NONCOERCIVE DIFFERENTIAL OPERATORS 
ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE 
AND THEIR GREEN FUNCTIONS

BY

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Abstract. We obtain upper and lower estimates for the Green function for a second order noncoercive differential operator on a homogeneous manifold of negative curvature.

1. Introduction and the main result. In this paper we study the Green function for a second order noncoercive differential operator $L$ on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semidirect product of a nilpotent Lie group $N$ and an abelian group $A = \mathbb{R}^+$. Moreover, for an $H$ belonging to the Lie algebra $\mathcal{A}$ of $A$, the eigenvalues of $\text{Ad}_{\exp H}|_N$ are all greater than 0. Conversely, every such group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [H]).

On $S$ we consider a second order left-invariant operator

$$L = \sum_{j=0}^{m} Y_j^2 + Y.$$ 

We assume that $Y_0, Y_1, \ldots, Y_m$ generate the Lie algebra $\mathcal{S}$ of $S$. Moreover, we can choose $Y_0, Y_1, \ldots, Y_m$ so that $Y_1(e), \ldots, Y_m(e)$ belong to the Lie algebra $\mathcal{N}$ of $N$. Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $L$ under $\pi$ is a second order left-invariant operator on $\mathbb{R}^+$,

$$(a\partial_a)^2 - \gamma a\partial_a,$$

where $\gamma \in \mathbb{R}$. The operator $L = L_\gamma$ is noncoercive (there is no $\varepsilon > 0$ such that $L + \varepsilon I$ admits the Green function) if and only if $\gamma = 0$.


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Finally, the operator we are interested in can be written in the form
\[(1.1) \quad \mathcal{L} = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2 \partial_a^2 + a \partial_a,\]
where \(X, X_1, \ldots, X_m\) are left-invariant vector fields on \(N\) and \(X_1, \ldots, X_m\) generate \(\mathcal{N}\), \(\Phi_a = \text{Ad}(\exp(\log a) Y_0) = e(\log a) \text{ad} Y_0 = e(\log a) D\) and \(D = \text{ad} Y_0\) is a derivation of the Lie algebra \(\mathcal{N}\) of the Lie group \(N\) such that the real parts \(d_j\) of the eigenvalues \(\lambda_j\) of \(D\) are positive. By multiplying \(\mathcal{L}\) by a constant we can make \(d_j\) arbitrarily large (see [DHU]).

Let \(G(xa, yb)\) be the Green function for \(\mathcal{L}\). It is (uniquely) defined by two conditions:

(i) \(\mathcal{L}G(\cdot, yb) = -\delta_{yb}\) as distributions (functions are identified with distributions via the right Haar measure),

(ii) for every \(yb \in S\), \(G(\cdot, yb)\) is a potential for \(\mathcal{L}\).

Let
\[(1.2) \quad G(x, a) = G(xa, e),\]
where \(e\) is the identity element of the group \(S\). In this paper we call \(G(x, a)\) the Green function for \(\mathcal{L}\).

For a positive \(\delta\) less than \(1/2\) define
\[(1.3) \quad T_\delta = \{(x, a) \in N \times \mathbb{R}^+ : 1 - \delta < a < 1 + \delta, \ |x| < \delta\},\]
where \(|\cdot|\) denotes the “homogeneous norm” (see Preliminaries).

Our aim is to prove the following result:

**Theorem 1.4.** For a given \(0 < \delta < 1/2\) there exists a positive constant \(C\) such that for \((x, a) \notin T_\delta\) we have the following estimate for the Green function \(G\) defined in (1.2):
\[(1.5) \quad C^{-1} w(x, a) \leq G(x, a) \leq C w(x, a),\]
where the function \(w\) is defined by
\[(1.6) \quad w(x, a) = \begin{cases} 1 & \text{if } |x| \leq 1, \ a \leq 1, \\ |x|^{-Q} & \text{if } |x| \geq 1, \ |x| \geq a, \\ a^{-Q} & \text{if } a \geq 1, \ a \geq |x|, \end{cases}\]
and \(Q = \sum d_j = \sum \text{Re} \lambda_j\).

The above result looks like the limit case (as \(\gamma\) tends to 0) of the estimate of the Green function for the operator \(\mathcal{L}_\gamma\) with positive \(\gamma\) (i.e. for a coercive operator). This has been proved by E. Damek [D] by means of Ancona’s theory. However, (1.5) cannot be obtained from Damek’s estimate by taking the limit and so requires essentially new methods. In this paper we make use of a probabilistic method introduced in [DH] and then developed e.g. in [DHZ], [DHU].
The structure of this paper is as follows. In Section 2 we state precisely notation and all necessary definitions.

In Section 3 we recall the basic properties of the Bessel process which appears as the “vertical” component of the diffusion generated by \( a^{-2}L \) on \( N \times \mathbb{R}^+ \) (cf. [DHU]).

In Section 4 we state the estimate of the transition probabilities of the evolution on \( N \) generated by an appropriate operator which appears as the “horizontal” component of the diffusion on \( N \times \mathbb{R}^+ \) mentioned above.

In Section 5 we prove the main lemmas, which are a crucial point in the proof of Theorem 1.4 given in Section 6.

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2. Preliminaries. Some of the notions which appear in this section have been introduced in the previous one. However, for the sake of completeness we state them precisely once again.

Let \( N \) be a connected and simply connected nilpotent Lie group. Let \( D \) be a derivation of the Lie algebra \( N \) of \( N \). For every \( a \in \mathbb{R}^+ \) we define an automorphism \( \Phi_a \) of \( N \) by

\[
\Phi_a = e^{(\log a)D}.
\]

Writing \( x = \exp X \) we have

\[
\Phi_a(x) := \exp \Phi_a(X).
\]

We assume that the real parts \( d_j \) of the eigenvalues \( \lambda_j \) of the matrix \( D \) are strictly greater than 0 and we define the number

\[
Q = \sum_j \text{Re} \lambda_j = \sum_j d_j.
\]

In this paper \( D = \text{ad} Y_0 \) (see Introduction). We consider a group \( S \) which is a semidirect product of \( N \) and the multiplicative group \( A = \mathbb{R}^+ = \{\exp ty_0 : t \in \mathbb{R}\} \):

\[
S = NA = \{xa : x \in N, a \in A\}
\]

with multiplication given by

\[
(xa)(yb) = (x\Phi_a(y)ab).
\]
In \( N \) we define the homogeneous norm \(| \cdot |\) ([DHZ], [DHU]). Let \((\cdot, \cdot)\) be a fixed inner product in \( N \). We define a new inner product
\[
\langle X, Y \rangle = \frac{1}{C_0^2} \int_0^1 (\Phi_a(X), \Phi_a(Y)) \, da
\]
and the corresponding norm
\[
\|X\| = \langle X, X \rangle^{1/2}.
\]
We put
\[
|X| = (\inf \{ a > 0 : \|\Phi_a(X)\| \geq 1 \})^{-1}.
\]
One can easily show that for every \( Y \neq 0 \) there exists precisely one \( a > 0 \) such that \( Y = \Phi_a(X) \) with \(|X| = 1\). Then we have \(|Y| = a\).

Finally, we define a homogeneous norm on \( N \). For \( x = \exp X \) we put
\[
|x| = |X|.
\]
Notice that if the action of \( A = \mathbb{R}^+ \) on \( N \) (given by \( \Phi_a \)) is diagonal, the norm we have just defined is the usual homogeneous norm on \( N \) (see [FS]).

And a final remark about notation: The letter \( C \) occurs in inequalities as a positive constant and may vary from statement to statement, even in the same calculation.

3. Bessel process. Let \( b_t \) denote the Bessel process with a parameter \( \alpha \geq 0 \) (cf. [RY]), i.e. a continuous Markov process with state space \([0, \infty)\) generated by
\[
\Delta = \partial_a^2 + \frac{2\alpha + 1}{a} \partial_a.
\]
The transition function with respect to the measure \( y^{2\alpha + 1} \, dy \) is given by (cf. [RY] again)
\[
(3.1) \quad p_t(x, y) = \begin{cases}
    c_\alpha \frac{1}{2t} \exp \left( \frac{-x^2 - y^2}{4t} \right) I_\alpha \left( \frac{xy}{2t} \right) \frac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\
    c_\alpha \frac{1}{(2t)^{\alpha + 1}} \exp \left( -\frac{y^2}{4t} \right) & \text{for } x = 0, y > 0,
\end{cases}
\]
where
\[
I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\alpha}}{k! \Gamma(k + \alpha + 1)}
\]
is the Bessel function (see [L]). Therefore for \( x \geq 0 \) and a measurable set \( B \subset (0, \infty) \),
\[
P_x(b_t \in B) = \int_B p_t(x, y) y^{2\alpha + 1} \, dy.
\]
The following lemmas concerning some properties of the Bessel process are very well known and their proofs are rather standard. Sketches of those proofs can be found in [DHU] or [U].

**Lemma 3.2.** Let $D, \gamma, a \geq 0$. There exists a positive constant $C$ such that for every $t > 0$,
\[
\sup_{a > 0} E_a \left( \frac{1}{0} b_s^{\gamma} ds \right)^{-D/2} < \infty.
\]
Moreover,
\[
E_a \left( \int_0^t b_s^{\gamma} ds \right)^{-D} \leq Ct^{-D(1+\gamma/2)}.
\]

**Lemma 3.3.** There exist constants $c_1, c_2$ such that for every $x \geq 0$, for every $\lambda > 0$ and for every $t > 0$,
\[
P_x \left( \sup_{s \in [0,t]} b_s > x + \lambda \right) \leq c_1 e^{-c_2 \lambda^2 / t}.
\]

**Lemma 3.4.** Let $0 < \eta < 1$. There exist constants $c_1, c_2$ such that for every $t > 0$,
\[
P_1 \left( \inf_{s \in [0,t]} b_s \leq 1 - \eta \right) \leq c_1 e^{-c_2 / t}.
\]

**Proof.** It is enough to rewrite the proof of Lemma 2.4 in [DHU].

By a straightforward computation, using the definition of the transition function $p_t(x,y)$ of the Bessel process (3.1) and the asymptotic behaviour of the Bessel function (see [L]):
\[
I_{\alpha}(x) \asymp \begin{cases} 
\frac{x^\alpha}{2^\alpha \Gamma(1+\alpha)}, & x \to 0, \\
\exp(x) / (2\pi x)^{1/2}, & x \to \infty,
\end{cases}
\]
we get

**Lemma 3.5.** There exists a constant $C$ independent of $x$ such that
\[
P_x(a-\eta \leq b_t \leq a+\eta) \leq Ct^{-(\alpha+1)m([a-\eta,a+\eta])},
\]
where $m(B) = \int_B y^{2\alpha+1} dy$.

**4. Evolutions.** For a multiindex $I = (i_1, \ldots, i_n)$, $i_j \in \mathbb{Z}^+$ and a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathcal{N}$ of $N$ we write $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. For $k = 0, 1, \ldots, \infty$ we define
\[
C^k = \{ f : X^I f \in C(N) \text{ for } |I| < k + 1 \}.
\]
and
\[ C^k = \{ f \in C^k : \lim_{x \to \infty} X^I f(x) \text{ exists for } |I| < k + 1 \}. \]

For \( k < \infty \) the space \( C^k_\infty \) is a Banach space with the norm
\[
\|f\|_{C^k_\infty} = \sum_{|I| \leq k} \|X^I f\|_{C(N)}.
\]

Let \( L_{\sigma(t)} = \sigma(t)^{-2} \left( \sum \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X) \right) \).

For a continuous function \( \sigma : [0, \infty) \to [0, \infty) \) let \( \{ U_{\sigma}(s,t) : 0 \leq s \leq t \} \) be the unique family of bounded operators on \( C_\infty = C_0^\infty \) which satisfy

(i) \( U_{\sigma}(s,s) = I \),
(ii) \( U_{\sigma}(s,r)U_{\sigma}(r,t) = U_{\sigma}(s,t), \) \( s < r < t \),
(iii) \( \partial_s U_{\sigma}(s,t) f = -L_{\sigma(s)} U_{\sigma}(s,t) f \) for every \( f \in C_\infty \),
(iv) \( \partial_t U_{\sigma}(s,t) f = U_{\sigma}(s,t) L_{\sigma(t)} f \) for every \( f \in C_\infty \),
(v) \( U_{\sigma}(s,t) : C_\infty^2 \to C_\infty^2 \).

\( U_{\sigma}(s,t) \) is a convolution operator. Namely, \( U_{\sigma}(s,t) f = f \ast p_{\sigma}(t,s) \), where \( p_{\sigma}(t,s) \) is a smooth density of a probability measure. By (ii) we have \( p_{\sigma}(t,r) \ast p_{\sigma}(r,s) = p_{\sigma}(t,s) \) for \( t > r > s \). Existence of the family \( U_{\sigma}(s,t) \) follows from [T].

In [DHU], using the Nash inequality, the following estimate of the evolution kernels \( p_{\sigma}(t,0) \) has been proved.

**Theorem 4.1.** For every compact set \( K \subset N \) which does not contain the identity \( e \) of \( N \), there exist positive constants \( C, \xi, \beta_1, \beta_2 \) and \( D \leq Q \) such that for every \( x \in K \) and for every \( t > 0 \),

\[
p_{\sigma}(t,0)(x) \leq C \left( \int_0^t \sigma^{-2(1-Q/D)}(u) \, du \right)^{-D/2} \exp \left( -\frac{\xi}{A(0,t)} \right),
\]

where \( A(s,t) = \int_s^t (\sigma^{\beta_1}(u) + \sigma^{\beta_2}(u)) \, du \).

In the proof of the above theorem the following estimate of the norm \( \|p_{\sigma}(t,s)\|_{L^\infty(N)} \) has been obtained:

**Theorem 4.2.** There exist positive constants \( C \) and \( D \leq Q \) such that for every \( s < t \),

\[
\|p_{\sigma}(t,s)\|_{L^\infty(N)} \leq C \left( \int_s^t \sigma^{-2(1-Q/D)}(u) \, du \right)^{-D/2}.
\]

5. **Main lemmas.** From now on we consider the Bessel process \( b_t \) with a parameter \( \alpha = 0 \). In this case \( b_t = \|w_t\| \), where \( w_t \) is a Brownian motion on \( \mathbb{R}^2 \).
In this section we prove some lemmas, which are our main tools in writing estimates for the Green function.

**Lemma 5.2.** Let $D, \gamma > 0$ and $dm(a) = ada$. For every $\delta > 0$ there exists a constant $C$ such that for every $a \leq 1 - \delta$,

$$\sup_{0 < \eta < \delta/2} \int_0^t \mathbf{E}_1 \left( \int_0^\eta b_s^\gamma \, ds \right)^{-D/2} m([a - \eta, a + \eta])^{-1} 1_{[a-\eta,a+\eta]}(b_t) \, dt \leq C.$$  

**Proof.** In order to simplify notation let $I_{a, \eta} = [a - \eta, a + \eta]$.

First we consider large time ($t \geq 1$):

$$\int_1^\infty \mathbf{E}_1 \left( \int_0^\eta b_s^\gamma \, ds \right)^{-D/2} m(I_{a, \eta})^{-1} 1_{I_{a, \eta}}(b_t) \, dt \leq \int_1^\infty \mathbf{E}_1 \left( \int_0^{t/2} b_s^\gamma \, ds \right)^{-D/2} m(I_{a, \eta})^{-1} 1_{I_{a, \eta}}(\theta_{t/2} b_{t/2}) \, dt,$$

where $\theta_s$ is the shift operator. Using the Markov property and Lemma 3.2 we get

$$\int_1^\infty \mathbf{E}_1 \left( \int_0^{t/2} b_s^\gamma \, ds \right)^{-D/2} \mathbf{E}_{b_{t/2}} m(I_{a, \eta})^{-1} 1_{I_{a, \eta}}(\sigma_{t/2}) \, dt$$

$$= \int_1^\infty \mathbf{E}_1 \left( \int_0^{t/2} b_s^\gamma \, ds \right)^{-D/2} m(I_{a, \eta})^{-1} \mathbf{P}_{b_{t/2}} (\sigma_{t/2} \in I_{a, \eta}) \, dt$$

$$\leq C \int_1^\infty t^{-(D/2)(1+\gamma/2)} m(I_{a, \eta})^{-1} \mathbf{P}_{b_{t/2}} (\sigma_{t/2} \in I_{a, \eta}) \, dt.$$  

By Lemma 3.5,

$$\mathbf{P}_x(\sigma_t \in I_{a, \eta}) \leq C t^{-1} m(I_{a, \eta})$$

with $C$ independent of the starting point $x$. Hence by (5.3) we get

$$\sup_{\eta > 0} \int_1^\infty \mathbf{E}_1 \left( \int_0^t b_s^\gamma \, ds \right)^{-D/2} m(I_{a, \eta})^{-1} 1_{I_{a, \eta}}(b_t) \, dt$$

$$\leq C \int_1^\infty t^{-(D/2)(1+\gamma/2)-1} \, dt \leq C_1.$$  

Now we consider $t \leq 1$. We divide the set of all trajectories of the Bessel process $b_t$ (with parameter 0) starting from 1 into two subsets:

$$A = \{ b : \sup_{s \in [0,t]} b_s > 2 \}, \quad B = \{ b : \sup_{s \in [0,t]} b_s \leq 2 \}.$$
Consider the set \( A \). Let \( T = \inf\{s : b_s = 2\} \). For \( n \geq 1 \), let
\[
A_n = \{b : t/2^n < T \leq t/2^{n-1}\}.
\]
Then the Markov property gives
\[
(5.6) \quad \int_0^t \mathbb{E}_1 \left( \int_0^t b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} \mathbb{I}_{I_{a,\eta}}(b_t) 1_A(b) \, dt
\]
\[
= \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^t b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} \mathbb{I}_{I_{a,\eta}}(b_t) 1_{A_n}(b) \, dt
\]
\[
\leq \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^t b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} \mathbb{I}_{I_{a,\eta}}(b_t) 1_{\{T \leq t/2^{n-1}\}}(b) \, dt
\]
\[
\leq \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^{t/2^n} b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} \mathbb{I}_{I_{a,\eta}}(b_t) 1_{\{T \leq t/2^{n-1}\}}(b) \, dt
\]
\[
= \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^{t/2^n} b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{\{T \leq t/2^{n-1}\}}(b)
\]
\[
\times \mathbb{E}_{b_t/2^{n-1}} \mathbb{I}_{\{\sigma : \sigma_{t-t/2^n-1} \in I_{a,\eta}\}}(\sigma) \, dt
\]
\[
\leq \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^{t/2^n} b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{\{b : \sup_{s \in [t/2^{n-1}]} b_s \geq 2\}}(b)
\]
\[
\times \mathbb{E}_{b_t/2^{n-1}} \mathbb{I}_{\{\sigma : \sigma_{t-t/2^n-1} \in I_{a,\eta}\}}(\sigma) \, dt.
\]
By (5.4) it follows that for \( n \geq 2 \),
\[
(5.7) \quad \mathbb{E}_{b_t/2^{n-1}} 1_{\{\sigma : \sigma_{t-t/2^n-1} \in I_{a,\eta}\}}(\sigma) \leq C(t - t/2^{n-1})^{-1} m(I_{a,\eta})
\]
\[
\leq C(t/2)^{-1} m(I_{a,\eta}).
\]
For \( n = 1 \) the expectation in (5.7) is equal to
\[
\mathbb{P}_{b_t}(\sigma_0 \in I_{a,\eta}) = \mathbb{P}_1(b_t \in I_{a,\eta})
\]
and by (5.4) we get (5.7) for \( n = 1 \).

Therefore using (5.7), Lemma 3.2, Lemma 3.3 and the Schwarz inequality we get
\[
(5.8) \quad \int \sum_{n=1}^\infty \mathbb{E}_1 \left( \int_0^t b_s^2 \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} \mathbb{I}_{I_{a,\eta}}(b_t) 1_{A_n}(b) \, dt
\]
\[
\leq C \int \sum_{n=1}^\infty t^{-1} \mathbb{E}_1 \left( \int_0^{t/2^n} b_s^2 \, ds \right)^{-D/2} 1_{\{b : \sup_{s \in [t/2^{n-1}]} b_s \geq 2\}}(b) \, dt
\]
\[ \leq C \int_0^1 \sum_{n=1}^\infty \left[ E_1 \left( \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D/2} \right] \, dt \]
\[ \times \left[ E_1 1_{\{b: \sup_{s \in [0,t/2^{n-1}]} b_s \geq 2\}} (b) \right]^{1/2} dt \]
\[ \leq C \int_0^1 \sum_{n=1}^\infty \left[ t^{-1}(t/2^n)^{-D/2}(1+\gamma/2)e^{-c2^{n-1}/t} \right] dt \leq C_2. \]

Now we consider the set \( B \). Let \( T = \inf \{ s : b_s = 1 - \delta/2 \} \). For \( n \geq 1 \), let
\[ A_n = \{ b : t/2^n < T \leq t/2^{n-1} \}. \]
Notice that
\[ T \leq t/2^{n-1} \text{ implies } \inf_{s \in [0,t/2^{n-1}]} b_s \leq 1 - \delta/2. \]
Moreover, by Lemma 3.4,
\[ P_1 \left( \inf_{s \in [0,t]} b_s \leq 1 - \delta/2 \right) \leq c_1 e^{-c_2/t}. \]
Then
\[ \int_0^1 E_1 \left( \int_0^t b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)1_B(b) \, dt \]
\[ = \int_0^1 \sum_{n=1}^\infty E_1 \left( \int_0^t b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)1_{A_n}(b) \, dt \]
\[ \leq \int_0^1 \sum_{n=1}^\infty E_1 \left( \int_0^T b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)1_{\{T \leq t/2^{n-1}\}}(b) \, dt \]
\[ \leq \int_0^1 \sum_{n=1}^\infty \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)1_{\{T \leq t/2^{n-1}\}}(b) \, dt \]
\[ = \int_0^1 \sum_{n=1}^\infty \left( \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{\{T \leq t/2^{n-1}\}}(b) \]
\[ \times E_{\delta t/2^{n-1}} 1_{\{\sigma: \sigma_{t-t/2^{n-1}} \in I_{a,\eta}\}}(\sigma) \, dt \]
\[ \leq \int_0^1 \sum_{n=1}^\infty \left( \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D/2} m(I_{a,\eta})^{-1} 1_{\{b: \inf_{s \in [0,t/2^{n-1}]} b_s \leq 1-\delta/2\}}(b) \]
\[ \times E_{\delta t/2^{n-1}} 1_{\{\sigma: \sigma_{t-t/2^{n-1}} \in I_{a,\eta}\}}(\sigma) \, dt \]
\[ \leq \int_0^1 \sum_{n=1}^\infty \left( \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D/2} 1_{\{b: \inf_{s \in [0,t/2^{n-1}]} b_s \leq 1-\delta/2\}}(b) t^{-1} \, dt, \]
where in the last inequality we have used (5.7) for \( n \geq 1 \) (see the remark after (5.7)). Now, as before, in order to estimate the expectation we use the Schwarz inequality. By Lemma 3.2 and (5.9) we have

\[
\int_0^1 E_1 \left( \int_0^t b_s^\gamma \, ds \right)^{-D/2} m(I_{a,n})^{-1} I_{a,n}(b_t) 1_B(b) \, dt
\]

\[
\leq C \int_0^1 t^{-1} \sum_{n=1}^\infty \left[ E_1 \left( \int_0^{t/2^n} b_s^\gamma \, ds \right)^{-D} \right]^{1/2} \times \left[ E_1 1 \{ b : \inf_{s \in [0,t/2^n-1]} b_s \leq 1 - \delta/2 \} \right]^{1/2} \, dt
\]

\[
\leq C \int_0^1 t^{-1} \sum_{n=1}^\infty (t/2^n)^{- (D/2)(1+\gamma/2)} \left[ E_1 1 \{ b : \inf_{s \in [0,t/2^n-1]} b_s \leq 1 - \delta/2 \} \right]^{1/2} \, dt
\]

\[
\leq C \int_0^1 t^{-1} \sum_{n=1}^\infty (t/2^n)^{- (D/2)(1+\gamma/2)} \left[ P_1 \left( \inf_{s \in [0,t/2^n-1]} b_s \leq 1 - \delta/2 \right) \right]^{1/2} \, dt
\]

\[
\leq C \sum_{n=1}^\infty t^{-1} (t/2^n)^{- (D/2)(1+\gamma/2)} e^{-c2^{n-1}/t} \, dt \leq C_3.
\]

Now (5.5), (5.8) and (5.10) complete the proof.  

**Lemma 5.11.** Let \( D, \gamma > 0 \) and \( dm(a) = ada \). For every \( 0 < \delta < 1/2 \) there exists a constant \( C \) such that for every \( x \leq 1/2 - \delta \) and every \( (1-\delta)/2 \leq a \leq 1/2 \),

\[
\sup_{0 < \eta < \delta/4} \int_0^t E_x \left( \int_0^s b_s^\gamma \, ds \right)^{-D/2} m([a - \eta, a + \eta])^{-1} 1_{[a - \eta, a + \eta]}(b_t) \, dt \leq C.
\]

**Proof.** For large time \( t \geq 1 \) it is enough to rewrite the proof of the previous lemma.

Let \( t \leq 1 \). We define \( T = \inf \{ s : b_s = 1/2 - 3\delta/4 \} \). For \( n \geq 1 \), let

\[
A_n = \{ b : t/2^n < T \leq t/2^{n-1} \}.
\]

Notice that

\[
T \leq t/2^{n-1} \quad \text{implies} \quad \sup_{s \in [0,t/2^n-1]} b_s \geq 1/2 - 3\delta/4.
\]

Then, since \( x \leq 1/2 - \delta \), by Lemma 3.3,

\[
P_x \left( \sup_{s \in [0,t/2^n-1]} b_s \geq 1/2 - 3\delta/4 \right)
\]
\[ = \mathbf{P}_x \left( \sup_{s \in [0, t/2^{n-1}]} b_s \geq (1/2 - 3\delta/4 - x) + x \right) \]
\[ \leq c_1 e^{-c_2(1/2 - 3\delta/4 - x)^2 2^{n-1} / t} \leq c_1 e^{-c_2(\delta/4)^2 2^{n-1} / t}. \]

Now, because of (5.12) it is enough to rewrite the end of the proof of Lemma 5.2 starting after (5.9). Namely, we have to change the starting point to \( x \) and instead of \( \{ b : \inf_{[0, t/2^{n-1}]} b_s \leq 1 - \delta/2 \} \) put \( \{ b : \sup_{s \in [0, t/2^{n-1}]} b_s \geq 1/2 - 3\delta/4 \} \).

The next lemma is taken from [DHU] (Lemma 5.18):

**Lemma 5.13.** Let \( D, \xi, \gamma > 0 \), \( dm(a) = ada \). For every \( a_1 > 0 \) there is a constant \( C \) such that for every \( x \leq a_1, 0 < a < 1 \),
\[ \sup_{0 < \eta < 1} \int_0^\infty \mathbb{E}(t) \left( \int_0^t b_s \gamma \right)^{-D/2} e^{-\xi/A(0,t)} m([a - \eta, a + \eta])^{-1} [a - \eta, a + \eta] (b_t) dt \leq C, \]
where \( A(0,t) \) is defined in Theorem 4.1.

**6. Proof of Theorem 1.4.** It turns out that it is very convenient to consider along with the operator \( L \) defined in (1.1) the corresponding operator \( L, \)
\[ (6.1) \quad L = a^{-2} L = a^{-2} \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + \frac{1}{a} \partial_a. \]

The Green function \( G \) for \( L \) is given by
\[ (6.2) \quad G(x, a; y, b) = \int_0^\infty \mathbb{P}_t(x, a; y, b) dt, \]
where \( T_t f(x, a) = \int f(y, b) \mathbb{P}_t(x, a; y, b) dy db \) is the heat semigroup on \( L^2(N \times \mathbb{R}^+, dy db) \) with infinitesimal generator \( L \).

In (6.2) we allow \( (x, a) \) to be \( (e, 0) \) since \( \lim_{(x, a) \to (e, 0)} G(x, a; y, b) \) exists (see [DHU]).

On \( N \times \mathbb{R}^+ \) we define *dilations*
\[ D_t(x, a) = (\Phi_t(x), ta), \quad t > 0. \]

It is not difficult to check that although the operator \( L \) is not left-invariant it has some homogeneity with respect to the family of dilations introduced above:
\[ L(f \circ D_t) = t^2 Lf \circ D_t. \]

This implies that
\[ (6.3) \quad G(x, a; y, b) = t^{-Q} G(D_t^{-1}(x, a); D_t^{-1}(y, b)). \]
It turns out (see (1.17) in [DHU]) that
\[ G(x,a) = G(x,a;e,1) = G^*(e,1;x,a), \]
where \( G^* \) is the Green function for the operator
\[ L^* = a^{-2} \sum \Phi_a(X_j)^2 - a^{-2}\Phi_a(X) + \partial_a^2 + a^{-1}\partial_a, \]
conjugate to \( L \) with respect to the measure \( adxda \). Moreover,
\[ G^*(e,1;x,a) = \lim_{\eta \to 0} \int_0^\infty \frac{1}{m([a-\eta,a+\eta])} 1_{[a-\eta,a+\eta]}(\sigma_t) \, dt, \]
where the expectation is taken with respect to the distribution of the Bessel process starting from 1 on the space \( C([0,\infty), (0,\infty)) \). All the above facts are proved in [DHU].

Now we are ready to give

Proof of Theorem 1.4. For \( r \geq 0 \), define
\[ V_r = \{(x,a) \in N \times \mathbb{R}^+ : |(x,a)| = r \}, \]
where \( |(x,a)| = |x| + a \). Let \( 0 < \delta < 1/2 \) be fixed.

Case 1. We consider the set
\[ S_1 = \{(x,a) \notin T_\delta : |x| \leq 1, a \leq 1 \}. \]
We have to show that there exists a positive constant \( C \) such that
\[ C^{-1} \leq G(x,a) = G^*(e,1;x,a) \leq C \]
for every \((x,a) \in S_1\).

It follows immediately from (6.4), Theorem 4.2, and Lemma 5.2 that we have the upper bound in (6.5) on \( \tilde{S}_1 = S_1 \cap \{(x,a) \in N \times \mathbb{R}^+ : a \leq 1 - \delta \} \). Therefore we are left with \((x,a) \in S_1 \setminus \tilde{S}_1\). But
\[ S_1 \setminus \text{Int} \tilde{S}_1 = \{(x,a) : N \times \mathbb{R}^+ : \delta \leq |x| \leq 1, 1 - \delta \leq a \leq 1 \} \]
is a compact set. Since \( G^* \) is a continuous function we get the upper bound on \( S_1 \). The lower bound in (6.5) is a consequence of Lemma 5.21 of [DHU].

Case 2. We consider the set
\[ S_2 = \{(x,a) \in N \times \mathbb{R}^+ : |x| \geq 1, |x| \geq a \}. \]
(Of course, \( S_2 \cap T_\delta = \emptyset \).)

Every element \((x,a) \in N \times \mathbb{R}^+ \) can be written as
\[ (x,a) = D_t(y,b), \quad \text{where } (y,b) \in V_1 \text{ and } t = |(x,a)| = |x| + a. \]
(Recall that \( D_t(x,a) = (\Phi_t(x), ta) \).) By homogeneity of \( G \) (see (6.3)), we get
\[ G^*(e,1;x,a) = G^*(D_t(e,t^{-1});D_t(y,b)) = t^{-Q}G^*(e,t^{-1};y,b) \]
\[ = |(x,a)|^{-Q}G^*(e,|(x,a)|^{-1};y,b) \]
Since \( (x, a) \in S_2 \) then the corresponding \((y, b) \in V_1\) has the property \(|y| \geq b\). Indeed, \(x = \Phi_t(y)\) and \(a = tb\), thus \(t|y| = |x| \geq a = tb\). The above property and \(|y| + b = 1\) imply that \(b \leq 1/2\). Therefore
\[
(y, b) \in V_1 \cap \{(x, a) \in N \times \mathbb{R}^+ : a \leq 1/2\} \subset V_1.
\]
Let \(\beta = |(x, a)|^{-1}\). For \((x, a) \in S_2\) we have \(\beta \leq 1\). Thus by (6.4), Theorem 4.1 and Lemma 5.13 we get
\[
G^*(e, \beta; x, a) \leq C \quad \text{for} \quad (x, a) \in S_2.
\]
Once again, Lemma 5.21 in [DHU] gives the lower bound
\[
G^*(e, \beta; x, a) \geq C^{-1}.
\]
Thus by (6.6) we get
\[
C^{-1}(|x| + a)^{-Q} \leq G(x, a) \leq C(|x| + a)^{-Q}.
\]
Since \(|x| \leq |x| + a \leq 2|x|\) for \((x, a) \in S_2\), the proof of the second case is complete.

**Case 3.** Finally we consider the set
\[
S_3 = \{(x, a) \notin T_5 : a \geq |x|, \ a \geq 1\}.
\]
Because \(V_1 \cap T_5 \neq \emptyset\) we write every element \((x, a) \in N \times \mathbb{R}^+\) as a dilation of some element from \(V_{1/2}\):
\[
(x, a) = D_t(y, b), \quad \text{where} \ (y, b) \in V_{1/2} \text{ and } t = 2|(x, a)| = 2|x| + 2a.
\]
By homogeneity, we can write, analogously to (6.6),
\[
(6.7) \quad G^*(e, 1; x, a) = 2^{-Q}(|x| + a)^{-Q}G(e, \tilde{\beta}; y, b),
\]
where \(\tilde{\beta} = 2^{-1}(|x| + a)^{-1}\). If \((x, a) \in S_3\) then the corresponding \((y, b) \in V_{1/2}\) has the property \(|y| \leq b\). Indeed, \(|x| = t|y| \leq a = tb\). This, together with \(|y| + b = 1/2\), implies that \(b \in [1/4, 1/2]\).

For \((x, a) \in S_3\) we have \(\tilde{\beta} \leq (2 + 2\delta)^{-1} = 1/2 - \tilde{\delta}\). Indeed, this is clear if \(a \geq 1 + \delta\). But if \(a < 1 + \delta\) then \(|x| \geq \delta\). Thus by (6.4), using Theorem 4.2 and Lemma 5.11 if \(b \geq (1 - \tilde{\delta})/2\), or Theorem 4.1 and Lemma 5.13 if \(b \leq (1 - \tilde{\delta})/2\) (then \(|y| \geq \delta/2\)), we find that there exists a constant \(C\) such that \(G^*(e, \tilde{\beta}; x, a)\) in (6.7) is less than or equal to \(C\). By Lemma 5.21 of [DHU], \(G^*(e, \tilde{\beta}; x, a)\) is also greater than or equal to \(C^{-1}\). Thus by (6.7),
\[
C^{-1}2^{-Q}(|x| + a)^{-Q} \leq G(x, a) \leq C^2 2^{-Q}(|x| + a)^{-Q}, \quad (x, a) \in S_3.
\]
Since \(a \leq |x| + a \leq 2a\) for \((x, a) \in S_3\), the proof is complete.

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