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BLOWUP RATES FOR NONLINEAR HEAT EQUATIONS WITH GRADIENT TERMS AND FOR PARABOLIC INEQUALITIES

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Abstract. Consider the nonlinear heat equation (E): $u_t - \Delta u = |u|^{p-1}u + b|\nabla u|^q$. We prove that for a large class of radial, positive, nonglobal solutions of (E), one has the blowup estimates $C_1(T-t)^{-1/(p-1)} \le \|u(t)\|_\infty \le C_2(T-t)^{-1/(p-1)}$. Also, as an application of our method, we obtain the same upper estimate if u only satisfies the nonlinear parabolic inequality $u_t - u_{xx} \ge u^p$. More general inequalities of the form $u_t - u_{xx} \ge f(u)$ with, for instance, $f(u) = (1+u)\log^p(1+u)$ are also treated. Our results show that for solutions of the parabolic inequality, one has essentially the same estimates as for solutions of the ordinary differential inequality $\dot{v} \ge f(v)$.

1. Introduction. The first aim of this article is to determine the blowup rates of nonglobal solutions for semilinear heat equations with gradient depending nonlinearities. As a typical example, we shall consider the equation

$$\text{(P)} \qquad \begin{cases} u_t - \Delta u = |u|^{p-1}u + b|\nabla u|^q, & 0 < t < T,\ x \in \Omega, \\ u(t,x) = 0, & 0 < t < T,\ x \in \partial\Omega, \\ u(0,x) = \phi(x), & x \in \Omega, \end{cases}$$

where $p > 1$, $q \ge 1$ and $b \in \mathbb{R}$ (see Remark 1 in §2.3 for more general gradient depending nonlinearities).

Many authors have studied the existence of global and nonglobal positive solutions to (P), especially for $b < 0$ (see [1, 5, 7, 8, 12–13, 18, 26, 27, 31–38]). Also, the associated elliptic problem was studied in [1, 4, 5, 7–9, 25, 29, 30, 39]. In particular, it is known [36] that finite time blowup occurs for large initial data whenever $p > q$, whereas all solutions are global and bounded if $q \ge p$, $b < 0$ and, for instance, if Ω is bounded (see [8, 27, 37, 34]).

A considerable amount of work has been devoted to the equation without gradient term, that is, (P) with $b = 0$. In this case, the blowup behavior

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of solutions is by now fairly well understood, and in particular there are precise results on the blowup rate of solutions (see §2.3 for details and some references).

On the other hand, relatively little is known on the blowup behavior of nonglobal solutions of (P). For some results on self-similar blowup profiles, see [35] in the case $q = 2p/(p+1)$, $b < 0$, and [12, 13] in the case $q = 2$, $b > 0$. Also, the blowup set was investigated in the latter case (see [12, 13, 18, 19]). Let us also mention the work [10], which gives blowup rate estimates when the nonlinear gradient term in (P) is replaced with a $(u^2)_x$ term of Burgers type ($N = 1$).

In the present paper, we consider nonnegative solutions u of (P) in a ball or in $\mathbb{R}^N$, such that u is nondecreasing in time, radially symmetric, and nonincreasing as a function of $|x|$. We will prove under certain assumptions on the parameters that the rate of blowup of u satisfies the estimate

$$C_1(T-t)^{-1/(p-1)} \leq \|u(t)\|_\infty \leq C_2(T-t)^{-1/(p-1)}. \tag{1}$$

Also, as an application of our method, we will prove that the upper estimate in (1) still holds if u only satisfies the nonlinear parabolic inequality

$$u_t - u_{xx} \geq u^p. \tag{2}$$

This result is for now unfortunately restricted to the case of one space dimension (see Remark 3.2). However, the method can also apply to general inequalities of the form

$$u_t - u_{xx} \geq f(u). \tag{3}$$

Under some mild assumptions on f, we obtain the estimate

$$\|u(t)\|_\infty \leq G^{-1}(C(T-t)) \quad \text{with} \quad G(s) = \int_s^\infty \frac{d\sigma}{f(\sigma)}. \tag{4}$$

For instance, for the inequality

$$u_t - u_{xx} \geq (1+u)\log^p(1+u), \tag{5}$$

we obtain the upper estimate $\|u(t)\|_\infty \leq \exp[C(T-t)^{-1/(p-1)}]$. Let us mention that the blowup properties for the corresponding equation were studied in [19, 12, 13].

We note that the blowup rate (4) is "natural" in the sense that solutions of the corresponding ordinary differential inequality $\dot{v} \geq f(v)$ (resp. $\dot{v} \leq f(v)$) satisfy $v(t) \leq G^{-1}(T-t)$ (resp. $v(t) \geq G^{-1}(T-t)$). Actually, the lower estimate $\|u(t)\|_\infty \geq G^{-1}(T-t)$ can be easily obtained when the inequality sign in (3) is reversed.

To our knowledge, there do not seem to be any results in the literature on upper blowup estimates for nonlinear parabolic inequalities. Furthermore, the proofs known for the case of problem (P) with $b = 0$ use in an essential way the equality sign in the equation, and do not carry over immediately to the unilateral case, nor to problem (P) for $b \neq 0$ (see § 2.3).

Let us indicate that the methods in the present article can be adapted to prove upper blowup estimates for coupled parabolic systems with no variational structure. This will be treated in a forthcoming publication of ours.

Finally, we would like to mention that, at the time we were completing this paper, we received the work [6], where upper blowup estimates were independently obtained for problem (P), under assumptions different from ours. More precisely, the estimate (7) is obtained in [6] without any symmetry or monotonicity assumptions on u, but only in $\Omega = \mathbb{R}^N$ and under more restrictive conditions on the parameters, namely, $1 \leq q < 2p/(p+1)$ and $p \leq 1 + 2/N$. Moreover, the method in [6], based on Fujita-type theorems, is completely different from ours.

The outline of the article is as follows. The results are stated in §2, along with some comments and further remarks. The upper blowup estimates are proved in §3. The additional results are proved in §4.

2. Statement of results

2.1. *Equations with gradient terms.* In what follows, we assume $p > 1$, $q \geq 1$, $b \in \mathbb{R}$, and either Ω is the ball B_R, of center 0 and radius R $(0 < R < \infty)$ in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N$. As for the initial data, we assume $\phi \geq 0$, and $\phi \in C^1(\overline{B}_R)$ with $\phi(x) = 0$ on ∂B_R if $\Omega = B_R$. If $\Omega = \mathbb{R}^N$, we suppose $\phi \in C^1(\mathbb{R}^N)$ with $\lim_{|x|\to\infty} \phi(x) = 0$, and the boundary condition in (P) is then understood in the sense $u(t,x) \to 0$, $|x| \to \infty$. Under these assumptions, there exists a unique, maximal in time, classical solution $u \geq 0$ of (P). Denote by $T = T(\phi) \in (0, \infty]$ the maximal existence time of u. If we assume in addition $q \leq 2$, then gradient blowup cannot occur, that is, we have $\lim_{t\to T} \|u(t)\|_\infty = \infty$ whenever $T < \infty$ (see, e.g., [20, 21, 1]).

We will consider solutions of (P) with the following properties:

$$(6) \quad u(t,x) = u(t,r) \geq 0, \quad r = |x|, \quad u_t \geq 0, \quad u_r \leq 0 \quad \text{in } (0,T) \times \Omega.$$

It is well known from previous work on problem (P) that the value $q = 2p/(p+1)$ plays a critical role in the study of this problem. We will distinguish between the cases where q is subcritical, critical, and supercritical.

THEOREM 1. *Let $\Omega = B_R$ or $\Omega = \mathbb{R}^N$. Let u be a solution of* (P) *such that u satisfies* (6) *and $T < \infty$. Assume that*

$$1 \leq q < 2p/(p+1), \quad b \in \mathbb{R} \quad \textit{and} \quad (N-2)p < N+2.$$

Then

$$(7) \qquad \limsup_{t\to T}\, (T-t)^{1/(p-1)} \|u(t)\|_\infty < \infty.$$

Theorem 2. *Let* $\Omega = B_R$ *or* $\Omega = \mathbb{R}^N$. *Let* u *be a solution of* (P) *such that* u *satisfies* (6) *and* $T < \infty$. *Assume that* $q = 2p/(p+1)$. *Then*:

(i) *There exists* $b_1 = b_1(p, N) > 0$ *such that* (7) *holds for all* $b \geq b_1$.

(ii) *If* $(N-2)p \leq N$, *then* (7) *holds for all* $b > 0$.

(iii) *If* $(N-2)p < N+2$, *then there exists* $b_0 = b_0(p, N) > 0$ *such that* (7) *holds for all* $-b_0 \leq b \leq b_0$.

Theorem 3. *Let* $\Omega = B_R$ *or* $\Omega = \mathbb{R}^N$. *Let* u *be a solution of* (P) *such that* u *satisfies* (6) *and* $T < \infty$. *Assume that*

$$q > 2p/(p+1), \quad b > 0 \quad \textit{and} \quad N = 1.$$

Then (7) *holds.*

If in Theorems 1–3 we relax the assumption $u_t \geq 0$, and if we assume $N = 1$, $b > 0$, $p > 1$ and $q \geq 1$, then u still satisfies the weaker estimate

$$\liminf_{t \to T} (T-t)^{1/(p-1)} \|u(t)\|_\infty < \infty.$$

This is a consequence of Theorem 4(ii) below. Theorems 1–3 are complemented with the following lower estimate.

Proposition 1. *Let* $p > 1$, $1 \leq q \leq 2$, $\Omega = B_R$ *or* $\Omega = \mathbb{R}^N$. *Let* $u \geq 0$ *be a radially symmetric solution of* (P), *with* $T < \infty$, *such that* $u_r \leq 0$. *Then*

$$\liminf_{t \to T} (T-t)^{1/(p-1)} u(t, 0) \geq \kappa \equiv (p-1)^{-1/(p-1)}. \tag{8}$$

It is not *a priori* clear if there actually exist initial data such that the corresponding solution of (P) satisfies the assumptions of Theorems 1–3. The next proposition provides such initial data.

Proposition 2. (i) *Let* $\Omega = B_R$, $p > 1$, $q \geq 1$, $b \in \mathbb{R}$, *and assume that* $\phi \geq 0$ *is radially symmetric nonincreasing*, *with* $\phi \in C^2(\overline{\Omega})$ *and* $\phi|_{\partial\Omega} = 0$. *Assume in addition that*

$$\Delta\phi + \phi^p + b|\nabla\phi|^q \geq 0 \quad \textit{in } \Omega. \tag{9}$$

Then the corresponding solution u *of* (P) *satisfies* (6).

(ii) *Let* $\Omega = B_R$, $p > 1$, $q \geq 1$ *and* $b \in \mathbb{R}$. *Assume in addition* $b > 0$ *if* $q > 2p/(p+1)$, *and* $b \geq -b_3$ *if* $q = 2p/(p+1)$, *where* $b_3 = b_3(N, p) > 0$ *is sufficiently small. Then there exist functions* ϕ *satisfying the assumptions of* (i) *and such that* $T(\phi) < \infty$. *Moreover, if* $b > 0$ *and* $q > 1$, *then for all* $\psi \geq 0$, *radially symmetric nonincreasing, with* $\psi \in C^2(\overline{\Omega})$, $\psi|_{\partial\Omega} = 0$ *and* $\psi_r(R) < 0$, *one may take* $\phi = \lambda\psi$ *for all sufficiently large* λ.

(iii) *Let* $\Omega = \mathbb{R}^N$, $p > 1$, $q \geq 1$, $b \in \mathbb{R}$ *and let* ϕ *be as in* (i) *for some* $R > 0$. *Then the solution of* (P) *with initial data*

$$\bar{\phi}(x) = \begin{cases} \phi(x), & |x| < R, \\ 0, & |x| \geq R, \end{cases}$$

satisfies (6) *and* $T < \infty$.

2.2. *Parabolic inequalities.* Concerning the nonlinear parabolic inequality (2), we obtain the following upper blowup estimate.

THEOREM 4. (i) *Let* $Q_T = (0,T) \times (-R,R)$ *and* $p > 1$. *Let* $u \in C^{1,2}(Q_T)$ *satisfy*

$$u_t - u_{xx} \geq u^p \quad \text{in } Q_T,$$

where u *is symmetric as a function of* $r = |x|$ *and satisfies* $u \geq 0$, $u_t \geq 0$, $u_r \leq 0$ *in* Q_T. *Then* u *satisfies the estimate* (7).

(ii) *If we relax the assumption* $u_t \geq 0$ *above, then* u *still satisfies the weaker estimate*

$$\liminf_{t \to T} (T-t)^{1/(p-1)} \|u(t)\|_\infty < \infty.$$

Now consider a general nonlinear parabolic inequality of the form

$$u_t - u_{xx} \geq f(u).$$

We assume that

(10) $\quad f : (a, \infty) \to (0, \infty)$ is of class C^1 for some $a \geq 0$, with $\displaystyle\int^\infty \frac{ds}{f(s)} < \infty$,

and we set $G(s) = \int_s^\infty d\sigma / f(\sigma)$. Note that $G^{-1}(y)$ is well defined for $y > 0$ small. Also, we assume that there exists a real $\alpha > 0$ such that

(11) $$f'G \geq 1 + \alpha \quad \text{for } s > a.$$

We obtain the following result.

THEOREM 5. (i) *Assume that* f *satisfies* (10) *and* (11). *Let* $Q_T = (0,T) \times (-R,R)$, *and let* $u \in C^{1,2}(Q_T)$ *satisfy*

$$u_t - u_{xx} \geq f(u) \quad \text{for all } (t,x) \in Q_T \text{ such that } u(t,x) > a,$$

where u *is symmetric as a function of* $r = |x|$ *and satisfies* $u_t \geq 0$, $u_r \leq 0$ *in* Q_T. *Then* u *satisfies the estimate*

$$\|u(t)\|_\infty \leq G^{-1}(C(T-t)) \quad \text{as } t \to T$$

for some $C > 0$.

(ii) *If we relax the assumption* $u_t \geq 0$ *above, and instead only assume* $u > a$ *in* Q_T, *then* u *still satisfies the weaker estimate*

$$\liminf_{t \to T} \frac{\|u(t)\|_\infty}{G^{-1}(C(T-t))} < \infty$$

for some $C > 0$.

The hypothesis (11) means, roughly speaking, that $f(s)$ does not grow faster than some power of s as $s \to \infty$, but it allows arbitrarily slow growth (provided $\int^\infty ds/f(s) < \infty$). It is satisfied, for instance, if f is as in (5), or if $f(s) = s \log s (\log\log s)^p$ for s large and some $p > 1$, or also if $f(s) = s^p \log^q s$ for s large and some $p > 1$, $q \geq 0$. It is not satisfied for $f(s) = e^s$. For the three aforementioned examples, after some tedious but elementary calculations, we deduce from Theorem 5 the following estimates: $\|u(t)\|_\infty \leq \exp[C(T-t)^{-1/(p-1)}]$, $\|u(t)\|_\infty \leq \exp\exp[C(T-t)^{-1/(p-1)}]$ and $\|u(t)\|_\infty \leq C((T-t)|\log(T-t)|^q)^{-1/(p-1)}$, respectively.

Theorems 4 and 5 are complemented with the following lower estimate.

Proposition 3. *Assume that f satisfies* (10) *and $u \in C^{1,2}((0,T)\times B_R)$ satisfies*

$$u_t - \Delta u \leq f(u) \quad \textit{for all } (t,x) \in Q_T \textit{ such that } u(t,x) > a,$$

with u radially symmetric, $u_r \leq 0$, and $\lim_{t\to T} u(t,0) = \infty$. Then u satisfies the lower estimate

$$u(t,0) \geq G^{-1}(T-t).$$

2.3. *Comments and remarks.* For problem (P) in the case $b = 0$, the upper bound (7) is well known for p subcritical, i.e. $(N-2)p < N+2$, when Ω is a smoothly bounded convex domain or $\Omega = \mathbb{R}^N$ (see [41, 14], and also [23, 24] for further recent results). If Ω is a smoothly bounded convex domain, it is valid for all $p > 1$ provided $u_t \geq 0$ (see [11], and also [22] for a partial result in the case $\Omega = \mathbb{R}^N$). However, (7) may fail for large p in high dimensions (see [16]). On the other hand, the lower bound (8) holds for all $p > 1$ (see, e.g., [11]).

As usual, the upper estimate will be much harder to obtain than the lower one. Among the classical techniques known for $b = 0$, the method in [11], relying on maximum principle arguments, and that in [14], using scaling and energy methods, do not seem applicable here. In particular, unlike in the case $b = 0$, the equation (P) has no variational structure. We will get back to the approach of [41], the first one historically, which relies mainly on scaling arguments. However, unlike [11] and [14], this method has the disadvantage of being limited to radial solutions.

In order to handle gradient terms or to treat the case of parabolic inequalities, new ideas in comparison with the proof in [41] are needed. One of them is to consider a *time-average* of the spatially rescaled solution (see Lemma 3.1 and formula (15)). By doing so, we actually improve the original proof of [41] even in the case $b = 0$, by relaxing the assumption that u_t be radially nonincreasing.

On the other hand, let us remark that in previous work, the proof of upper blowup estimates relies on a reduction to some Liouville-type theorem for an autonomous elliptic equation (or ODE) in the whole space, after deriving some a priori estimates on ∇u and applying a compactness argument (see, e.g., [41, 14]). Instead of that, we here use a reduction to a (nonautonomous) ODE or differential inequality on a *finite* interval (see Lemma 3.2 and Proposition 3.3). An advantage is that we need no compactness argument, and consequently much less a priori estimates. And indeed, in the present case, it does not seem possible to obtain suitable a priori estimates in order to apply the usual procedure.

Concerning Theorem 5, let us point out that it can handle nonlinearities f which do not enjoy any homogeneity properties (see the examples after Theorem 5), although the method indirectly relies on scaling. If we compare with the method of [11], which also works for general nonlinearities when $u_t \geq 0$ (but only for equations), the latter has the advantage of being applicable also in higher dimensions and in nonradial situations. However, it requires f not growing up too slowly, in order that the blowup set be a compact subset of Ω. For instance, it does not apply when $f(u) = (1+u)\log^p(1+u)$ with $1 < p < 2$, in which case blowup is known to occur globally in Ω (see [19]).

REMARKS. 1. The result of Theorem 1 remains valid for the more general equation

$$u_t - \Delta u = |u|^{p-1}u + F(u, |\nabla u|),$$

where F is locally Lipschitz continuous and satisfies the growth condition

$$|F(u, |\nabla u|)| \leq C(1 + |u|^q) + \varepsilon |\nabla u|^{2p/(p+1)}$$

with $1 \leq q < p$, $C > 0$, $\varepsilon > 0$ sufficiently small, and $p < (N+2)/(N-2)$ if $N \geq 3$.

2. The proofs of Theorems 1–3 show that the $\limsup$ appearing in the estimate (7) is bounded independently of the solution u. Under the assumptions of Theorems 1–3, we also obtain the following information on the blowup profile: there exists a constant $C > 0$ (independent of u) such that

$$\frac{u(t, |y|\sqrt{T-t})}{u(t,0)} \geq 1 - C|y|$$

for t close to T and $|y|$ sufficiently small. This follows from formulae (12), (13) and Lemma 3.5.

3. Let us recall that the value $q = 2p/(p+1)$ is critical with respect to scaling arguments (see [5, 1, 35, 31] and formula (13) below). In particular, equation (P) for $q = 2p/(p+1)$ enjoys the same scale invariance as for $b = 0$. This partly explains the difference in our results of Theorems 1–3. Also, we note that when $q \geq 2p/(p+1)$, the gradient term is not scaled out from the equation (see formula (13)), so that our results are not mere perturbations of the case $b = 0$.

4. One can be more precise concerning the admissible values of $b < 0$ in Theorem 2 ($q = 2p/(p+1)$), if we further assume $N = 1, 2$ or $p < N/(N-2)$. Namely, we may then take $-b_{p,N} < b < 0$, where

$$b_{p,N} = (p+1)\left(\frac{N}{2p} - \frac{N-2}{2}\right)^{p/(p+1)}.$$

Moreover, if also $N \geq 2$, we may take $b = -b_{p,N} - \varepsilon$ with $\varepsilon \geq 0$ small. (See the proof of Proposition 3.3 in §3.3.)

5. In the nonmonotone case (ii) of Theorem 4, we actually prove a bit more. Namely, setting $\overline{t} = (t+T)/2$, we have

$$\frac{1}{\overline{t}-t}\int_t^{\overline{t}} u(s,0)\,ds \leq C(T-t)^{-1/(p-1)} \qquad \text{as } t \to T.$$

In other words, the upper estimate (7) is satisfied "on average".

6. The conclusion of Proposition 1 actually holds for any solution (not necessarily radial) in any domain, with $\|u(t)\|_\infty$ instead of $u(t,0)$. This can be proved by the technique of [11, Theorem 4.5] (see also [40] for a different approach). On the other hand, the hypothesis $\phi \in C^2(\overline{\Omega})$ in Proposition 2 can be weakened to $\phi \in C^1(\overline{\Omega})$ with (9) being then understood in the weak sense.

3. Proofs of the upper estimates

3.1. *Proof of Theorems 1 and 2.* We define the following auxiliary functions:

$$\alpha(t) = [u(t,0)]^{(p-1)/2} \quad \text{and} \quad v(t,r) = \frac{u(t,r\alpha^{-1}(t))}{u(t,0)}, \tag{12}$$

for $0 < t < T$ and $0 \leq r < R\alpha(t)$ (with $R = \infty$ if $\Omega = \mathbb{R}^N$). Under the assumptions of Theorem 1, we may obviously suppose that $\lim_{t\to T}\alpha(t) = \infty$, since otherwise (7) is trivially satisfied. Moreover, we have

$$0 \leq v(t,r) \leq 1, \quad v_r(t,r) \leq 0, \quad v(t,0) = 1 \quad \text{and} \quad v_r(t,0) = 0.$$

We first observe that

$$a(t,r) \equiv \Delta v + v^p + b\alpha^{-m}(t)|\nabla v|^q = \frac{u_t(t,r\alpha^{-1}(t))}{u^p(t,0)} \geq 0, \tag{13}$$

where $m = (2p - q(p+1))/(p-1) \geq 0$.

A key step of the proofs of Theorems 1, 2 and 4 is the following lemma, which enables one to relate the blowup rate of u with a time-average of the right-hand side of (13).

LEMMA 3.1. *For all $t \in (0,T)$ and all $r \in [0, R\alpha(t))$, we have*

$$\frac{1}{T-t}\int_t^T \frac{u_t(s, r\alpha^{-1}(s))}{u^p(s,0)}\,ds \le \frac{p}{p-1}\,\frac{u^{1-p}(t,0)}{T-t} \equiv g(t).$$

Proof. For a fixed r, let $\gamma(t) = u(t, r\alpha^{-1}(t))$, which is defined for all t such that $R\alpha(t) > r$. We note that

$$\gamma'(t) = u_t(t, r\alpha^{-1}(t)) - \frac{r\alpha'(t)}{\alpha^2(t)}\,u_r(t, r\alpha^{-1}(t)) \ge u_t(t, r\alpha^{-1}(t)),$$

since $u_t \ge 0$ and $u_r \le 0$. Let $\tau \in (t,T)$. Integrating by parts and using $\gamma(s) \le u(s,0)$ (since $u_r \le 0$), we obtain

$$\begin{aligned}\int_t^\tau \frac{u_t(s, r\alpha^{-1}(s))}{u^p(s,0)}\,ds &\le \int_t^\tau \frac{\gamma'(s)}{u^p(s,0)}\,ds = \left[\frac{\gamma(s)}{u^p(s,0)}\right]_t^\tau + p\int_t^\tau \frac{\gamma(s)\,u_t(s,0)}{u^{p+1}(s,0)}\,ds \\ &\le \frac{\gamma(\tau)}{u^p(\tau,0)} + p\int_t^\tau \frac{u_t(s,0)}{u^p(s,0)}\,ds \\ &\le u^{1-p}(\tau,0) - \frac{p}{p-1}[u^{1-p}(s,0)]_t^\tau \le \frac{p}{p-1}u^{1-p}(t,0).\end{aligned}$$

The lemma then follows by letting $\tau \to T$. ∎

From Lemma 3.1 we next deduce the following lemma, which allows us to reduce the proof of Theorems 1 and 2 to a nonexistence result for a certain ODE.

LEMMA 3.2. *For all $R' > 0$ and all $t \in (0,T)$ sufficiently close to T, there exists $t' \in (t,T)$ such that $w(r) \equiv v(t', r)$, $h(r) \equiv a(t', r)$ and $\beta = \alpha^{-m}(t')b$ satisfy*

$$\begin{cases} w_{rr} + \dfrac{N-1}{r}w_r + |w|^p + \beta|w_r|^q = h(r), \qquad 0 < r < R', \\ w(0) = 1, \qquad w_r(0) = 0, \\ w \ge 0, \qquad w_r \le 0, \qquad 0 < r < R', \end{cases} \tag{14}$$

with $h \ge 0$ and $\int_0^{R'} h(r)\,dr \le R'g(t)$.

Proof. From (13) and Lemma 3.1, we deduce that

$$\frac{1}{T-t}\int_t^T\int_0^{R'} a(s,r)\,dr\,ds \le R'g(t)$$

for all $t \in (0,T)$ such that $R\alpha(t) > R'$. Therefore there exists $t' \in (t,T)$ such that $\int_0^{R'} a(t', r)\,dr \le R'g(t)$ and the lemma follows. ∎

The proof of Theorems 1 and 2 then relies on the following ODE result, whose proof is postponed to §3.3.

PROPOSITION 3.3. *Let $p > 1$, $q \geq 1$, R', $\varepsilon > 0$, $\beta \in \mathbb{R}$, and let $h \in C([0, R'])$ satisfy $\int_0^{R'} |h(r)|\,dr \leq \varepsilon$. Then there does not exist any solution $w \in C^2([0, R'])$ of* (14), *under each of the following circumstances*:

(a) $-b_0 < \beta < b_0, \quad p < \dfrac{N+2}{N-2} \quad (\textit{if } N > 2), \quad R' \geq R_0, \quad \varepsilon \leq \varepsilon_0;$

(b) $\beta > 0, \quad q > 1, \quad q \leq \dfrac{N}{N-1} \quad (\textit{if } N > 1), \quad h \geq 0, \quad R' \geq R_1, \quad \varepsilon \leq \varepsilon_1;$

(c) $\beta \geq b_1, \quad q > 1, \quad R' \geq 4, \quad \varepsilon \leq \varepsilon_2,$

where the numbers $b_0, R_0, \varepsilon_0, b_1 > 0$ depend only on p, q, N, the numbers $R_1, \varepsilon_2 > 0$ depend only on p, q, N, β, and the number $\varepsilon_1 > 0$ depends only on p, q, N and R'.

End of proof of Theorems 1 and 2. If $q < 2p/(p+1)$ (hence $m > 0$) and $b \in \mathbb{R}$, or if $q = 2p/(p+1)$ and $-b_0 < \beta < b_0$, we have $-b_0 < b < b_0$ for t close to T. If we assume $p < (N+2)/(N-2)$ if $N \geq 3$, and choose $R' = R_0(p, q, N)$, it follows from Lemma 3.2 and Proposition 3.3(a) that we must have $R'g(t) > \varepsilon_0(p, q, N)$ for t close to T. The results of Theorems 1 and 2(iii) follow.

Assume $q = 2p/(p+1)$ and $b = \beta \geq b_1$. By choosing $R' = R_2(p, q, N)$, it follows from Proposition 3.3(c) that we must have $R'g(t) > \varepsilon_2(p, q, N, b)$ for t close to T. The result of Theorem 2(i) follows.

The result of Theorem 2(ii) follows in the same way from Proposition 3.3(b). (Note that when $q = 2p/(p+1)$ and $N > 2$, $p \leq N/(N-2)$ is equivalent to $q \leq N/(N-1)$.) ■

REMARK 3.1. If the number ε_2 in Proposition 3.3(c) were independent of β, then this would give the result of Theorem 3 ($q > 2p/(p+1)$) for all $N \geq 1$. Actually, we will prove Theorem 3 (only for $N = 1$) in the next section by a different argument.

3.2. *Proof of Theorems 3, 4 and 5*

Proof of Theorem 4(i). We first treat the case $u_t \geq 0$. In addition to α and v (see formula (12)), we define the auxiliary function

$$z(t, r) = \frac{1}{\bar{t} - t} \int_t^{\bar{t}} v(s, r)\,ds, \tag{15}$$

where $\bar{t} = (t+T)/2$, which is defined for $0 < t < T$ and $0 \leq r < R\alpha(t)$.

From the assumptions of Theorem 2, v satisfies

$$\Delta v + v^p \le \frac{u_t(t, r\alpha^{-1}(t))}{u^p(t,0)}.$$

By using Lemma 3.1 and Hölder's (or Jensen's) inequality, it follows easily that z satisfies

$$\begin{cases} \Delta z + z^p \le \dfrac{2p}{p-1}\dfrac{u^{1-p}(t,0)}{T-t} = 2g(t), & 0 < r < R\alpha(t), \\ z(0) = 1, \quad z_r(0) = 0. \end{cases}$$

We now assume $N = 1$. The result of Theorem 4(i) is then an immediate consequence of the following differential inequality lemma.

LEMMA 3.4. *Let $p \ge 1$ and $\varepsilon, R' > 0$, and consider the problem*

$$\begin{cases} w_{rr} + w^p \le \varepsilon, & 0 < r < R', \\ w \ge 0, \quad w_r \le 0, & 0 < r < R', \\ w(0) = 1, \quad w_r(0) = 0. \end{cases} \tag{16}$$

Then there does not exist any solution $w \in C^2([0, R'])$ of (16) whenever $R' \ge R_3$ and $\varepsilon \le \varepsilon_3$, where R_3 and $\varepsilon_3 > 0$ depend only on p.

Proof. From (16), we have

$$w_r(r) + \int_s^r w^p(\tau)\,d\tau \le w_r(s) + \varepsilon(r-s), \quad 0 \le s < r < R'. \tag{17}$$

By further integrating, we get

$$-1 \le w(r) - w(s) \le (r-s)w_r(s) + \frac{\varepsilon}{2}(r-s)^2, \quad 0 \le s < r < R',$$

hence

$$|w_r(s)| \le \frac{1}{r-s} + \frac{\varepsilon}{2}(r-s), \quad 0 \le s < r < R'.$$

Now assume $\varepsilon \le 1/2$ and $R' \ge 2/\varepsilon$. By choosing $r = s + 1/\varepsilon$ in the above inequality, it follows that, for all $s \in [0, 1/2]$, we have $|w_r(s)| \le \varepsilon + 1/2 \le 1$, hence $w(s) \ge 1/2$. But then applying (17) with $s = 0$, we deduce that

$$w_r(r) \le \varepsilon r - \int_0^r w^p(\tau)\,d\tau \le \varepsilon r - \frac{1}{2^{p+1}} \le -\frac{1}{2^{p+2}} \quad \text{for all } r \in \left[\frac{1}{2}, \frac{1}{2^{p+2}\varepsilon}\right],$$

where we have assumed $\varepsilon \le 1/2^{p+1}$. One more integration yields

$$w\left(\frac{1}{2^{p+2}\varepsilon}\right) \le 1 - \frac{1}{4^{p+2}\varepsilon},$$

a contradiction for $\varepsilon < 1/4^{p+2}$. Lemma 3.4 follows. ∎

Proof of Theorem 4(ii). Fix $t_0 \in (0,T)$ and let

$$V(t,r) = \frac{1}{T-t_0}\int_{t_0}^{t} u(s,r)\,ds.$$

For $t \in (t_0,T)$, we compute

$$\begin{aligned} V_t - \Delta V &= \frac{1}{T-t_0}\Big(u(t,r) - \int_{t_0}^{t} \Delta u(s,r)\,ds\Big) \\ &\geq \frac{1}{T-t_0}\int_{t_0}^{t} (u_t(s,r) - \Delta u(s,r))\,ds \geq \frac{1}{T-t_0}\int_{t_0}^{t} u^p(s,r)\,ds \\ &\geq \left(\frac{T-t_0}{t-t_0}\right)^{p-1} V^p(t,r) \geq V^p(t,r), \end{aligned}$$

where we used Jensen's inequality. Note that $V_t \geq 0$ and $V_r \leq 0$. By setting

$$\overline{g}(t) = \frac{p}{p-1}\frac{V^{1-p}(t,0)}{T-t} \quad \text{and} \quad \overline{\alpha}(t) = V^{(p-1)/2}(t,0)$$

it thus follows from the proof of case (i) and from Lemma 3.4 that either $R\overline{\alpha}(t) \leq R_3$ or $2\overline{g}(t) \geq \varepsilon_3$. In other words, we have

$$\begin{aligned} V(t,0) &\leq \max\left(\left(\frac{R_3}{R}\right)^{2/(p-1)}, \left(\frac{2p}{(p-1)\varepsilon_3}\right)^{1/(p-1)}(T-t)^{-1/(p-1)}\right) \\ &\leq C(T-t)^{-1/(p-1)}, \end{aligned}$$

with C independent of t_0. In particular, we get

$$\begin{aligned} \inf_{t_0<t<T} u(t,0) &\leq \frac{2}{T-t_0}\int_{t_0}^{(t_0+T)/2} u(s,0)\,ds \\ &= 2V\left(\frac{t_0+T}{2},0\right) \leq 2C\left(\frac{T-t_0}{2}\right)^{-1/(p-1)}, \end{aligned}$$

and the conclusion follows. ∎

Theorem 3 is an immediate consequence of Theorem 4(i).

Proof of Theorem 5. (i) We may assume that $\lim_{t\to T} u(t,0) = \infty$, since otherwise the conclusion is immediate. In particular, there exists $t_0 \in (0,T)$ such that $u(t_0,0) \geq a+1$. By continuity, it follows that for some $R' \in (0,R)$, we have $u(t_0,x) > a$ whenever $|x| < R'$. Since $u_t \geq 0$, up to replacing R with R' and $(0,T)$ with (t_0,T), we may therefore assume that $u > a$ in Q_T.

Noting that G is strictly decreasing and maps (a,∞) onto $(0,c)$ with $0 < c = G(a+0) \leq \infty$, we may set

$$\gamma = 1/\alpha, \quad h(s) = G^{-1}(\alpha/s^\gamma) \quad \text{and} \quad H(y) = 1/G^\alpha(y) = \gamma^\alpha h^{-1}(y),$$

and in particular $h^{-1}(y)$ is well defined for $y > a$. The functions G, h and H are of class C^2 and we have

$$G' = \frac{-1}{f} < 0, \qquad H' = -\alpha \frac{G'}{G^{\alpha+1}} = \frac{\alpha}{fG^{\alpha+1}} > 0,$$

hence $h' > 0$. Moreover, H'' has the same sign as $-(fG^{\alpha+1})'$, that is, the same sign as $-f'G-(\alpha+1)fG' = \alpha+1-f'G$, which is ≤ 0 by assumption. It follows that H is concave increasing, so that h is convex. On the other hand, we have $G \circ h(s) = \alpha/s^\gamma$, hence

$$\frac{1}{s^{1+\gamma}} = -G' \circ h(s) \cdot h'(s) = \frac{h'(s)}{f \circ h(s)}.$$

Now setting $U = h^{-1}(u)$, and using $h' > 0$ and $h'' \ge 0$, we obtain

$$\begin{aligned} f(h(U)) &\le u_t - u_{xx} = h'(U)(U_t - U_{xx}) - h''(U)(U_x)^2 \\ &\le h'(U)(U_t - U_{xx}), \end{aligned}$$

hence

$$U_t - U_{xx} \ge \frac{f(h(U))}{h'(U)} = U^{1+\gamma}.$$

Since $h' > 0$, the assumptions on u entail $U_r \le 0$ and $U_t \ge 0$. It follows from Theorem 4(i), applied to U with $p = 1+\gamma$, that $\|U(t)\|_\infty \le C(T-t)^{-1/\gamma}$ as $t \to T$. We then conclude that $\|u(t)\|_\infty \le h[C(T-t)^{-1/\gamma}] = G^{-1}[\alpha C^{-\gamma}(T-t)]$ as $t \to T$.

(ii) This follows similarly from Theorem 4(ii). ■

Remark 3.2. The only reason why our results on parabolic inequalities (and also Theorem 3) are restricted to one space dimension is that we are unable to prove the higher-dimensional analogue of Lemma 3.4 (with an additional $((N-1)/r)w_r$ term on the left-hand side of $(14)_1$).

Note that the nonexistence of nonnegative nontrivial C^2 solutions to $\Delta u + u^p \le 0$ in the whole space was proved in [3] under the assumption $(N-2)p \le N$. But it does not seem possible to extend the proof therein to obtain a higher-dimensional analogue of Lemma 3.4.

Actually, we can prove that the conclusion of Theorem 3 remains valid in dimension $N \ge 2$, provided $q > N$. However, since then $q > 2$, one cannot discard the possibility of gradient blowup, i.e. $\|u(t)\|_\infty$ remaining bounded while $\lim_{t\to T} \|u_r(t)\|_\infty = \infty$, in which case the estimate (7) has no interest.

3.3. *Proof of Proposition 3.3.* The proof is rather technical. We have to consider the three cases (a), (b), (c) separately.

CASE (a). Denote by $\overline{w}$ the solution of

$$\begin{cases} \overline{w}_{rr} + \dfrac{N-1}{r}\overline{w}_r + |\overline{w}|^{p-1}\overline{w} = 0, & r > 0, \\ \overline{w}(0) = 1, \quad \overline{w}_r(0) = 0. \end{cases} \tag{18}$$

Since $E(r) = \overline{w}_r^2/2 + |\overline{w}|^{p+1}/(p+1)$ is nonincreasing, $\overline{w}$ exists for all $r > 0$. Under the assumption $p < (N+2)/(N-2)$ (if $N > 2$), it is well known (see [17] or [15, Theorem 4]) that there exists $r_0 > 0$ such that $\overline{w}(r_0) < 0$. The conclusion follows by continuous dependence, with $R_0 = r_0$.

To prove the assertion of Remark 4 concerning the case $q = 2p/(p+1)$, $b < 0$, we may use the sharp results of [5, 9, 39]. These results state that the solution of (18) with an additional $b|\overline{w}_r|^q$ term on the left-hand side achieves some negative values, provided b, p, N are as described in Remark 4. (See [5, Proposition 4.11], [9, Theorem 1] and [39, Theorem IV.1].)

CASE (b). We will need the following lemma (which, in turn, is used in Remark 2).

LEMMA 3.5. *Assume that w solves* (14) *with $h \geq 0$ and $\beta \in \mathbb{R}$. Then there exists $r_1 = r_1(p,q,\beta) > 0$ such that $|w_r| \leq 1$ and $w \geq 1/2$ on $[0,r_1]$. Moreover, r_1 is bounded away from* 0 *when β remains bounded above.*

Proof. Let $E(r) = w_r^2/2 + w^{p+1}/(p+1)$. Multiplying by $w_r \leq 0$, we get

$$E_r(r) = (w_{rr} + w^p)w_r \leq \beta|w_r|^{q+1} \leq \beta(2E)^{(q+1)/2}.$$

Since $E(0) = 1/(p+1) < 1/2$, we deduce that $w_r^2 \leq 2E(r) \leq 1$ for $r \leq r_1$ where $r_1 = r_1(p,q,\beta) > 0$ is small (bounded away from 0 when β remains bounded above). Since $w(0) = 1$, by further assuming $r_1 \leq 1/2$, it follows that $w \geq 1/2$ on $[0, r_1]$. ∎

Let

$$\Theta(r) = \int_0^r s^{N-1}(w^p + \beta|w_r|^q)(s)\,ds.$$

By integrating (14) and using the assumption on h, we obtain

$$r^{N-1}|w_r(r)| \geq \Theta(r) - r^{N-1}\varepsilon, \quad 0 < r < R'.$$

On the other hand, Lemma 3.5 implies that $\Theta(r_1) \geq r_1^N 2^{-p} N^{-1}$. Now assume $\varepsilon \leq \varepsilon_1 \equiv r_1^N 2^{-(p+1)} N^{-1} R'^{1-N}$. Then

$$\Theta_r(r) \geq \beta r^{N-1}|w_r(r)|^q \geq \beta r^{-(N-1)(q-1)} 2^{-q} \Theta^q, \quad r_1 < r < R'.$$

By integrating, it follows that

$$\frac{1}{q-1}\left(\frac{r_1^N}{2^p N}\right)^{1-q} \geq \frac{\Theta^{1-q}(r_1)}{q-1} \geq \int_{r_1}^{r} \frac{\Theta_r(s)}{\Theta^q(s)}\,ds$$
$$\geq \beta 2^{-q} \int_{r_1}^{r} s^{-(N-1)(q-1)}\,ds, \quad r_1 \leq r < R'.$$

Since $(N-1)(q-1) \leq 1$ by assumption, this last integral diverges as $r \to \infty$. Therefore, we must have $R' \leq R_1$ for some $R_1(N,p,q,\beta) > 0$, and the conclusion follows.

CASE (c). This is more involved. We first need the following lemma.

LEMMA 3.6. *Assume that w solves*

$$\begin{cases} w_{rr} + \dfrac{N-1}{r} w_r + |w|^p + \beta |w_r|^q = 0, & 0 < r < R^*, \\ w(0) = 1, \quad w_r(0) = 0, \end{cases} \tag{19}$$

with $q > 1$ and $R^* \in (0,\infty]$ the maximum existence time of w. Then there exists $b_1 = b_1(N,p,q) > 0$ such that if $\beta \geq b_1$, then $R^* < 2$, and w satisfies

$$\lim_{r \to R^*} w_r(r) = -\infty \quad \text{and} \quad \int_0^{R^*} |w_r(s)|^q\,ds = \infty. \tag{20}$$

Proof. STEP 1. We first prove (20) assuming $R^* < \infty$. We have

$$-r^{N-1} w_r(r) = \int_0^r s^{N-1}(|w|^p + \beta |w_r|^q)(s)\,ds, \tag{21}$$

hence $w_r < 0$ for all $r \in (0,R^*)$. Moreover, $r^{N-1}|w_r(r)|$ is nondecreasing. Therefore, if $R^* < \infty$, we must have $w_r(r) \to -\infty$ as $r \to R^*$. On the other hand, $\int_0^{R^*} |w_r(s)|^q\,ds = \infty$, since otherwise w would be bounded on $[0,R^*)$, and (21) would imply w_r bounded, a contradiction.

Since for $N = 1$ the result of Lemma 3.6 follows from Lemma 3.4, we may now assume $N > 1$.

STEP 2. We claim that $w_{rr} \leq 0$ for all $r \in (0,R^*)$ such that $r \leq ((N-1)/p)^{1/2}$. To prove this, first note that $w_{rr}(0) = -1/N < 0$. By differentiating $(19)_1$, we obtain

$$w_{rrr} + \left(\frac{N-1}{r} - \beta q |w_r|^{q-1}\right) w_{rr} + \left(p|w|^{p-2}w - \frac{N-1}{r^2}\right) w_r = 0.$$

Assume that there is a first $r > 0$ such that $w_{rr}(r) = 0$. Then $w_{rrr}(r) \geq 0$, hence $(N-1)/r^2 \leq p|w(r)|^{p-2}w(r) \leq p$, which proves the claim.

STEP 3. Let $r_2 = \min(1, ((N-1)/p)^{1/2})$. By (19), we have

$$|w_r(r)| \geq \Theta(r), \qquad r_2 \leq r \leq 2r_2, \tag{22}$$

where

$$\Theta(r) = \int_0^{r_2} \left(\frac{s}{2r_2}\right)^{N-1} (|w|^p + \beta|w_r|^q)(s)\, ds + \frac{\beta}{2^{N-1}} \int_{r_2}^{r} |w_r(s)|^q\, ds.$$

Assume $\beta \geq 1$. We then claim that $\Theta(r_2) \geq \eta(N,p,q) > 0$. If $w(r_2/2) \geq 1/2$, then $\Theta(r_2) \geq 2^{-1-(2N+p)} N^{-1} r_2$. Otherwise, by the Mean Value Theorem, there exists $r' \in (0, r_2/2)$ such that $|w_r(r')| \geq 1/r_2$. But then $|w_r| \geq 1/r_2$ on $[r_2/2, r_2]$ by Step 2, so that $\Theta(r_2) \geq 2^{1-2N} r_2^{1-q}$.

STEP 4. Thanks to (22) and Step 3, Θ satisfies

$$\begin{cases} \Theta_r = \dfrac{\beta}{2^{N-1}} |w_r|^q \geq \dfrac{\beta}{2^{N-1}} \Theta^q, \qquad r_2 \leq r \leq 2r_2, \\ \Theta(r_2) \geq \eta(N,p,q) > 0. \end{cases}$$

It follows easily that if $\beta \geq b_1$, where $b_1 = b_1(p,q,N)$ is sufficiently large, then Θ has to blow up before $r = 2r_2$. This contradiction concludes the proof of Lemma 3.6. ∎

We can now complete Case (c). Let $\beta \geq b_1$ and let w_0 be the corresponding solution of (19). Let $A > 1$ to be fixed later. By Lemma 3.6, the maximal existence time R^* of w_0 satisfies $R^* < 2$, and there exists $\overline{r} \in (R^*/2, R^*)$ such that $\int_{R^*/2}^{\overline{r}} \beta|w_{0,r}(s)|^q\, ds \geq 2^N A + 1$.

Assume that w exists for $r \in [0, 2R^*]$. By continuous dependence, there exists $\varepsilon_2 = \varepsilon_2(N,p,q,\beta) \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_2]$,

$$\int_{R^*/2}^{\overline{r}} \beta|w_r(s)|^q\, ds \geq \int_{R^*/2}^{\overline{r}} \beta|w_{0,r}(s)|^q\, ds - 1 \geq 2^N A.$$

It follows that for all $r \in [R^*, 2R^*]$,

$$\begin{aligned} |w_r(r)| &\geq \int_0^r \left(\frac{s}{r}\right)^{N-1} (w^p(s) + \beta|w_r(s)|^q)\, ds - \varepsilon \\ &\geq \frac{1}{2^{N-1}} \int_{R^*/2}^{\overline{r}} \beta|w_r(s)|^q\, ds - \varepsilon + \frac{\beta}{4^{N-1}} \int_{\overline{r}}^{r} |w_r(s)|^q\, ds \\ &\geq A + \frac{\beta}{4^{N-1}} \int_{\overline{r}}^{r} |w_r(s)|^q\, ds \equiv \Lambda(r), \end{aligned}$$

hence

$$\begin{cases} \Lambda_r \geq \dfrac{\beta}{4^{N-1}}\Lambda^q, & R^* \leq r \leq 2R^*, \\ \Lambda(R^*) \geq A. \end{cases}$$

But if A is taken sufficiently large (depending on N, q, R^*, β), then Λ has to blow up before $r = 2R^*$, a contradiction. The result follows. ∎

4. Proof of Propositions 1, 2 and 3

Proof of Proposition 1. The assumptions on u imply $\nabla u(t,0) = 0$ and $\Delta u(t,0) \leq 0$. Therefore,

$$u_t(t,0) \leq u^p(t,0). \tag{23}$$

Moreover, since $q \leq 2$, gradient blowup is excluded, so that $\lim_{t\to T} u(t,0) = \infty$. The conclusion then follows immediately by integrating the differential inequality (23) between t and T. ∎

Proof of Proposition 3. From the assumptions, we have $u(t,0) \geq a$ for t close to T and $\Delta u(t,0) \leq 0$. Therefore, $u_t(t,0) \leq G(u(t,0))$ and the conclusion follows by integrating between t and T. ∎

Proof of Proposition 2. (i) This is proved e.g. in [2, Theorem 3.3 and p. 64] in the case $b = 0$. The proof immediately carries over to the case $b \neq 0$.

(ii) We may assume $b < 0$, the case $b \geq 0$ being much easier. For all $\delta \in (0, R/4)$ it is clear that there exists a unique function $\chi \in C^2([0,R])$ such that

$$\chi''(r) = \begin{cases} -2 & \text{in } [0,\delta], \\ 0 & \text{in } [2\delta, R-2\delta], \\ 2 & \text{in } [R-\delta, R], \end{cases}$$

χ'' is linear on $[\delta, 2\delta]$ and on $[R-2\delta, R-\delta]$, and $\chi'(R) = \chi(R) = 0$. One easily verifies that $\chi'(0) = 0$, $\chi(r) \geq 0$, $-4\delta < \chi'(r) \leq 0$, $\chi''(r) \geq -2$ on $[0,R]$, $-4\delta < \chi'(r) \leq -2\delta$ on $[\delta, R-\delta]$, hence $\chi(r) \geq R\delta$ on $[0, R/4]$, $\chi(r) = (R-r)^2$ on $[0, R-\delta]$, and $\chi(r) \geq \delta^2$ on $[R-\delta, R]$. Setting $\phi(x) = k\chi(r)$, $r = |x|$, with $k > 0$, we get, for $r \in [R-\delta, R]$,

$$\Delta\phi + \phi^p + b|\nabla\phi|^q = k\left[2 + \frac{N-1}{r}2(r-R) - |b|(2(R-r))^q k^{q-1} + k^{p-1}\chi^p\right].$$

By choosing $\delta = \frac{1}{2}|b|^{-1/q}k^{(1-q)/q}$ with k large enough if $q > 1$, or δ small with any $k > 0$ if $q = 1$, it follows that

$$\Delta\phi + \phi^p + b|\nabla\phi|^q \geq k\left[2 - \frac{N-1}{r}2\delta - |b|(2\delta)^q k^{q-1}\right] \geq 0, \qquad r \in [R-\delta, R].$$

For $r \in [0, R-\delta)$, the assumptions on χ imply that $\Delta\chi \geq N \min \chi'' \geq -2N$ and that

$$\begin{aligned}\Delta\phi + \phi^p + b|\nabla\phi|^q &\geq k[-2N - |b|(4\delta)^q k^{q-1} + k^{p-1}\delta^{2p}] \\ &= k[-2N - 2^q + (2|b|^{1/q})^{-2p} k^{p-1-2p(q-1)/q}].\end{aligned}$$

The latter expression is positive either if $p-1-2p(q-1)/q > 0$, that is, $q < 2p/(p+1)$, and k is large, or if $p-1-2p(q-1)/q = 0$, that is, $q = 2p/(p+1)$, and $|b|$ is small enough (depending on p, N). It follows that ϕ satisfies (9).

On the other hand, we observe that $\phi(r) \geq Rk\delta$ on $[0, R/4]$, with $Rk\delta = \frac{1}{2}R|b|^{-1/q}k^{1/q}$ if $q > 1$. The fact that $T(\phi) < \infty$ for k large is then a consequence of [36, Theorem 1].

If $b > 0$, with ψ as in the statement of Proposition 2(ii), we have $|\psi'(r)| \geq \varepsilon > 0$, $R-\eta \leq r < R$, for some ε, $\eta > 0$. Thus we get $\Delta\phi + \phi^p + b|\nabla\phi|^q \geq \lambda(\Delta\psi + b\varepsilon^q\lambda^{q-1})$ for $r \in [R-\eta, R]$, and $\Delta\phi + \phi^p + b|\nabla\phi|^q \geq \lambda(\Delta\psi + \psi^p(R-\eta)\lambda^{p-1})$ for $r \in [0, R-\eta)$. Since $p > 1$ and $q > 1$, the conclusion follows when λ is sufficiently large, the fact that $T(\lambda\psi) < \infty$ being again a consequence of [36, Theorem 1].

(iii) Since $\overline{\phi}$ still satisfies (9) in the weak sense, the fact that $u_t \geq 0$ follows from a straightforward modification of the proof of [2, Theorem 3.1]. On the other hand, if $\underline{u}$ denotes the solution of (P) in B_R with initial data ϕ, by the maximum principle we have $u(t,x) \geq \underline{u}(t,x)$ for all $x \in B_R$ as long as u and $\underline{u}$ exist. Since $\|\underline{u}(t)\|_\infty \to \infty$ as $t \to T(\phi) < \infty$, we conclude that $T(\overline{\phi}) < \infty$. ∎

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