deleting the vertices a and b, and $M = \operatorname{rad} V/\operatorname{soc} V$, then it follows from the definition of canonical two-parametric tilted algebras that both the extension $B[M]$ and the coextension $[M]B$ are canonical. In particular, B is a tilted algebra of Dynkin type.

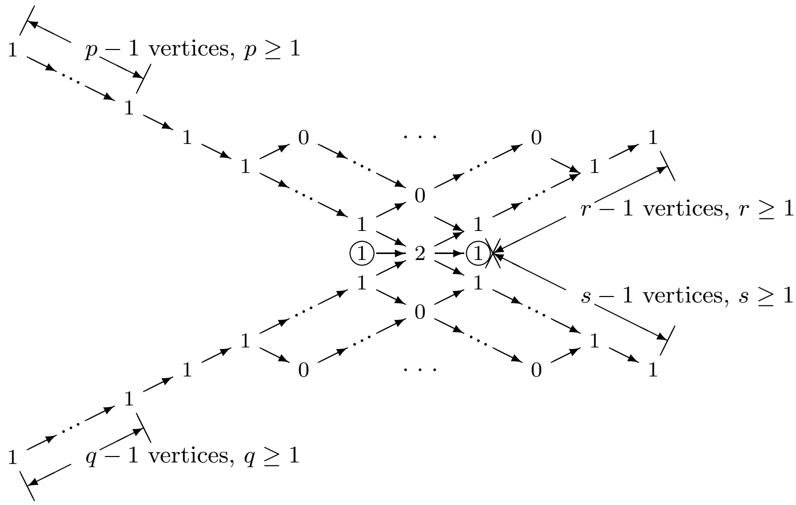
Assume first that B is tilted of type $\mathbb{A}_m$, $m \geq 1$. Then D is tilted of type $\widetilde{\widetilde{\mathbb{A}}}_m$ and it follows (see [20]) that D is of the form $A(p,q,r,s)$ for some positive integers p, q, r, s.

Let now B be of type $\mathbb{D}_n$, $n \geq 4$. It follows from the proof of Proposition 2.1 and our assumption that V is sincere, that the convex subquiver of Γ_B containing all the injective modules is a convex subquiver of



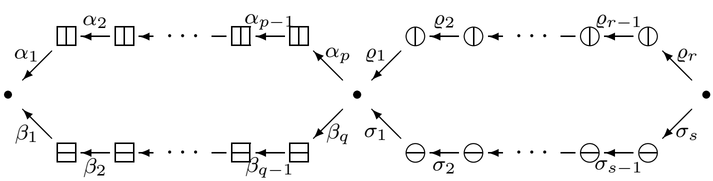
containing exactly one of the encircled vertices and exactly one of the vertices in squares, and all the vertices without frames. Here and later on, each vertex is replaced by the dimension of the space of B-homomorphisms from M to the corresponding module. Hence, it follows that D, or its opposite algebra, belongs to one of the families $D'(p,r)$ or $D''(p,r)$.

Suppose that B is tilted of one of the types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. Again, we infer from the proof of Proposition 2.1 that the convex subquiver of Γ_B containing all the injective modules is a convex subquiver of

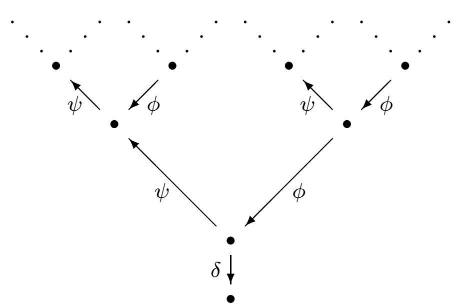


containing exactly one encircled vertex and all the vertices which are not encircled. In addition the numbers p, q, r, s must satisfy the condition $(p+r, q+s) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}$, and then D, or its opposite algebra, belongs to one of the families $E'(p,r,s)$ and $E''(p,q,r,s)$. ∎

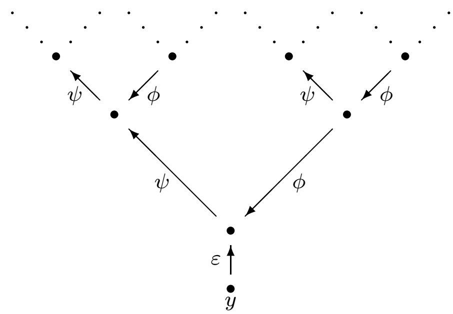
Now we define the families $\mathbf{A}(u,v,w,t)$, $\mathbf{D}'(u,w)$, $\mathbf{D}''(u,w)$, $\mathbf{E}'(u,w,t)$, $\mathbf{E}''(u,v,w,t)$ of algebras announced in Section 1. First, we denote by $\boldsymbol{\Delta}_1(u,v,w,t)$, $u,v,w,t \geq 1$, the quiver



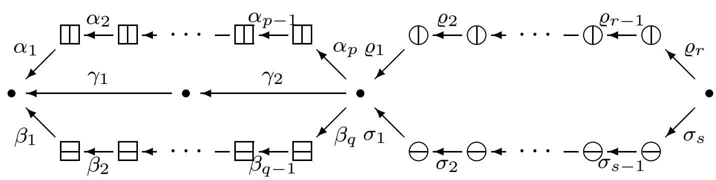
where each vertex denoted by the squares ⊟ and ⎅ can be replaced by a connected subquiver of the following infinite quiver:



containing the vertex x, and the original vertex is then identified with the vertex x. Similarly, each vertex denoted by the circles ⊖ and ⦶ can be replaced by a connected subquiver of



containing the vertex y, and the original vertex is then identified with the vertex y. The integer u is the sum of p and the number of arrows in the quivers denoted by squares with a horizontal bar and the integer v is the sum of q and the number of arrows in the quivers denoted by squares with a vertical bar. Analogously, using r (respectively, s) and quivers denoted by circles with a horizontal (respectively, vertical) bar we define w (respectively, t). Similarly we define the quiver $\boldsymbol{\Delta}_2(u, v, w, t)$, $u, v \geq 2$, $w, t \geq 1$, as



where the meaning of semigraphical sings is the same as above. We also define the numbers u, v, w, t in the same way as before.

The families of algebras $\mathbf{A}(u, v, w, t)$, $\mathbf{D}'(u, w)$, $\mathbf{D}''(u, w)$, $\mathbf{E}'(u, w, t)$ and $\mathbf{E}''(u, v, w, t)$ are defined in a similar way as $A(p, q, r, s)$, $D'(p, r)$, $D''(p, r)$, $E'(p, r, s)$ and $E''(p, q, r, s)$. Namely, for $u, v, w, t \geq 1$, we set

$$\mathbf{A}(u, v, w, t) = K\boldsymbol{\Delta}_1(u, v, w, t)/\mathbf{I}(u, v, w, t),$$

where $\mathbf{I}(u, v, w, t)$ is the ideal in $K\boldsymbol{\Delta}_1(u, v, w, t)$ generated by the elements $\varrho_1\alpha_p$, $\sigma_1\beta_q$, $\sigma_s \ldots \sigma_1\alpha_p \ldots \alpha_1 - \varrho_r \ldots \varrho_1\beta_q \ldots \beta_1$ and all the elements of the form $\delta\alpha_i$, $\delta\beta_i$, $\varrho_i\varepsilon$, $\sigma_i\varepsilon$ and $\phi\psi$. If $u, w \geq 1$, $u + w \geq 3$, we set

$$\mathbf{D}'(u, w) = K\boldsymbol{\Delta}_1(u, 2, w, 2)/\mathbf{I}'(u, w),$$

where the ideal $\mathbf{I}'(u, w)$ is generated by the elements $\varrho_1\alpha_p$, $\sigma_1\alpha_p \ldots \alpha_1 - \sigma_1\beta_2\beta_1$, $\varrho_r \ldots \varrho_1\beta_2 - \sigma_2\sigma_1\beta_2$ and all the elements $\delta\alpha_i$, $\varrho_i\varepsilon$ and $\phi\psi$. Further,

$$\mathbf{D}''(u, w) = K\boldsymbol{\Delta}_2(u, 2, w, 1)/\mathbf{I}''(u, w),$$

$u \geq 2$, $w \geq 1$, and $\mathbf{I}''(u, w)$ is generated by $\alpha_p \ldots \alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1$, $\varrho_1\alpha_p$, $\sigma_1\beta_q$, $\varrho_r \ldots \varrho_1\gamma_1 - \sigma_1\gamma_1$ and all the elements $\delta\alpha_i$, $\varrho_i\varepsilon$ and $\phi\psi$. For $u, w \geq 1$

and t such that $(u+w,t)=(4,3),(5,3),(6,3),(4,4),(4,5)$, we define

$$\mathbf{E}'(u,w,t)=K\boldsymbol{\Delta}_1(u,2,w,t)/\mathbf{J}'(u,w,t),$$

where the generators of $\mathbf{J}'(u,w,t)$ are $\varrho_1\alpha_p$, $\sigma_1\alpha_p\ldots\alpha_1-\sigma_1\beta_2\beta_1$, $\varrho_r\ldots\varrho_1\beta_2-\sigma_s\ldots\sigma_1\beta_2$ and all the elements of the form $\delta\alpha_i$, $\varrho_i\varepsilon$, $\sigma_i\varepsilon$ and $\phi\psi$. Finally, for $u,v\geq 2$ and $w,t\geq 1$ with $(u+w,v+t)=(4,4),(4,5),(4,6)$, we put

$$\mathbf{E}''(u,v,w,t)=K\boldsymbol{\Delta}_2(u,v,w,t)/\mathbf{J}''(u,v,w,t),$$

where the ideal $\mathbf{J}''(u,v,w,t)$ is generated by $\alpha_p\ldots\alpha_1+\beta_q\ldots\beta_1+\gamma_2\gamma_1$, $\varrho_1\alpha_p$, $\sigma_1\beta_q$, $\varrho_r\ldots\varrho_1\gamma_1-\sigma_s\ldots\sigma_1\gamma_1$ and all the elements of the form $\delta\alpha_i$, $\delta\beta_i$, $\varrho_i\varepsilon$, $\sigma_i\varepsilon$ and $\phi\psi$. We have the following classification of canonical two-parametric tilted algebras.

PROPOSITION 2.5. *An algebra D is a canonical two-parametric tilted algebra if and only if D or D^{op} is isomorphic to one of the algebras $\mathbf{A}(u,v,w,t)$, $u,v,w,t\geq 1$, $\mathbf{D}'(u,w)$, $u,w\geq 1$, $u+w\geq 3$, $\mathbf{D}''(u,w)$, $u\geq 2$, $w\geq 1$, $\mathbf{E}'(u,w,t)$, $u,w\geq 1$, where $(u+w,t)=(4,3),(5,3),(6,3),(4,4),(4,5)$ or $\mathbf{E}''(u,v,w,t)$, $u,v\geq 2$, $w,t\geq 1$, $(u+w,v+t)=(4,4),(4,5),(4,6)$.*

Proof. Let D be a canonical two-parametric tilted algebra and X be a unique indecomposable projective-injective D-module. It follows from Lemma 2.3 that $D'=\operatorname{supp}X$ is also a canonical two-parametric tilted algebra. Moreover, the left end algebra D_1 of D is a domestic tubular extension of its unique convex tame concealed subcategory C_1. However, C_1 is also a unique convex tame concealed subcategory of the left end algebra D_1' of D', and we conclude from Proposition 2.4 that C_1 is a canonical tame concealed algebra. Now the claim follows, since a complete description of domestic tubular extensions of canonical tame concealed algebras is a direct consequence of [22, (3.7), (4.7), (4.9)].

On the other hand, if D or D^{op} belongs to one of the above families, then obviously there exists a unique indecomposable projective-injective D-module V. Let a and b be the vertices of Q_D such that $V=P(a)=I(b)$. Then the algebra D_1 obtained from D by deleting the vertex a is a domestic tubular extension of a canonical tame concealed algebra. In particular, D_1 is tilted of Euclidean type, and this implies that the connecting component of Γ_D is almost regular and D_1 is the left end algebra of D. Moreover, $D_1=[M]B$, where $M=\operatorname{rad}V/\operatorname{soc}V$ and B is the algebra obtained from D_1 by deleting the vertex b. Notice that there exists a vertex x of Q_B such that $\dim_K\operatorname{Hom}_B(P_B(x),M)=2$, hence the coextension $[M]B$ is canonical by Proposition 2.1, since M is indecomposable. Thus D is a canonical two-parametric tilted algebra. ∎

We have the following consequence of the above proposition.

COROLLARY 2.6. *If B is a domestic tubular extension* (*respectively*, *coextension*) *of a canonical tame concealed algebra*, *then B is the left* (*respectively*, *right*) *end algebra of a canonical two-parametric tilted algebra.*

REMARK 2.7. An alternative proof of Proposition 2.5 can be done using the Bongartz–Happel–Vossieck list of tame concealed algebras [7], [15]. Namely, Proposition 2.1 and Corollary 2.2 imply that the convex tame concealed subcategories of canonical two-parametric tilted algebras are not Schurian, and hence have to be canonical. Thus we may classify all canonical two-parametric tilted algebras without considering the sincere case. However, since the Bongartz–Happel–Vossieck list is established using computer programs, we have decided to present here a longer but direct proof. Recall that an algebra C is called *Schurian* if $\dim_K \operatorname{Hom}_C(P_1, P_2) \leq 1$ for any indecomposable projective C-modules P_1 and P_2.

REMARK 2.8. Proposition 2.5 is extended in [4] to a complete description of tame tilted algebras with an almost regular connecting component.

REMARK 2.9. It is known that if X is a directing module over an algebra B then the support algebra D of X is a tilted algebra and X lies in a connecting component of $\mathit{\Gamma}_D$. We have proved in [5] that the affine variety $\mathrm{mod}_B(\mathbf{d})$ of all B-modules of dimension-vector $\mathbf{d} = \mathbf{dim}\, X$ is a complete intersection with at most two irreducible components, and $\mathrm{mod}_B(\mathbf{d})$ is not irreducible (respectively, is not normal) if and only if D is one of the algebras exhibited in Proposition 2.4.

3. Proofs of the main results. In order to prove our main results we need a more detailed structure of the module categories of repetitive algebras of tilted algebras of Euclidean type. For a triangular algebra B and a vertex i of Q_B we denote by $T_i^+ B$ the one-point extension $B[I_B(i)]$. If in addition i is a sink of Q_B then we denote by $S_i^+ B$ the full subcategory of $T_i^+ B$ formed by all objects except the object i, called the *reflection* of B at i [16]. The quiver of $S_i^+ B$, denoted by $\sigma_i^+ Q_B$, is obtained from Q_B by replacing the sink i be a source i' (the extension vertex of $B[I_B(i)]$). A reflection sequence of sinks $i_1, \ldots, i_t$ is a sequence of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \ldots \sigma_{i_1}^+ Q_B$ for any $1 \leq s \leq t$.

Let B be a domestic tubular extension of a tame concealed algebra C. It follows from [22, (4.9)] that the preinjective component of $\mathit{\Gamma}_B$ admits a Euclidean section $\mathit{\Delta}$ whose number of vertices is the number of vertices in Q_B. Applying [1, (4.3)] we conclude that there exists a reflection sequence of sinks $i_1, \ldots, i_r, i_{r+1}, \ldots, i_s, i_{s+1}, \ldots, i_t, i_{t+1}, \ldots, i_n$ in Q_B, where n is the rank of $K_0(B)$, $1 \leq r \leq s < t \leq n$, and the following statements hold:

(1) $B_1 = S^+_{i_r} \dots S^+_{i_1} B$ is a domestic tubular coextension of a tame concealed algebra C_1,

(2) $B_2 = S^+_{i_s} \dots S^+_{i_{r+1}} B_1$ is a domestic tubular extension of C_1, and $B_2 = B_1 = C_1$ for $r = s$,

(3) $B_3 = S^+_{i_t} \dots S^+_{i_{s+1}} B_2$ is a domestic tubular coextension of a tame concealed algebra C_2,

(4) $B_4 = S^+_{i_n} \dots S^+_{i_{t+1}} B_3$ is a domestic tubular extension of C_2, and $B_4 = B_3 = C_2$ for $t = n$,

(5) $C_2 \simeq \nu_{\widehat{B}}(C)$ and $B_4 \simeq \nu_{\widehat{B}}(B)$,

(6) $\Gamma_{\widehat{B}} = \bigvee_{p\in\mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_p)$, where $\mathcal{X}_{2q} = \nu^q_{\widehat{B}}(\mathcal{X}_0)$, $\mathcal{X}_{2q+1} = \nu^q_{\widehat{B}}(\mathcal{X}_1)$, $\mathcal{R}_{2q} = \nu^q_{\widehat{B}}(\mathcal{R}_0)$, $\mathcal{R}_{2q+1} = \nu^q_{\widehat{B}}(\mathcal{R}_1)$, for any $q \in \mathbb{Z}$,

(7) $\mathcal{X}_0$ is a nonperiodic component with the stable part $\mathbb{Z}\Delta$ and consists of $T^+_{i_r} \dots T^+_{i_1} B$-modules,

(8) $\mathcal{R}_0$ is a $\mathbb{P}_1(K)$-family of components whose stable parts are tubes and consist of $T^+_{i_s} \dots T^+_{i_{r+1}} B_1$-modules,

(9) $\mathcal{X}_1$ is a nonperiodic component with the stable part $\mathbb{Z}\Delta$ and consists of $T^+_{i_t} \dots T^+_{i_{s+1}} B_2$-modules,

(10) $\mathcal{R}_1$ is a $\mathbb{P}_1(K)$-family of components whose stable parts are tubes and consist of $T^s_{i_n} \dots T^s_{i_{t+1}} B_3$-modules.

Proof of Theorem 1. Let, as above, B be a domestic tubular extension of a tame concealed algebra C. Since $\mathcal{X}_{p+2} = \nu_{\widehat{B}}(\mathcal{X}_p)$ for $p \in \mathbb{Z}$, all nonperiodic components of $\Gamma_{\widehat{B}}$ are almost regular if and only if the components $\mathcal{X}_0$ and $\mathcal{X}_1$ are almost regular. Assume $\mathcal{X}_0$ is almost regular. Then B is the left end algebra of a two-parametric tilted algebra D with an almost regular connecting component. Moreover, C_1 is a convex tame concealed subcategory of the right end algebra of D. Observe now that the projective-injective modules in $\mathcal{X}_1$ consists of the indecomposable injective $\widehat{B}$-modules $I_{\widehat{B}}(i_{s+1}), \dots, I_{\widehat{B}}(i_t)$, $\{i_{s+1}, \dots, i_t\}$ is the set of common objects of C and B_2 and clearly it contains the set of common objects of C and C_1. Therefore, if $\mathcal{X}_1$ is almost regular then C and C_1 have at most one common object, and applying Proposition 2.1 and Corollary 2.2 we conclude that D is a canonical two-parametric tilted algebra and C is a canonical tame concealed algebra. On the other hand, if D is canonical then a direct application of the reflection procedure leading from B to $S^+_{i_n} \dots S^+_{i_1} B = \nu_{\widehat{B}}(B)$ shows that the injective envelope in $\operatorname{mod} \widehat{B}$ of the simple $\widehat{B}$-module given by the unique sink (respectively, source) of C is the unique indecomposable projective-injective module in $\mathcal{X}_0$ (respectively, in $\mathcal{X}_1$), and consequently all nonperiodic components of $\Gamma_{\widehat{B}}$ are almost regular. Hence, the statements (i) and (iv) are equivalent. The equivalence of (iv) and (v) follows from the fact that the repetitive algebras of B and $B_3 = S^+_{i_t} \dots S^+_{i_1} B$ are isomorphic (see [16]). The implication (iv)⇒(iii) follows from the fact that if B is a domestic tubular extension of a canonical tame concealed algebra Λ then B is the left end algebra of a canonical two-parametric tilted algebra, by Corollary 2.6. The implications (iii)⇒(ii) and (ii)⇒(i) are obvious. ■

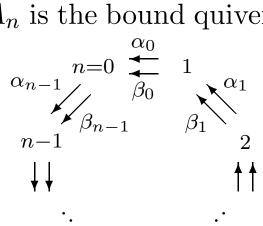
Proofs of Theorems 2 and 3. Let B be a domestic tubular extension of a tame concealed algebra C, G an admissible group of K-linear automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$. It follows from [24, (2.9)] and the description of $\Gamma_{\widehat{B}}$ presented above that Γ_A admits two (respectively, three) nonperiodic components if and only if G is infinite cyclic generated by $\varphi\nu_{\widehat{B}}$ for some positive (respectively, strictly positive) automorphism φ of $\widehat{B}$. Recall also that a tilted algebra B given by a tilting module without nonzero preinjective (respectively, preprojective) direct summand is tame if and only if B is of Euclidean type (see [17]). The required equivalences in Theorems 2 and 3 are then direct consequences of Theorem 1 and [27, Theorem 5.5 and Corollary 5.6]. ■

4. Examples. In this final section we present some examples illustrating our considerations.

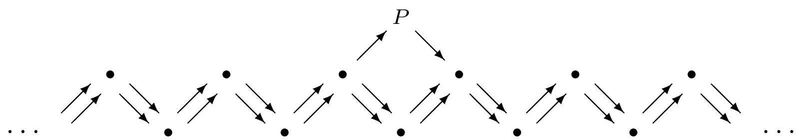
Example 4.1. Let B be the path algebra of the quiver $0 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 1$. Then the repetitive algebra (category) $\widehat{B}$ of B is the bound quiver category given by the quiver

$$\cdots \leftleftarrows -2 \underset{\beta_{-2}}{\overset{\alpha_{-2}}{\leftleftarrows}} -1 \underset{\beta_{-1}}{\overset{\alpha_{-1}}{\leftleftarrows}} 0 \underset{\beta_{0}}{\overset{\alpha_{0}}{\leftleftarrows}} 1 \underset{\beta_{1}}{\overset{\alpha_{1}}{\leftleftarrows}} 2 \underset{\beta_{2}}{\overset{\alpha_{2}}{\leftleftarrows}} 3 \leftleftarrows \cdots$$

and the ideal generated by $\alpha_{i+1}\alpha_i$, $\beta_{i+1}\beta_i$, and $\alpha_{i+1}\beta_i - \beta_{i+1}\alpha_i$ for all $i \in \mathbb{Z}$, and the Nakayama shift $\nu_{\widehat{B}}$ is given by $\nu_{\widehat{B}}(i) = i+2$, $\nu_{\widehat{B}}(\alpha_i) = \alpha_{i+2}$ and $\nu_{\widehat{B}}(\beta_i) = \beta_{i+2}$ for all $i \in \mathbb{Z}$. Moreover, we have the shift $\psi : \widehat{B} \to \widehat{B}$ given by $\psi(i) = i+1$, $\psi(\alpha_i) = \alpha_{i+1}$ and $\psi(\beta_i) = \beta_{i+1}$ for all $i \in \mathbb{Z}$, and clearly $\psi^2 = \nu_{\widehat{B}}$. For any positive integer n, consider the selfinjective algebra $A_n = \widehat{B}/(\psi^n)$. Note that A_n is the bound quiver algebra given by the quiver

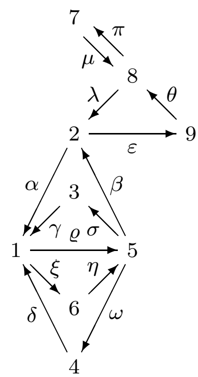


and the ideal generated by all elements $\alpha_{i+1}\alpha_i$, $\beta_{i+1}\beta_i$ and $\alpha_{i+1}\beta_i - \beta_{i+1}\alpha_i$ for $0 \le i \le n-1$, where $\alpha_n = \alpha_0$ and $\beta_n = \beta_0$. Then the Auslander–Reiten quiver Γ_{A_n} consists of n $\mathbb{P}_1(K)$-families of stable tubes of rank 1 and n nonperiodic components, each of them of the same shape

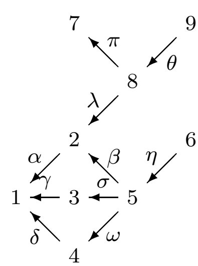


where P is one of the projective-injective modules $P(i)$, $0 \le i \le n-1$. Clearly, all nonperiodic components of Γ_{A_n} are almost regular.

EXAMPLE 4.2. Let A be the algebra given by the quiver

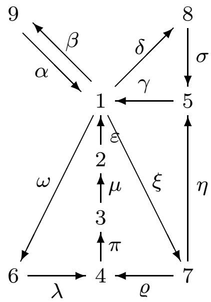


and the ideal generated by the elements $\beta\alpha + \sigma\gamma + \omega\delta$, $\beta\eta$, $\xi\eta\sigma - \varrho\sigma$, $\varrho\omega$, $\varrho\beta\alpha - \xi\eta\omega\delta$, $\alpha\xi$, $\delta\varrho$, $\gamma\xi\eta - \gamma\varrho$, $\alpha\varrho\beta - \varepsilon\theta\lambda$, $\mu\lambda$, $\lambda\alpha$, $\lambda\varepsilon\theta - \pi\mu$, $\theta\pi$. Then A is the trivial extension $B \ltimes D(B) = \widehat{B}/(\nu_{\widehat{B}})$ of the tilted algebra B of type $\widetilde{\mathbb{E}}_8$ given by the quiver

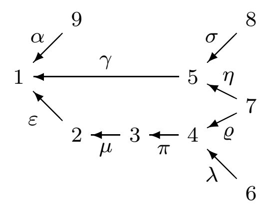


and the ideal generated by $\beta\alpha + \sigma\gamma + \omega\delta$, $\lambda\alpha$, $\theta\pi$, $\eta\beta$, which is a domestic tubular extension of the canonical tame concealed convex subcategory given by the vertices 1, 2, 3, 4, 5. It follows from [1] and our main results that Γ_A has exactly two nonperiodic components, both of them almost regular, one containing the projective module $P(1)$, and the other containing the projective module $P(5)$. Moreover, Γ_A has two $\mathbb{P}_1(K)$-families of quasi-tubes (the stable parts are tubes), one of them containing $P(2)$, $P(3)$, $P(4)$, $P(7)$, and the other containing $P(6)$, $P(8)$, $P(9)$.

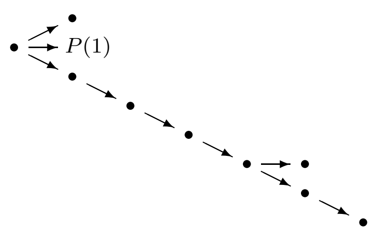
EXAMPLE 4.3. We now present an example of a selfinjective algebra of Euclidean tilted type whose Auslander–Reiten quiver admits a nonperiodic almost regular component (even with an extended Euclidean section) and also a nonperiodic component which is not almost regular. Let A be the bound quiver algebra given by the quiver



and the ideal generated by $\beta\alpha - \delta\sigma\gamma$, $\xi\eta\gamma - \omega\lambda\pi\mu\varepsilon$, $\delta\sigma - \xi\eta$, $\xi\varrho - \omega\lambda$, $\alpha\delta$, $\alpha\xi$, $\alpha\omega$, $\gamma\beta$, $\gamma\omega$, $\varepsilon\beta$, $\varepsilon\delta$. Then A is of the form $\widehat{B}/(\nu_{\widehat{B}}) = B \ltimes D(B)$, where B is the bound quiver algebra given by the quiver



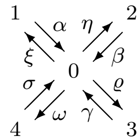
and the ideal generated by $\eta\gamma - \varrho\pi\mu\varepsilon$. A simple inspection of the Bongartz–Happel–Vossieck list of the tame concealed algebras [7], [15] shows that B is a tame concealed algebra of type $\widetilde{\mathbb{E}}_8$. Hence A is a selfinjective algebra of Euclidean tilted type $\widetilde{\mathbb{E}}_8$. Applying [1] we deduce that Γ_A contains exactly two nonperiodic components, one of them almost regular with a section



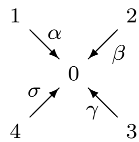
of extended Euclidean type $\widetilde{\widetilde{\mathbb{D}}}_8$, and the second one containing the projective modules $P(6)$, $P(7)$, $P(8)$, $P(9)$ such that $\operatorname{rad} P(6) = \tau_A^5 \operatorname{rad} P(7)$, $\operatorname{rad} P(8) = \tau_A^4 \operatorname{rad} P(6)$, $\operatorname{rad} P(9) = \tau_A^5 \operatorname{rad} P(9)$, and the τ_A-orbit of the radicals of $P(6)$, $P(7)$, $P(8)$, $P(9)$ intersects any Euclidean section (of type $\widetilde{\mathbb{E}}_8$) of the stable part of this component in the vertex whose deletion leads to a Dynkin quiver of type $\mathbb{E}_8$. Note that the tame concealed algebra B is not canonical.

Our next example shows that there exists a selfinjective algebra of Euclidean tilted type whose indecomposable projective-injective modules all lie on sections of nonperiodic components.

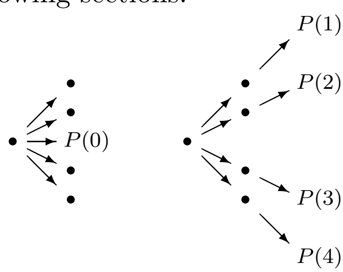
EXAMPLE 4.4. Let A be the bound quiver algebra given by the quiver



and the ideal generated by $\xi\alpha - \eta\beta$, $\eta\beta - \varrho\gamma$, $\varrho\gamma - \omega\sigma$, $\alpha\eta$, $\alpha\varrho$, $\alpha\omega$, $\beta\xi$, $\beta\varrho$, $\beta\omega$, $\gamma\xi$, $\gamma\eta$, $\gamma\omega$, $\sigma\xi$, $\sigma\eta$, $\sigma\varrho$. Then $A = \widehat{B}/(\nu_{\widehat{B}}) = B \ltimes D(B)$ where B is the path algebra of the Euclidean quiver of type $\widetilde{\mathbb{D}}_4$



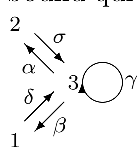
Applying again [1] we conclude that Γ_A admits exactly two nonperiodic components with the following sections:



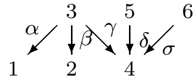
Observe that one of these nonperiodic components is almost regular, and clearly the other is not.

We end the paper with an example showing the other extreme. Namely, there exist selfinjective algebras of Euclidean tilted type all of whose indecomposable projective modules lie on one section of a nonperiodic component.

EXAMPLE 4.5. Consider the bound quiver algebra A given by the quiver

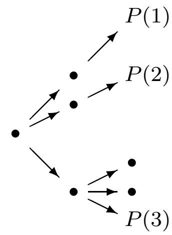


and the ideal generated by $\alpha\sigma - \gamma^2$, $\beta\delta - \gamma^2$, $\delta\alpha$, $\delta\gamma$, $\sigma\beta$, $\gamma\beta$, $\gamma\alpha$, $\sigma\gamma$. Then $A = \widehat{B}/(\psi)$, where B is the path algebra of the quiver



of Euclidean type $\widetilde{\mathbb{D}}_5$ and ψ is a K-linear automorphism of $\widehat{B}$ such that $\psi^2 = \nu_{\widehat{B}}$, and so A is of Euclidean tilted type $\widetilde{\mathbb{D}}_5$. Further, according to [24],

Γ_A admits exactly one nonperiodic component, and it is easy to see that this component has a section of the form



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Received 16 May 2000;
revised 29 June 2000 (3932)