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NONCOMMUTATIVE POINCARÉ RECURRENCE THEOREM

BY

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Abstract. Poincaré's classical recurrence theorem is generalised to the noncommutative setup where a measure space with a measure-preserving transformation is replaced by a von Neumann algebra with a weight and a Jordan morphism leaving the weight invariant. This is done by a suitable reformulation of the theorem in the language of L^∞-space rather than the original measure space, thus allowing the replacement of the commutative von Neumann algebra L^∞ by a noncommutative one.

Introduction. Poincaré's celebrated recurrence theorem (see e.g. [2]) states that given a probability space $(\Omega, \mathcal{F}, \mu)$, a measure-preserving transformation T and any set E in $\mathcal{F}$, the set of those points ω in E for which $T^n\omega$, $n = 1, 2, \ldots$, does not return to E infinitely often has measure zero. An equivalent (though seemingly weaker) version says that the set of those ω in E for which $T^n\omega$, $n = 1, 2, \ldots$, is never in E has measure zero. This amounts to saying that for almost all ω in E, $T^n\omega$ belongs to E for some n, which in turn means that for almost all ω in Ω we have $\chi_E(T^n\omega) = 1$ for some n whenever $\chi_E(\omega) = 1$, χ_E being the indicator function of the set E. This property can be stated in the form

$$\chi_E(\omega) \le \sup_n \chi_E(T^n\omega) \quad \text{for almost all } \omega.$$

Now if we look at this problem from the "L^∞ point of view", and denote by $\widehat{T^n}$ the transformation induced in $L^\infty(\Omega, \mathcal{F}, \mu)$ by T^n, i.e. $(\widehat{T^n} f)(\omega) = f(T^n\omega)$, we get the inequality

$$\chi_E \le \sup_n \widehat{T^n}\chi_E. \tag{1}$$

The last formulation allows a straightforward generalisation to the noncommutative context by replacing $L^\infty(\Omega, \mathcal{F}, \mu)$ by a von Neumann algebra with a finite trace, the transformation $\widehat{T}$ by a $*$-automorphism α of this algebra leaving the trace invariant, and the function χ_E, which is a projection in

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$L^\infty(\Omega,\mathcal{F},\mu)$, by a projection p in the von Neumann algebra. The inequality (1) then reads

$$p \leq \bigvee_{n=1}^{\infty} \alpha^n(p).$$

Now the more customary version of Poincaré's recurrence theorem in the noncommutative setup would be: Let M be a von Neumann algebra with a finite trace, and let α be a $*$-automorphism of M leaving the trace invariant. Then for each projection p in M we have

$$p \leq \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} \alpha^n(p). \tag{2}$$

It may appear a little surprising that this formulation allows a further generalisation in which the trace is replaced by a faithful weight, and the automorphism by a Jordan morphism. However, in this case inequality (2) is proved only for projections with orthogonal complement "not too big" as measured by the weight, and "not too big" themselves as measured by values of the Jordan morphism on the identity of the algebra.

1. Preliminaries and notation. Throughout, M will stand for a von Neumann algebra with identity $\mathbf{1}$. For a projection p in M we denote by $p^\perp$ the orthogonal complement of p, i.e. $p^\perp = \mathbf{1} - p$.

A *weight* ψ on M is an additive and positively homogeneous map from the positive part M_+ of M into $[0,\infty]$. We call ψ *faithful* if for each x in M_+ the condition $\psi(x) = 0$ implies $x = 0$. A fairly elaborated account of the theory of (normal) weights can be found in [3]; however, for our purposes we shall not need any of the usually assumed properties of such weights as, for instance, normality or semifiniteness.

A *Jordan morphism* α on M is a linear map from M into M satisfying

(i) $\alpha(x^*) = \alpha(x)^*$ for $x \in M$;

(ii) $\alpha(xy + yx) = \alpha(x)\alpha(y) + \alpha(y)\alpha(x)$ for $x, y \in M$.

In particular, we have $\alpha(x^2) = \alpha(x)^2$, so α is positive and transforms projections into projections. Moreover, using (ii) consecutively with $y = x^2$, $y = x^3, \ldots$ we get $\alpha(x^n) = \alpha(x)^n$ for every positive integer n.

α is called *normal* if it is continuous in the σ-weak topology on M.

LEMMA 1. *Let α be a normal Jordan morphism on M. Then for each sequence $\{p_n\}$ of projections in M we have*

$$\alpha\Big(\bigwedge_{n=1}^{\infty} p_n\Big) = \bigwedge_{n=1}^{\infty} \alpha(p_n).$$

Proof. Let p and q be projections in M. Since

$$\lim_{n\to\infty}(pq)^n = p\wedge q \qquad \sigma\text{-weakly},$$

we have

$$\lim_{n\to\infty}(pqp)^n = \lim_{n\to\infty}(pq)^n p = p\wedge q. \tag{3}$$

For all $x, y \in M$,

$$\alpha(xyx) = \alpha(x)\alpha(y)\alpha(x)$$

(cf. e.g. [1; p. 208]), thus

$$\alpha((pqp)^n) = \alpha(pqp)^n = [\alpha(p)\alpha(q)\alpha(p)]^n,$$

and taking limits on both sides yields, on account of the normality of α and (3),

$$\alpha(p\wedge q) = \alpha(p)\wedge\alpha(q).$$

Consequently, for a sequence $\{p_n\}$ of projections, we have

$$\alpha\Big(\bigwedge_{n=1}^{k} p_n\Big) = \bigwedge_{n=1}^{k}\alpha(p_n),$$

and again passing to the limit as $k\to\infty$ gives the conclusion. ■

2. Noncommutative Poincaré recurrence theorem. Recalling our discussion from the introduction, we can now formulate the first version of this theorem.

THEOREM 2. *Let α be a normal Jordan morphism on a von Neumann algebra M with a faithful weight ψ such that $\psi\cdot\alpha = \psi$. For each projection p in M satisfying* (i) $\psi(p^\perp) < \infty$ *and* (ii) $p \le \bigwedge_{n=1}^{\infty}\alpha^n(\mathbf{1})$, *we have*

$$p \le \bigvee_{n=1}^{\infty}\alpha^n(p).$$

Proof. Put $e_n = \alpha^n(\mathbf{1})$. Since $e_1 \le \mathbf{1}$, we have $e_{n+1}\le e_n$. Let

$$q = \bigwedge_{n=1}^{\infty}\alpha^n(p^\perp) = \bigwedge_{n=1}^{\infty}[e_n - \alpha^n(p)].$$

Then

$$q \le \alpha(p^\perp) = e_1 - \alpha(p) \le \alpha(p)^\perp, \tag{4}$$

so

$$\psi(q) \le \psi(\alpha(p^\perp)) = \psi(p^\perp) < \infty.$$

Furthermore,

$$\alpha(q) = \bigwedge_{n=2}^{\infty}\alpha^n(p^\perp) \ge q,$$

and the finiteness of $\psi(q) = \psi(\alpha(q))$ yields $\psi(\alpha(q) - q) = 0$. Since $\alpha(q) - q \geq 0$, we infer, by the faithfulness of ψ, that $\alpha(q) = q$.

Now put

$$x = pqp.$$

Then

$$\alpha(x) = \alpha(p)\alpha(q)\alpha(p) = \alpha(p)q\alpha(p) = 0,$$

by inequality (4). Consequently, $\psi(x) = \psi(\alpha(x)) = 0$, which, again by the faithfulness of ψ, means that $x = 0$, and thus $pq = 0$. Hence

$$p \leq q^{\perp} = \bigvee_{n=1}^{\infty} [e_n - \alpha^n(p)]^{\perp} = \bigvee_{n=1}^{\infty} [e_n^{\perp} + \alpha^n(p)]. \tag{5}$$

For each positive integer m assumption (ii) yields

$$\alpha^m(p) \leq \bigwedge_{n=1}^{\infty} \alpha^m(e_n) = \bigwedge_{n=m+1}^{\infty} e_n = \bigwedge_{n=1}^{\infty} e_n,$$

where the last equality follows from the fact that the sequence $\{e_n\}$ of projections is nonincreasing. Thus we get

$$\bigvee_{n=1}^{\infty} \alpha^n(p) \leq \bigwedge_{n=1}^{\infty} e_n,$$

which means that

$$0 = \Big(\bigvee_{n=1}^{\infty} \alpha^n(p)\Big)\Big(\bigwedge_{n=1}^{\infty} e_n\Big)^{\perp} = \Big(\bigvee_{n=1}^{\infty} \alpha^n(p)\Big)\Big(\bigvee_{n=1}^{\infty} e_n^{\perp}\Big).$$

Accordingly,

$$\bigvee_{n=1}^{\infty} [e_n^{\perp} + \alpha^n(p)] = \bigvee_{n=1}^{\infty} e_n^{\perp} + \bigvee_{n=1}^{\infty} \alpha^n(p) = \Big(\bigwedge_{n=1}^{\infty} e_n\Big)^{\perp} + \bigvee_{n=1}^{\infty} \alpha^n(p),$$

so (5) reads

$$p \leq \Big(\bigwedge_{n=1}^{\infty} e_n\Big)^{\perp} + \bigvee_{n=1}^{\infty} \alpha^n(p).$$

Since $p \leq \bigwedge_{n=1}^{\infty} e_n$, the above inequality yields

$$p \leq \bigvee_{n=1}^{\infty} \alpha^n(p). \ \blacksquare$$

REMARK. Obviously, assumption (i) of the theorem is satisfied if ψ is a positive linear functional on M, and assumption (ii) if α is unital.

Observe that (ii) also holds for $p \in \bigcap_{n=1}^{\infty} \alpha^n(M)$. Indeed, if $p \in \alpha^n(M)$, then $p = \alpha^n(x)$ for some $x \in M$, and putting

$$z = \frac{x^* x + x x^*}{2} \geq 0,$$

we get $\alpha^n(z) = p$. Now $z \leq \|z\|\mathbf{1}$, so

$$p = \alpha^n(z) \leq \|z\|\alpha^n(\mathbf{1}),$$

and since $\alpha^n(\mathbf{1})$ is a projection, it follows that $p \leq \alpha^n(\mathbf{1})$, giving condition (ii).

The second version of noncommutative Poincaré recurrence theorem is

Theorem 3. *Let α and ψ be as in Theorem* 2, *and assume that a projection p satisfies conditions* (i) *and* (ii) *of Theorem* 2. *Then*

$$p \leq \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} \alpha^n(p).$$

Proof. Fix a positive integer k. Then α^k is a normal Jordan morphism leaving ψ invariant, and

$$\bigwedge_{n=1}^{\infty} (\alpha^k)^n(\mathbf{1}) = \bigwedge_{n=1}^{\infty} \alpha^{kn}(\mathbf{1}) \geq \bigwedge_{n=1}^{\infty} \alpha^n(\mathbf{1}) \geq p,$$

and thus we can apply Theorem 2 to α^k, which gives

$$p \leq \bigvee_{n=1}^{\infty} (\alpha^k)^n(p) = \bigvee_{n=1}^{\infty} \alpha^{kn}(p) \leq \bigvee_{n=k}^{\infty} \alpha^n(p).$$

As k is arbitrary, we get

$$p \leq \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} \alpha^n(p). \ \blacksquare$$

REFERENCES

[1] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer, Berlin, 1979.

[2] J. R. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York, 1976.

[3] Ş. Strătilă, *Modular Theory in Operator Algebras*, Editura Academiei, Bucureşti, and Abacus Press, Tunbridge Wells, 1981.

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