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DIFFERENTIATION AND SPLITTING FOR LATTICES OVER ORDERS

BY

WOLFGANG RUMP (Eichstätt)

Abstract. We extend our module-theoretic approach to Zavadskiĭ's differentiation techniques in representation theory. Let R be a complete discrete valuation domain with quotient field K, and Λ an R-order in a finite-dimensional K-algebra. For a hereditary monomorphism $u : P \hookrightarrow I$ of Λ-lattices we have an equivalence of quotient categories $\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\mathcal{H}] \xrightarrow{\sim} \delta_u\Lambda\text{-}\mathbf{lat}/[B]$ which generalizes Zavadskiĭ's algorithms for posets and tiled orders, and Simson's reduction algorithm for vector space categories. In this article we replace u by a more general type of monomorphism, and the derived order $\delta_u\Lambda$ by some over-order $\partial_u\Lambda \supset \delta_u\Lambda$. Then $\widetilde{\partial}_u$ remains an equivalence if $\delta_u\Lambda$-**lat** is replaced by a certain subcategory of $\partial_u\Lambda$-**lat**. The extended differentiation comprises a splitting theorem that implies Simson's splitting theorem for vector space categories.

Introduction. In a previous article [19] we generalized Zavadskiĭ's differentiation algorithm [26–28] for representations of posets to lattices over orders Λ in a finite-dimensional algebra A over a field K with a complete discrete valuation. Instead of a pair of points in a poset, our differentiation depends on a *hereditary* monomorphism $u : P \hookrightarrow I$ of Λ-lattices, that is, I/P is of finite length and satisfies

$$\mathrm{Hom}_\Lambda(P, I/P) = \mathrm{Ext}_\Lambda(I/P, I) = \mathrm{Ext}_\Lambda(H, L) = 0$$

for Λ-lattices H, L between P and I, and

(P) $\qquad P$ and I^* are projective.

Then the isomorphism classes of Λ-lattices between P and I can be represented by a finite set $\mathcal{H}_u$. With each (left) Λ-lattice E, we associate a pair $\partial_u E = \binom{E^+}{E_-}$ of Λ-lattices with $E_- \subseteq E \subseteq E^+$. Dually, the hereditary monomorphism $u^* : I^* \hookrightarrow P^*$ yields a pair $\binom{F^-}{F_+}$ of right Λ-lattices with $F_+ \subseteq F \subseteq F^-$ for any given right Λ-lattice F. Then we can form the *derived order*

$$\delta_u\Lambda := \begin{pmatrix} \Lambda^+ & \Lambda^+\Lambda^- \\ \Lambda_- & \Lambda^- \end{pmatrix} \subseteq \mathrm{M}_2(A)$$

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of Λ, and ∂_u becomes a functor

$$\partial_u : \Lambda\text{-}\mathbf{lat} \to \delta_u\Lambda\text{-}\mathbf{lat}$$

between Λ- and $\delta_u\Lambda$-lattices. Since $\Lambda_+ = \Lambda_-$, the definition of $\delta_u\Lambda$ is self-dual.

In [19] we proved that ∂_u induces an equivalence of quotient categories

$$\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\mathcal{H}_u] \xrightarrow{\sim} \delta_u\Lambda\text{-}\mathbf{lat}/\left[\binom{I}{P}\right], \tag{0}$$

which generalizes known versions of Zavadskiĭ's algorithm, e.g. Simson's algorithm for vector space categories [21–23] in case Λ is subhereditary, and Zavadskiĭ's algorithm for tiled orders [28] in case P and I are tame irreducible with I/P of length one ([19], §3).

In the present article we show that a modified version of (0) remains valid when the projectivity condition (P) is dropped. To this end we consider *pre-hereditary* monomorphisms $u : P \to I$, i.e. such that $U := I/P$ is length-finite with

$$\partial_u P = \partial_u I = \begin{pmatrix} I \\ P \end{pmatrix}, \tag{C}$$

$\mathrm{End}_\Lambda(I) \to \mathrm{End}_\Lambda(U)$ surjective, and U is a *Zavadskiĭ module* [19] over $B := \Lambda/\Lambda_-$, that is, a module ${}_BU$ with the property that each submodule is U-projective and each factor module U-injective. The closure condition (C) implies that

$$\partial_u\Lambda := \begin{pmatrix} \Lambda^+ & \Lambda^{+-} + \Lambda^{-+} \\ \Lambda_- & \Lambda^- \end{pmatrix} \subseteq \mathrm{M}_2(A)$$

is an over-order of $\delta_u\Lambda$. If u is pre-hereditary, ∂_u induces an equivalence (Theorem 1)

$$\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\mathcal{H}_u] \xrightarrow{\sim} \partial_u\Lambda\text{-}\mathbf{lat}^s/\left[\binom{I}{P}\right], \tag{0$'$}$$

where $\partial_u\Lambda\text{-}\mathbf{lat}^s$ consists of the $\partial_u\Lambda$-lattices $\binom{F}{G}$ with $F \supseteq G^+$ and $G \subseteq F_-$. Moreover, $\partial_u\Lambda\text{-}\mathbf{lat}^s$ coincides with $\partial_u\Lambda\text{-}\mathbf{lat}$ if

$$ {}_{\Lambda^-}P \text{ and } I^*{}_{\Lambda^+} \text{ are projective.} \tag{P$^\circ$}$$

When the stronger projectivity condition (P) holds, the orders $\partial_u\Lambda$ and $\delta_u\Lambda$ coincide.

If $u : P \hookrightarrow I$ is pre-hereditary, then any decomposition of I/P induces a decomposition of u. The functor ∂_u does not change if multiplicities of indecomposable direct summands of u are reduced to one. For $u = u_1 \oplus \ldots \oplus u_n$ with $u_1, \ldots, u_n$ indecomposable and pairwise non-isomorphic, $u_1' := \partial_{u_2 \oplus \ldots \oplus u_n}(u)$ is pre-hereditary, and the functor ∂_u is equivalent to the composition $\partial_{u_1'}\partial_{u_2 \oplus \ldots \oplus u_n}$. Therefore, we may assume u to be indecomposable. In this case, I/P is uniserial.

Apart from the various Zavadskiĭ algorithms mentioned above, the modified equivalence $(0')$ generalizes D. Simson's splitting theorem ([24], Theorem 17.53) which extends previous results of Nazarova & Roĭter ([24], Lemma 8.1), and Dlab & Ringel ([2], Lemma 8.4). The splitting theorem has served as a basic tool in the theory of representation-finite Schurian vector space categories [7].

For our splitting theory (§5) which we are going to explain now, the use of $\partial_u \Lambda$ instead of $\delta_u \Lambda$ is indispensable (see §7, Example 6).

In dealing with orders in not necessarily semisimple algebras A, the concept of *generalized over-order* Γ of Λ introduced (for A semisimple) by the Kiev school (e.g. [3]) is important. Such a Γ is given by a ring homomorphism $\Lambda \to \Gamma$ with R-torsion cokernel. A pre-hereditary monomorphism $u : P \hookrightarrow I$ with $S := KP = KI$ simple and $\Delta := \operatorname{End}_\Lambda(P) = \operatorname{End}_\Lambda(I)$ the (unique) maximal order in the skew field $D := \operatorname{End}_A(S)$ will be called *splitting* if $A = \operatorname{End}_D(S) \times A'$ and $\operatorname{Hom}_\Delta(I, P\Pi) \subseteq \Lambda$. Our fundamental splitting lemma (Proposition 18) then says that in this case, the maximal order Γ_0 in $\mathrm{M}_2(\operatorname{End}_D(S))$ with indecomposable representation $\binom{I}{P}$ satisfies $\operatorname{Rad} \Gamma_0 \subseteq \partial_u \Lambda$. (Hence $\partial_u \Lambda$ is subhereditary whenever A is simple.) Remarkably, that inclusion does not hold for $\delta_u \Lambda$ instead of $\partial_u \Lambda$.

In order to apply this result, we define a *splitting* of Λ as a pair of generalized over-orders Λ_1, Λ_2 such that $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ is an order, and each indecomposable Λ-lattice is a Λ_i-lattice for some $i \in \{1, 2\}$. (Here, the product $\Lambda_1 \Lambda_2$ is an R-lattice in $K\Lambda_1 \otimes_{K\Lambda} K\Lambda_2$.) The importance of this notion comes from the fact (Proposition 17) that a splitting is tantamount to an equivalence of categories

$$\Lambda_1\text{-}\mathbf{lat}/[\Gamma] \times \Lambda_2\text{-}\mathbf{lat}/[\Gamma] \to \Lambda\text{-}\mathbf{lat}/[\Gamma]$$

with $\Gamma := \Lambda_1 \Lambda_2$. Of particular interest is the case where Γ is hereditary. We then speak of a *hereditary* splitting. Under some extra assumption, a splitting pre-hereditary monomorphism u gives rise to a hereditary splitting (Theorem 4). For orders Λ in a simple K-algebra, this result is equivalent to Simson's splitting theorem ([24], §17.53).

A special case of hereditary splitting will be characterized in Theorem 3: Here, ${}_\Lambda\Lambda$ admits a decomposition $\Lambda = P_1 \oplus P_2$ which yields an equivalence

$$\Omega_1\text{-}\mathbf{lat}/[\Gamma_1] \times \Omega_2\text{-}\mathbf{lat}/[\Gamma_2] \xrightarrow{\sim} \Lambda\text{-}\mathbf{lat}/[\Gamma]$$

of categories with $\Omega_i := (\operatorname{End}_\Lambda P_i)^{\mathrm{op}}$ and $\Gamma_i := (\operatorname{End}_\Gamma \Gamma P_i)^{\mathrm{op}}$. Such type of splitting arises for generalized Brauer tree orders (Example 5 of §7).

For an R-order Λ, there always exist proper monomorphisms $u : P \hookrightarrow I$ with $S := KP = KI$ simple, $\operatorname{End}_\Lambda(P) = \operatorname{End}_\Lambda(I) =: \Delta$ maximal, and I/P uniserial with pairwise non-isomorphic composition factors. Then u is pre-hereditary if and only if $P \ncong I$. For $P \cong I$, however, there are cases where

(0′) still holds. Namely, if the identical morphism $1 : I \to I$ is splitting, and the projection of Λ into $\mathrm{End}_{K\Delta}(S)$ is a hereditary order Λ_0, Proposition 20 yields an equivalence

$$(0'') \qquad \Lambda\text{-}\mathbf{lat}/[\Lambda_0] \xrightarrow{\sim} \Lambda'\text{-}\mathbf{lat}$$

for some R-order Λ'. If $\partial_u\Lambda$ is an order (which is not always true here since (C) is no longer valid), then Λ'-**lat** coincides with $\partial_u\Lambda\text{-}\mathbf{lat}/[\partial_u P]$, and the equivalence is given by $\widetilde{\partial}_u$. Only the weak form (P°) of the projectivity condition (P) is satisfied in that case.

Equivalences of type (0″) with Λ_0 not necessarily hereditary have recently been studied by Iyama [5] who defines Λ' in terms of the Auslander–Reiten quiver of Λ. The question arises whether a similar generalization of (0) or even (0′) is possible. We shall take up this problem in [20].

Some examples are collected in §7, chosen as small as possible, to illustrate the results of the paper.

1. The derivative. Throughout this article, let R be a complete discrete valuation domain with quotient field K, and Λ an R-order in a finite-dimensional K-algebra A; that is to say, Λ is an R-subalgebra of A which is finitely generated over R such that $K\Lambda = A$. Unless otherwise stated, modules over a ring S will be assumed to be left modules. By S-**mod** we denote the category of finitely generated S-modules.

A Λ-submodule E of a left A-module M is said to be a (full) Λ-*lattice* in M if ${}_RE$ is finitely generated and $KE = M$. Since M can be identified with $K \otimes_R E$, the embedding $E \hookrightarrow M$ is determined by the Λ-module E, which is also called a Λ-*representation*. Every homomorphism $f : E \to F$ of Λ-lattices has a unique A-linear extension $KE \to KF$, which we again denote by f. Therefore, the inverse image $f^{-1}(F)$ will be regarded as a Λ-submodule of KE which may strictly contain E. The category of Λ-lattices is denoted by Λ-**lat**. Recall that a (left) Λ-lattice E is said to be *injective* if the right Λ-lattice $E^* := \mathrm{Hom}_R(E, R)$ is projective. When ${}_\Lambda E$ is projective and injective, then E is also called *bijective*. Moreover, a Λ-lattice E is said to be *irreducible* if KE is a simple A-module. If KE decomposes into two simple A-modules, we call E *binomial*. An irreducible Λ-lattice E with $\mathrm{End}_\Lambda(E)$ a maximal order in $\mathrm{End}_A(KE)$ is said to be *tame*. For the general theory of lattices over orders we refer to [12].

Let $u : P \hookrightarrow I$ be a monomorphism of Λ-lattices with $KP = KI$. In [19] we defined for any Λ-lattice E the u-*trace* and u-*cotrace*:

$$\mathrm{trc}_u E := \sum\{f(I) \mid f \in \mathrm{Hom}_\Lambda(P, E)\},$$
$$\mathrm{ctr}_u E := \bigcap\{f^{-1}(P) \mid f \in \mathrm{Hom}_\Lambda(E, I)\}.$$

Thus $\mathrm{trc}_u E$ is R-finite, and $\mathrm{ctr}_u E$ is *full* in KE, i.e. $K(\mathrm{ctr}_u E) = KE$. Hence

(1) $$E^+ := E + \mathrm{trc}_u E, \quad E_- := E \cap \mathrm{ctr}_u E$$

are Λ-lattices in KE with $E_- \subseteq E \subseteq E^+$. Dually, with respect to the monomorphism $u^* : I^* \hookrightarrow P^*$ of Λ^{op}-lattices, for $F \in \Lambda^{\mathrm{op}}$-**lat** we define

(2) $$F^- := F + \mathrm{trc}_{u^*} F, \quad F_+ := F \cap \mathrm{ctr}_{u^*} F.$$

Then $F_+ \subseteq F \subseteq F^-$, and

(3) $$(E^+)^* = (E^*)_+, \quad (E_-)^* = (E^*)^-.$$

Since every homomorphism ${}_\Lambda\Lambda \to I$ is of the form $a \mapsto ax$ with $x \in I$, we obtain $\Lambda_- = \{a \in \Lambda \mid aI \subseteq P\} = \{a \in \Lambda \mid P^*a \subseteq I^*\}$ and thus

(4) $$\Lambda_- = \Lambda_+,$$

which is a (two-sided) ideal of Λ.

The following *closure condition*:

(C) $$\boxed{I^+ = I, \quad P_- = P}$$

has been introduced in [19]. Since the identity $1 : P \to P$ carries I to I, we have $I \subseteq P^+$. On the other hand, $P \hookrightarrow I$ gives $P^+ \subseteq I^+$. Therefore, condition (C) implies that P and I determine each other:

(5) $$P^+ = I, \quad I_- = P.$$

Note, however, that (C) does not imply the *minimality condition*

(M) $$I = \Lambda^+ P, \quad P = \mathrm{Hom}_\Lambda(\Lambda^-, I),$$

which states that there are no Λ^+- or Λ^--lattices strictly between P and I. Moreover, we shall see that (C) does not even imply the *weak minimality condition*

(M°) $$I = \Lambda^{-+} P, \quad P = \mathrm{Hom}_\Lambda(\Lambda^{+-}, I).$$

Here, the second equations in (M) and (M°) assume that P is identified with $\mathrm{Hom}_\Lambda(\Lambda, P)$.

In [19] we proved the following

Proposition 1. *If $I^+ = I$ (resp. $P_- = P$), then Λ^+ (resp. Λ^-) is an over-order of Λ, and for any Λ-lattice E we have $E^+ = \Lambda^+ E^+$ (resp. $E_- = \Lambda^- E_-$). Moreover,* (C) *implies $\Lambda_- E^+ \subseteq E_-$.*

Proposition 2. *If* (C) *is satisfied, then $\Lambda_- E^+ \subseteq E_- \subseteq (\Lambda^{+-} + \Lambda^{-+}) E_- \subseteq E^+$ for every Λ-lattice E.*

Proof. The inclusion $\Lambda^{-+} E_- \subseteq E^+$ follows since Λ^{-+} is mapped into E^+ by each homomorphism $\Lambda^- \to E_-$. Dually, $(E^*)_+ \Lambda^{+-} \subseteq (E^*)^-$ and thus $(E^+)^* \Lambda^{+-} \subseteq (E_-)^*$, which gives $\Lambda^{+-} E_- \subseteq E^+$. ∎

In particular, (C) implies:

$$\Lambda^-\Lambda_-\Lambda^+ = \Lambda_-, \quad \Lambda^+\Lambda^{-+}\Lambda^- = \Lambda^{-+}, \qquad (6)$$
$$\Lambda^{-+}\Lambda_- \subseteq \Lambda^+, \quad \Lambda_-\Lambda^{-+} \subseteq \Lambda^-.$$

Here the first equation follows by (4) and Proposition 1; the second follows since the functor $(\ )^+$ respects right Λ^--lattices; thirdly, $\Lambda^{-+}\Lambda_- \subseteq (\Lambda^-\Lambda_-)^+ \subseteq \Lambda^+$, and the fourth equation follows by Proposition 2. By duality, the last three equations also hold for Λ^{+-} instead of Λ^{-+}.

Thus under the assumption (C) we can define the *u-derivative* of Λ as the R-order:

$$\Lambda' = \partial_u\Lambda := \begin{pmatrix} \Lambda^+ & \Lambda^{+-} + \Lambda^{-+} \\ \Lambda_- & \Lambda^- \end{pmatrix} \subseteq \mathrm{M}_2(A). \qquad (7)$$

Then a Λ'-lattice is suitably given by a column $\binom{F}{G}$ with $F \in \Lambda^+$-**lat**, $G \in \Lambda^-$-**lat**, and $\Lambda_-F \subseteq G \subseteq (\Lambda^{+-} + \Lambda^{-+})G \subseteq F$. Hence, the map $E \mapsto \binom{E^+}{E_-}$ gives rise to a functor

$$\partial_u : \Lambda\text{-}\mathbf{lat} \to \Lambda'\text{-}\mathbf{lat}^s \qquad (8)$$

into the full subcategory

$$\Lambda'\text{-}\mathbf{lat}^s := \left\{ \begin{pmatrix} F \\ G \end{pmatrix} \in \Lambda'\text{-}\mathbf{lat} \;\middle|\; F \supseteq G^+,\ G \subseteq F_- \right\} \qquad (9)$$

of Λ'-**lat**. We shall call (8) the *differentiation functor* with respect to u, or simply the *u-differentiation*. (For representations of partially ordered sets, a similar functor is known as "refinement functor"; see [24], Definition 9.14.) Note that the order $\partial_u\Lambda$ has to be distinguished from the Λ-lattice $\partial_u({}_\Lambda\Lambda)$, which is a proper direct summand of ${}_\Lambda(\partial_u\Lambda)$.

Let us call $u : P \hookrightarrow I$ *pre-hereditary* (cf. [19], §2) if the following holds:

(Z) Condition (C) is valid, and for Λ-lattices H, H', L, L' with $P \subseteq H' \subseteq H \subseteq I$ and $P \subseteq L' \subseteq L \subseteq I$, every isomorphism $\overline{h} : H/H' \xrightarrow{\sim} L/L'$ is induced by a homomorphism $h : H \to L$ with $h(H') \subseteq L'$.

An analysis of this condition will be given in §§2–3.

For a class $\mathcal{C}$ of objects in an additive category, let $[\mathcal{C}]$ denote the ideal of morphisms which factor through a finite direct sum of objects in $\mathcal{C}$. By $\operatorname{add}\mathcal{C}$ we denote the full subcategory consisting of direct summands of finite direct sums of objects isomorphic to those in $\mathcal{C}$. In particular, define

$$\mathcal{H}_u := \operatorname{add}\{H \in \Lambda\text{-}\mathbf{lat} \mid P^s \subseteq H \subseteq I^s \text{ for some } s \in \mathbb{N}\}. \qquad (10)$$

As usual, $\operatorname{ind}\Lambda$ denotes a representative system of isomorphism classes of indecomposable Λ-lattices. The following theorem generalizes [19], Theorem 2:

THEOREM 1. *If* $u : P \hookrightarrow I$ *is pre-hereditary, then the* u*-differentiation* (8) *induces an equivalence of categories*

$$\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\mathcal{H}_u] \xrightarrow{\sim} \Lambda'\text{-}\mathbf{lat}^s/\big[\big(\begin{smallmatrix} I \\ P \end{smallmatrix}\big)\big]$$

Moreover, $\Lambda'\text{-}\mathbf{lat}^s = \Lambda'\text{-}\mathbf{lat}$ *if and only if the weak minimality condition* (M°) *is satisfied.*

We shall prove in §3 that (M°) follows by the *weak projectivity condition*

(P°) $\qquad {}_{\Lambda^-}P$ and $I^*{}_{\Lambda^+}$ are projective,

and that (P°) and (M°) are equivalent whenever u has no direct summands $u_1 : P_1 \hookrightarrow I_1$ with $u_1(P_1) = I_1 \neq 0$.

Thus if (M°) holds, the theorem yields a bijection:

$$\operatorname{ind}\Lambda \setminus \operatorname{ind}\mathcal{H}_u \xrightarrow{\sim} \operatorname{ind}\Lambda' \setminus \operatorname{ind}\operatorname{add}\big\{\big(\begin{smallmatrix} I \\ P \end{smallmatrix}\big)\big\}.$$

Here, $\operatorname{ind}\operatorname{add}\big\{\big(\begin{smallmatrix} I \\ P \end{smallmatrix}\big)\big\}$ consists of the indecomposable direct summands of $\big(\begin{smallmatrix} I \\ P \end{smallmatrix}\big)$. An explicit determination of $\operatorname{ind}\mathcal{H}_u$ will be given in §2.

REMARK. If the u-derivative (7) of Λ is replaced by the suborder (see [19])

(11) $$\delta_u\Lambda := \begin{pmatrix} \Lambda^+ & \Lambda^+\Lambda^- \\ \Lambda_- & \Lambda^- \end{pmatrix},$$

we gain some simplification in return for a slightly weaker statement of the main theorem. Then a $\delta_u\Lambda$-lattice is just given by a pair $\big(\begin{smallmatrix} F \\ G \end{smallmatrix}\big)$ with $\Lambda_- F \subseteq G \subseteq F$, and the u-differentiation (8) induces an equivalence $\Lambda\text{-}\mathbf{lat}/[\mathcal{H}_u] \xrightarrow{\sim} \delta_u\Lambda\text{-}\mathbf{lat}/\big[\big(\begin{smallmatrix} I \\ P \end{smallmatrix}\big)\big]$ if and only if the (strong) minimality condition (M) holds. In analogy with the above, (M) is a consequence of the (strong) *projectivity condition*

(P) $\qquad P$ and I^* are projective Λ-lattices.

In the presence of this condition, the collection of concepts related with u attains its simplest form (§3). Thus (11) seems to be more natural than the definition (7) of the u-derivative. On the other hand, all the results of §5 depending on Proposition 18 are no longer valid if $\partial_u\Lambda$ is replaced by $\delta_u\Lambda$. In §3 we shall prove that (P) implies $\delta_u\Lambda = \partial_u\Lambda$.

2. Pre-hereditary monomorphisms. The proof of Theorem 1 will be divided into three parts showing that $\widetilde{\partial}_u$ is faithful, full, and dense, respectively. For this purpose, we shall prove that a pre-hereditary monomorphism u satisfies three conditions which will be used in order to conclude each of the partial assertions on $\widetilde{\partial}_u$. For any Λ-lattice E, Proposition 2 implies that

E^+/E_- is a module over the artinian ring

(12) $$B := \Lambda/\Lambda_-.$$

This notation will be maintained throughout the paper. The three conditions mentioned are:

(C) $$I^+ = I, \qquad P_- = P.$$

(L) Condition (C) holds, and for $M, M' \in B$-**mod** and $H, H' \in \mathcal{H}_u$, each diagram

$$\begin{array}{ccc} H & \dashrightarrow & H' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow{f} & M' \end{array}$$

with $q(H_-) = 0$ and $q'(H'_-) = 0$ can be completed.

(H) Condition (C) holds, B is (left) hereditary, and I/P is a bijective B-module.

The fundamental condition (C) has already been introduced. Together with (C), (L), and (H), we shall discuss the following related properties. Firstly, there are two stronger versions of (C):

(C′) $$\mathrm{Ext}_\Lambda(I/P, I) = \mathrm{Hom}_\Lambda(P, I/P) = 0.$$

(C″) $P/\mathrm{Rad}\, P$ and $\mathrm{Rad}^\circ I/I$ have no common composition factors with I/P.

Here, $\mathrm{Rad}\, P = (\mathrm{Rad}\, \Lambda)P$ denotes the Jacobson radical, and the *upper radical* Rad° is defined for any $E \in \Lambda$-**lat** by

$$(\mathrm{Rad}^\circ E)^* = \mathrm{Rad}\, E^*.$$

Stronger than the *lifting condition* (L) is the *extension property*:

(E) $$\text{(C) holds, and } \mathrm{Ext}_\Lambda(H, L) = 0 \text{ for } H, L \in \mathcal{H}_u;$$

weaker is the *restricted lifting condition*:

(R) $$\text{(C) holds, and } \mathrm{End}_\Lambda(I) \to \mathrm{End}_\Lambda(I/P) \text{ is surjective.}$$

In §3, the rôle of the *projectivity conditions*

(P°) $${}_{\Lambda^-}P \text{ and } I^*_{\Lambda^+} \text{ are projective,}$$

(P) $${}_{\Lambda}P \text{ and } I^*_{\Lambda} \text{ are projective,}$$

and their relationship to the *minimality conditions*

(M°) $$I = \Lambda^{-+}P, \qquad P = \mathrm{Hom}_\Lambda(\Lambda^{+-}, I),$$

(M) $$I = \Lambda^+P, \qquad P = \mathrm{Hom}_\Lambda(\Lambda^-, I)$$

will be clarified.

Let us show first that all these conditions (including (Z)) are self-dual. This is obvious in all cases except (L), (H), and (C′). For the *heredity condition* (H) this follows by (4) and the fact that $\mathrm{Ext}_R(-, R)$ gives a duality in B-**mod**. In particular,

$$\mathrm{Ext}_R(I/P, R) \cong P^*/I^*.$$

In order to verify that (L) is self-dual, note that $q(H_-) = 0$ signifies that $M \cong H/L$ with $H_- \subseteq L \subseteq H$. Thus if we identify M with H/L and M' with H'/L' for some $L' \supset H'_-$, we can assume q, q' to be the natural epimorphisms. Hence the dual diagram is

$$\begin{array}{ccc} L^* & \leftarrow\text{-----} & L'^* \\ \downarrow & & \downarrow \\ L^*/H^* & \xleftarrow{f^*} & L'^*/H'^* \end{array}$$

with $f^* = \mathrm{Ext}_R(f, R)$ and $L^* \subseteq (H^*)^-$ by (3). Hence, (L) is self-dual.

For a finitely generated R-torsion Λ-module V and $F \in \Lambda$-**lat** define $\mathrm{Ext}^{\mathrm{lat}}_\Lambda(V, F)$ as the subset of extensions $F \rightarrowtail E \twoheadrightarrow V$ in $\mathrm{Ext}_\Lambda(V, F)$ with $E \in \Lambda$-**lat**.

Lemma 1. *If U runs through the submodules of V, there is a natural partition of sets*:

$$\mathrm{Ext}_\Lambda(V, F) = \coprod_{U \subseteq V} \mathrm{Ext}^{\mathrm{lat}}_\Lambda(V/U, F).$$

Proof. For any $\varepsilon : F \hookrightarrow E \twoheadrightarrow V$ in $\mathrm{Ext}_\Lambda(V, F)$, the R-torsion part $\mathrm{T}(E)$ is mapped bijectively onto a submodule U of V which yields an exact sequence $\varepsilon_0 : F \hookrightarrow E_0 \twoheadrightarrow V/U$ with $E_0 = E/\mathrm{T}(E)$. The diagram

$$\begin{array}{rcccc} \varepsilon_0 : & F & \hookrightarrow & E_0 & \twoheadrightarrow & V/U \\ & \| & & \uparrow & \text{PB} & \uparrow \\ \varepsilon : & F & \hookrightarrow & E & \twoheadrightarrow & V \\ & & & \uparrow & & \uparrow \\ & & & \mathrm{T}(E) & \xrightarrow{\sim} & U \end{array}$$

shows that ε and ε_0 determine each other since PB is a pullback square. ∎

As a consequence, we find that (C′) is self-dual:

$$\mathrm{Ext}_\Lambda(I/P, I) = 0 \iff \mathrm{Hom}_\Lambda(I^*, P^*/I^*) = 0.$$

In fact, by the lemma, $\mathrm{Ext}_\Lambda(I/P, I) = 0$ says that any overlattice E of I with E/I isomorphic to a factor module of I/P must coincide with I. Therefore, we get the implications

$$\text{(C}''\text{)} \Rightarrow \text{(C}'\text{)} \Rightarrow \text{(C)}. \tag{13}$$

As an immediate consequence of (1), we obtain

$$(14)\qquad \text{(C)} \;\Leftrightarrow\; \mathrm{Hom}_\Lambda(I,I)=\mathrm{Hom}_\Lambda(P,I)=\mathrm{Hom}_\Lambda(P,P).$$

Next we shall derive an equivalent formulation of (C″). Firstly, we have

PROPOSITION 3. *A simple Λ-module is annihilated by Λ_- if and only if it occurs as a composition factor in I/P.*

Proof. By the definition of Λ_- we have $\Lambda_- I \subseteq P$. Conversely, [19], Lemma 4, implies that B is finitely cogenerated by I/P. Hence, the simple B-modules occur as composition factors in I/P. ∎

The proposition yields an alternative formulation of (C″):

$$(15)\qquad (\mathrm{C}'') \;\Leftrightarrow\; (\Lambda_- P = P \text{ and } I^*\Lambda_- = I^*).$$

Here, the condition $I^*\Lambda_- = I^*$ can be replaced by virtue of the equivalence

$$(16)\qquad I^*\Lambda_- = I^* \;\Leftrightarrow\; \mathrm{Hom}_\Lambda(\Lambda_-, I) = I,$$

where $\mathrm{Hom}_\Lambda(\Lambda_-, I)$ is identified with $\{x \in KI \mid \Lambda_- x \subseteq I\}$.

Next we turn our attention to the lifting condition (L). Define

$$(17)\qquad \mathfrak{p} := \mathrm{Rad}\, R, \qquad \mathfrak{k} := R/\mathfrak{p}.$$

Then [19], Proposition 9, implies that B is a finite-dimensional $\mathfrak{k}$-algebra. Whenever (C) holds, let us consider two full subcategories of B-**mod**:

$$(18)\qquad \mathfrak{B}^+ := \{H^+/H \mid H \in \mathcal{H}_u\}, \qquad \mathfrak{B}^- := \{H/H_- \mid H \in \mathcal{H}_u\}.$$

LEMMA 2. *If* (L) *is satisfied, and $H \in \mathcal{H}_u$ is indecomposable, then H^+ and H^- are indecomposable.*

Proof. Suppose $H^+ = I_1 \oplus I_2$ with I_1 indecomposable, and let $q : H^+ \twoheadrightarrow I_1$ be the natural projection. If $P_1 := (I_1)_-$ and $H_1 := q(H) \supseteq P_1$, then (L) implies that the natural epimorphism $r : H_1 \twoheadrightarrow H_1/P_1$ can be lifted along the epimorphism $r \circ q|_H : H \to H_1 \twoheadrightarrow H_1/P_1$, i.e. there is an $s : H_1 \to H$ with $rq \circ s = r$. Hence, $1 - qs \in \mathrm{End}_\Lambda(H_1)$ factors through $P_1 \hookrightarrow H_1$. Now if $H_1 = P_1$, then P_1 is a direct summand of H, whence $H = P_1$ and $H^+ = I_1$ is indecomposable. Otherwise, qs is an isomorphism, i.e. H_1 is a direct summand of H and thus $H = H_1$. ∎

For a module $M \in B$-**mod**, let $\mathit{Gen}(M)$ be the class of B-modules which are finitely *generated* by M, i.e. are epimorphic images of finite direct sums M^s of M. Similarly, $\mathit{Cog}(M)$ denotes the class of B-modules finitely *cogenerated* by M, i.e. submodules of M^s, $s \in \mathbb{N}$. If (C) holds, then

$$(19)\qquad \mathfrak{B}^+ = \mathit{Gen}(I/P), \qquad \mathfrak{B}^- = \mathit{Cog}(I/P).$$

PROPOSITION 4. *If* (L) *is valid, then the functors $Q^+ : \mathcal{H}_u \to \mathfrak{B}^+$ and $Q^- : \mathcal{H}_u \to \mathfrak{B}^-$ with $Q^+(H) = H^+/H$ and $Q^-(H) = H/H_-$ yield equiva-*

lences of categories:

$$\mathcal{H}_u/[I] \xrightarrow{\sim} \mathfrak{B}^+, \qquad \mathcal{H}_u/[P] \xrightarrow{\sim} \mathfrak{B}^-.$$

Proof. A morphism $f : H \to L$ in $\mathcal{H}_u$ factors through some I^s if and only if f extends to H^+. But this is tantamount to $Q^+(f) = 0$. Thus Q^+ is faithful modulo $[I]$. It is also full by virtue of (L), and dense by (18). Hence, Q^+ induces an equivalence. The remaining assertion follows by duality. ∎

As an immediate consequence, we get

COROLLARY. *If* (L) *is valid, and* $H \in \mathcal{H}_u$ *has no direct summand in* add$\{I\}$ (*resp.* add$\{P\}$), *then* H *is indecomposable if and only if* H^+/H (*resp.* H/H_-) *is indecomposable.*

PROPOSITION 5. *If* (L) *is satisfied, and* $H \in \mathcal{H}_u$ *is indecomposable, then*

$$H/H_- \in \mathfrak{B}^+ \;\Leftrightarrow\; H^+/H \in \mathfrak{B}^- \;\Leftrightarrow\; H \in \operatorname{add}\{P \oplus I\}.$$

Proof. $H \in \operatorname{add}\{P \oplus I\}$ says that $H = H^+$ or $H = H_-$. If $H/H_- \in \mathfrak{B}^+$ and $H \neq H_-$, then we have an isomorphism $h : H/H_- \xrightarrow{\sim} L^+/L$ with $L \in \mathcal{H}_u$, and by the above corollary, we may assume L to be indecomposable. Thus by the symmetry of this assumption, it remains to prove that $H = H^+$ and $L = L_-$. Now (L) implies that h lifts to an $f : H \to L^+$ with $f(H_-) \subseteq L$. Then f extends to H^+, whence H/H_- is a direct summand of H^+/H_-. By Lemma 2 we infer that H^+, hence also H^+/H_-, is indecomposable. Consequently, $H = H^+$. Similarly, h factors through L^+/L_-, which yields $L = L_-$. ∎

In particular, (L) implies

$$\mathfrak{B}^+ \cap \mathfrak{B}^- = \operatorname{add}\{I/P\}. \tag{20}$$

Our next result holds without the assumption (L). Let B-**proj** (resp. B-**inj**) denote the full subcategory of projective (resp. injective) modules in B-**mod**.

PROPOSITION 6. *If* (C) *is valid, then every module* $M \in B$-**mod** *is of the form* $M = H/L$ *with* $P^s \subseteq L \subseteq H \subseteq I^s$ *for some* $s \in \mathbb{N}$. *Moreover,* B-**proj** $\subseteq \mathfrak{B}^-$ *and* B-**inj** $\subseteq \mathfrak{B}^+$.

Proof. By [19], Lemma 4, every finitely generated free B-module is isomorphic to some H/P^s with $P^s \subseteq H \subseteq I^s$. Hence M is of the desired form. If M is projective, then M is a direct summand of some $B^t \cong H/P^s \in \mathfrak{B}^-$, and if $M = H/L$ is injective, then $H/L \hookrightarrow L^+/L$ splits, whence $M \in \mathfrak{B}^+$. ∎

Concluding the analysis of (L), we show

$$\text{(E)} \Rightarrow \text{(L)}. \tag{21}$$

In fact, if we put $L := \operatorname{Ker} q'$ in the diagram of (L), then $L \in \mathcal{H}_u$, and the exact sequence

$$\operatorname{Hom}_\Lambda(H, L) \hookrightarrow \operatorname{Hom}_\Lambda(H, H') \xrightarrow{q'_*} \operatorname{Hom}_\Lambda(H, M') \to \operatorname{Ext}_\Lambda(H, L)$$

yields (21).

Now let us focus our attention upon the heredity condition (H). Since ${}_BB \in \mathfrak{B}^-$, we have

$$\text{(22)} \qquad \text{(H)} \Leftrightarrow ((\text{C}) \ \& \ \mathfrak{B}^+ = B\text{-}\mathbf{inj} \ \& \ \mathfrak{B}^- = B\text{-}\mathbf{proj}).$$

Moreover, the following characterization of (H) is valid. Recall ([19], §1) that a B-module M is called a *Zavadskiĭ module* if each submodule is M-projective, and each factor module M-injective.

Proposition 7. (H) *is satisfied if and only if* (C) *holds and I/P is a Zavadskiĭ module.*

Proof. Suppose (H). Then every submodule of I/P is projective, and every factor module of I/P is injective, whence I/P is a Zavadskiĭ module. Conversely, suppose (C) holds and I/P is a Zavadskiĭ module. Then Proposition 6 (with [1], 16.12.f) implies that a module $M \in B$-**mod** is projective (resp. injective) if and only if M is I/P-projective (resp. I/P-injective). By [19], Proposition 2, $(I/P)^s$ is a Zavadskiĭ module for any $s \in \mathbb{N}$. Hence, every submodule of ${}_BB$ is projective, i.e. B is left hereditary. Moreover, I/P is bijective, whence (H). ■

Now we are able to prove

Theorem 2. (Z) $\Leftrightarrow$ ((H) & (R)) $\Leftrightarrow$ (L).

Proof. (Z) $\Rightarrow$ ((H) & (R)). By (C), the homomorphism h in condition (Z) induces an endomorphism of I/P, whence I/P is a Zavadskiĭ module. By Proposition 7, this implies (H). In order to verify (R), suppose $\bar{f} \in \operatorname{End}_\Lambda(I/P)$. Then there are Λ-lattices H, L between I and P with $\bar{f} : I/P \twoheadrightarrow I/L \xrightarrow{\sim} H/P \hookrightarrow I/P$, and (Z) yields a homomorphism $f : I \to H$ with $f(L) \subseteq P$ which induces the isomorphism $I/L \xrightarrow{\sim} H/P$. By (C), the endomorphism $\bar{f}$ is also induced by f.

((H) & (R)) $\Rightarrow$ (L). Under the hypothesis (H) we shall reduce (L) to (R). Consider the diagram for (L) and replace H' by L. The conditions $q(H_-) = 0$ and $q'(L_-) = 0$ imply that q and q' factor through the natural epimorphisms $H \twoheadrightarrow H/H_-$ and $L \twoheadrightarrow L/L_-$. By (22), $H/H_- \in \mathfrak{B}^-$ is a projective B-module. Hence, f lifts to a map $g : H/H_- \to L/L_-$, and it remains to prove that the diagram

$$\begin{array}{ccc} H & \dashrightarrow & L \\ \downarrow & & \downarrow \\ H/H_- & \xrightarrow{g} & L/L_- \end{array}$$

can be completed. Considering the pullback

$$\begin{array}{ccc} L & \hookrightarrow & L^+ \\ \downarrow & & \downarrow \\ L/L_- & \hookrightarrow & L^+/L_- \end{array}$$

we may assume without loss of generality that $L = L^+$. But then L/L_- is injective, whence g factors through $H/H_- \hookrightarrow H^+/H_-$. Therefore, it suffices to complete a diagram

$$\begin{array}{ccc} I_1 & \dashrightarrow & I_2 \\ \downarrow & & \downarrow \\ I_1/P_1 & \longrightarrow & I_2/P_2 \end{array}$$

with $I_1, I_2 \in \operatorname{add}\{I\}$ and $P_i = (I_i)_-$ for $i \in \{1,2\}$. Then I_1, I_2 may be assumed to be indecomposable, and thus (R) yields the desired lifting.

The remaining implication (L) ⇒ (Z) is trivial. ■

COROLLARY. *$u : P \hookrightarrow I$ is pre-hereditary if and only if* (R) *holds*, *and I/P is a Zavadskiĭ module.*

Let us investigate which modifications of $u : P \hookrightarrow I$ preserve the property (Z). Firstly, we have:

PROPOSITION 8. *Property* (Z) *remains valid if u is replaced by a finite direct sum $u^s : P^s \hookrightarrow I^s$. If $u_1 : P_1 \hookrightarrow I_1$ and $u_2 : P_2 \hookrightarrow I_2$ satisfy* (Z), *and the modules I_1/P_1 and I_2/P_2 have no composition factor in common*, *then $u_1 \oplus u_2 : P_1 \oplus P_2 \hookrightarrow I_1 \oplus I_2$ is pre-hereditary if it satisfies* (C).

Proof. Clearly, the restricted lifting property (R) carries over to u^s and $u_1 \oplus u_2$ under the given hypothesis, and (C) carries over to u^s. By [19], Theorem 1, I^s/P^s and $I_1 \oplus I_2/P_1 \oplus P_2$ are Zavadskiĭ modules, whence the above corollary gives the desired result. ■

If (C) holds, then by (14), any decomposition of P or I gives rise to a decomposition of $u : P \hookrightarrow I$, say,

$$u = u_1 \oplus \ldots \oplus u_n, \qquad u_i : P_i \hookrightarrow I_i. \tag{23}$$

The trace and cotrace of a Λ-lattice E are then given by

$$\operatorname{trc}_u E = \sum_{i=1}^{n} \operatorname{trc}_{u_i} E, \qquad \operatorname{ctr}_u E = \bigcap_{i=1}^{n} \operatorname{ctr}_{u_i} E, \tag{24}$$

and similarly, the u-differentiation ∂_u is calculated by means of the ∂_{u_i}. If two different summands u_i and u_j in (23) are equivalent, i.e. if there is an isomorphism $f : I_i \xrightarrow{\sim} I_j$ with $f(P_i) = P_j$, then ∂_u does not change if the direct summand u_j in (23) is cancelled. On the other hand, if u is an isomorphism, then $E^+ = E_- = E$. Such monomorphisms will be called *trivial*. Clearly, ∂_u also does not change if a trivial direct summand of u is cancelled. Therefore, we shall say that u is *reduced* if there are neither multiple nor trivial summands in a decomposition (23). Thus if (Z) is satisfied for a reduced monomorphism (23), then each I_i/P_i is an indecomposable Zavadskiĭ module, and the composition factors of I/P are pairwise non-isomorphic. Hence each submodule of I/P is of the form $M_1 \oplus \ldots \oplus M_n$ with submodules M_i of I_i/P_i. The following result is easily verified:

PROPOSITION 9. *If $u : P \hookrightarrow I$ is reduced pre-hereditary, then each $u' : P' \hookrightarrow I'$ with Λ-lattices P', I', and $P \subseteq P' \subseteq I' \subseteq I$, is again pre-hereditary.*

By [19], Proposition 5, we have

PROPOSITION 10. *If* (Z) *is satisfied, then $B = \Lambda/\Lambda_-$ is Morita equivalent to a product of triangular matrix algebras over finite-dimensional division algebras over $\mathfrak{k}$.*

The indecomposable B-modules are thus of the form H_1/H_2 with indecomposable $H_1, H_2 \in \mathcal{H}_u$ and $H_1 \subseteq H_2 \subseteq H_1^+$. This also follows by Proposition 6 and the structure of Zavadskiĭ modules ([19], §1).

3. The projectivity conditions. In the known versions [28, 26, 21, 19] of Zavadskiĭ's algorithm, if considered as special cases of Theorem 1, the projectivity condition

(P) $\qquad\qquad$ P is projective, I is injective

is satisfied. We shall demonstrate in this section how the relationship between the various conditions on $u : P \hookrightarrow I$ is simplified in the presence of (P).

Firstly, the implications (13) are turned into equivalences:

$$(\mathrm{P}) \Rightarrow ((\mathrm{C}'') \Leftrightarrow (\mathrm{C}') \Leftrightarrow (\mathrm{C})). \tag{25}$$

Namely, if I/P and $P/\operatorname{Rad} P$ had a common composition factor, (P) would yield a homomorphism $P \to I$ with image not in P.

Secondly, we have

$$(\mathrm{P}) \Rightarrow ((\mathrm{L}) \Leftrightarrow (\mathrm{E})). \tag{26}$$

Indeed, suppose (P) and (L) are satisfied, and $H, L \in \mathcal{H}_u$. Then $L \hookrightarrow L^+ \overset{q}{\twoheadrightarrow} L^+/L$ induces an exact sequence

$$\operatorname{Hom}_\Lambda(H, L^+) \xrightarrow{q_*} \operatorname{Hom}_\Lambda(H, L^+/L) \to \operatorname{Ext}_\Lambda(H, L) \to \operatorname{Ext}_\Lambda(H, L^+),$$

where $\mathrm{Ext}_\Lambda(H, L^+) = 0$ since L^+ is injective; moreover, for each homomorphism $H \to L^+/L$, the composition $g : H_- \hookrightarrow H \to L^+/L$ factors through $L^+ \twoheadrightarrow L^+/L$ by the projectivity of H_-. Hence $g = 0$, and we infer that q_* is surjective by virtue of (L). In conjunction with (21), the equivalence (26) follows.

Thirdly, let us focus our attention upon the minimality condition

$$\text{(M)} \qquad I = \Lambda^+ P, \qquad P = \mathrm{Hom}_\Lambda(\Lambda^-, I).$$

PROPOSITION 11. *Let* (C) *be satisfied. Then* (M) *is equivalent to each of the following properties*:

(a) $E^+ = \Lambda^+ E$ *and* $E_- = \mathrm{Hom}_\Lambda(\Lambda^-, E)$ *for every* Λ*-lattice* E.
(b) $(\Lambda^+)_+ = \Lambda^+$ *and* $(\Lambda^-)_- = \Lambda^-$.

Proof. (M) $\Rightarrow$ (a). For any morphism $f : P \to E$ in Λ-**lat**, we have $f(I) = f(\Lambda^+ P) \subseteq \Lambda^+ E \subseteq E^+$. Hence $E^+ = \Lambda^+ E$, i.e. E^+ is the smallest Λ^+-overlattice of E. Therefore, $E_- = \mathrm{Hom}_\Lambda(\Lambda^-, E)$ follows by duality.

(a) $\Rightarrow$ (b) $\Rightarrow$ (M). The equality $(\Lambda^-)_- = \Lambda^-$ states that $\mathrm{Hom}_\Lambda(\Lambda^-, I)$ coincides with $\mathrm{Hom}_\Lambda(\Lambda^-, P) = P$, that is, the second assertion of (a) with $E = I$. By duality, the first assertion of (a) implies $(\Lambda^+)_+ = \Lambda^+$. The latter equation is equivalent to $I = \Lambda^+ P$. ∎

In particular, the proposition implies that if (C) and (M) are satisfied, then $\partial_u \Lambda$ coincides with the simplified u-derivative $\delta_u \Lambda$ defined in (11), and

$$(27) \qquad E^{++} = E^+, \quad E_{--} = E_-$$

for each $E \in \Lambda$-**lat**. Clearly, this also follows by (C$''$).

If in the definition (1) of E^+, the morphisms $P \to E$ are restricted to those which factor through a free Λ-lattice, then $\Lambda^+ E$ is obtained instead of E^+. Similarly, if $E \in \Lambda^-$-**lat**, and we restrict ourselves to homomorphisms $P \to E$ in $[\Lambda^-]$, we get $\Lambda^{-+} E$ instead of E^+. Therefore, the implications

$$(28) \qquad \text{(P)} \Rightarrow \text{(M)}, \quad (\text{P}^\circ) \Rightarrow (\text{M}^\circ)$$

hold in general. Under the hypothesis of Theorem 1, the converse is also true:

PROPOSITION 12. *If* $u : P \hookrightarrow I$ *is reduced pre-hereditary*, *then the equivalences* (P) $\Leftrightarrow$ (M) *and* (P$^\circ$) $\Leftrightarrow$ (M$^\circ$) *are valid.*

Proof. (M) $\Rightarrow$ (P). By duality it suffices to prove that $I = \Lambda^+ P$ implies the projectivity of P. Let P_1 be any indecomposable direct summand of P. Then $I = \Lambda^+ P$ implies $\Lambda^+ P_1 = P_1^+$. Therefore, an epimorphism $g : \Lambda^n \twoheadrightarrow P_1$ maps $(\Lambda^+)^n$ onto P_1^+. Since by assumption $P_1^+ \neq P_1$, there exists a direct summand P_2 of P together with a homomorphism $f : P_2 \to \Lambda^n$ such that $gf(P_2^+) \not\subseteq P_1$. By [19], Proposition 9, we conclude that $gf : P_2^+ \to P_1^+$ is an isomorphism. Hence $gf : P_2 \to \Lambda^n \to P_1$ is an isomorphism, and thus P_1 is projective. Analogously, $(\mathrm{M}^\circ) \Rightarrow (\mathrm{P}^\circ)$ follows. ∎

Remark. By the above implications (25), (26), we obtain [19], Theorem 2, as a special case of Theorem 1.

4. Proof of Theorem 1. The fundamental condition (C) already suffices to prove that the u-differentiation (8) induces a faithful functor of quotient categories:

Proposition 13. *Let* (C) *be satisfied. Then ∂_u induces a faithful functor $\widetilde{\partial}_u$.*

Proof. Clearly, the ideal $[\mathcal{H}_u]$ is mapped into $\left[\binom{I}{P}\right]$. Hence $\widetilde{\partial}_u$ is well defined. For any $E \in \Lambda$-**lat** we have

$$\operatorname{Hom}_{\Lambda'}\left(\begin{pmatrix} E^+ \\ E_- \end{pmatrix}, \begin{pmatrix} I \\ P \end{pmatrix}\right) = \operatorname{Hom}_\Lambda(E, I),$$
$$\operatorname{Hom}_{\Lambda'}\left(\begin{pmatrix} I \\ P \end{pmatrix}, \begin{pmatrix} E^+ \\ E_- \end{pmatrix}\right) = \operatorname{Hom}_\Lambda(P, E).$$

Now let $f : E \to F$ be a morphism in Λ-**lat** such that $\partial_u f$ has a factorization

$$\partial_u f : \begin{pmatrix} E^+ \\ E_- \end{pmatrix} \xrightarrow{g} \begin{pmatrix} I^s \\ P^s \end{pmatrix} \xrightarrow{h} \begin{pmatrix} F^+ \\ F_- \end{pmatrix}.$$

Then $f = h \circ g$ with $g : E \to I^s$ and $h : P^s \to F$. Hence, f factors through $g(E) + P^s \in \mathcal{H}_u$. ∎

For the proof of Theorem 1 we need a criterion which decides for a Λ'-lattice in Λ'-$\mathbf{lat}^s$ whether it has a direct summand in common with $\binom{I}{P}$:

Proposition 14. *Let $u : P \hookrightarrow I$ be reduced pre-hereditary. Then $\binom{F}{G} \in \Lambda'$-$\mathbf{lat}^s$ has a direct summand in* $\operatorname{add}\left\{\binom{I}{P}\right\}$ *if and only if $G^+ \not\subseteq F_-$.*

Proof. This follows by the proof of [19], Proposition 12. ∎

Lemma 3. *If* (C) *is satisfied, then for each Λ-lattice E,*

$$\Lambda^{+-}E_- \subseteq \Lambda^{-+}E, \qquad \operatorname{Hom}_\Lambda(\Lambda^{+-}, E) \subseteq \operatorname{Hom}_\Lambda(\Lambda^{-+}, E^+).$$

Proof. The first inclusion is equivalent to $(\Lambda^{-+}E)^*\Lambda^{+-} \subseteq (E_-)^*$. Now $(\Lambda^{-+}E)^*$ is a right Λ^+-lattice. Hence, every homomorphism $\Lambda^+ \to (\Lambda^{-+}E)^*$ of right Λ^+-lattices maps Λ^{+-} into $(\Lambda^{-+}E)^{*-}$, i.e. $(\Lambda^{-+}E)^*\Lambda^{+-} \subseteq (\Lambda^{-+}E)^{*-} \subseteq E^{*-} = (E_-)^*$. The second inclusion is dual to the first. ∎

Proof of Theorem 1. An obvious modification of the proof of [19], Theorem 2, using Proposition 14 above, shows that $\widetilde{\partial}_u$ is full and dense, hence an equivalence by virtue of Proposition 13.

If (M°) is satisfied, then each homomorphism $P \to G \in \Lambda^-$-**lat** carries $I = \Lambda^{-+}P$ into $\Lambda^{-+}G$. Hence $G^+ \subseteq \Lambda^{-+}G$, and dually, $\operatorname{Hom}_\Lambda(\Lambda^{+-}, F) \subseteq$

F_- for every Λ^+-lattice F. Hence Λ'-$\mathbf{lat}^s$ coincides with Λ'-$\mathbf{lat}$. Conversely, if Λ'-$\mathbf{lat}^s$ coincides with Λ-$\mathbf{lat}$, then Lemma 3 implies that $\binom{\Lambda^{-+}P}{P}$ is a Λ'-lattice, and thus $I = P^+ \subseteq \Lambda^{-+}P$. By duality, we obtain (M°). ■

Let us add some remarks on the subcategory Λ'-$\mathbf{lat}^s$ of Λ'-$\mathbf{lat}$. If we assume that (C) is valid, there are two monomorphisms in Λ'-$\mathbf{lat}$ which are naturally associated with u:

$$u^+ : \begin{pmatrix} I \\ P \end{pmatrix} \hookrightarrow \begin{pmatrix} I \\ \mathrm{Hom}_\Lambda(\Lambda^{+-}, I) \end{pmatrix}, \quad u^- : \begin{pmatrix} \Lambda^{-+}P \\ P \end{pmatrix} \hookrightarrow \begin{pmatrix} I \\ P \end{pmatrix}. \tag{29}$$

Then the inclusion

$$\mathrm{trc}_{u^-} E' \subseteq \mathrm{ctr}_{u^+} E' \tag{30}$$

holds for each Λ'-lattice E', and for $E' = \binom{F}{G}$ we have

$$F \supseteq G^+ \;\Leftrightarrow\; \mathrm{trc}_{u^-} E' \subseteq E', \qquad G \subseteq F_- \;\Leftrightarrow\; \mathrm{ctr}_{u^+} E' \supseteq E'. \tag{31}$$

Hence there is a functor

$$\sigma_u : \Lambda'\text{-}\mathbf{lat} \to \Lambda'\text{-}\mathbf{lat}^s \tag{32}$$

given by

$$\sigma_u E' := (E' + \mathrm{trc}_{u^-} E') \cap \mathrm{ctr}_{u^+} E' = (E' \cap \mathrm{ctr}_{u^+} E') + \mathrm{trc}_{u^-} E'. \tag{33}$$

Explicitly, we have

$$\sigma_u \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F + G^+ \\ G \cap F_- \end{pmatrix}, \tag{34}$$

and therefore, σ_u operates identically on the objects of Λ'-$\mathbf{lat}^s$. This gives an intrinsic characterization of Λ'-$\mathbf{lat}^s$:

$$E' \in \Lambda'\text{-}\mathbf{lat}^s \;\Leftrightarrow\; \sigma_u E' \cong E'. \tag{35}$$

PROPOSITION 15. *If* (C) *is satisfied*, *then the functor* (32) *induces a faithful dense functor* $\widetilde{\sigma}_u : \Lambda'\text{-}\mathbf{lat}/[\mathcal{H}'_u] \to \Lambda'\text{-}\mathbf{lat}^s/\big[\binom{I}{P}\big]$, *where*

$$\mathcal{H}'_u := \mathrm{add}\left\{ \begin{pmatrix} H \\ L \end{pmatrix} \in \Lambda'\text{-}\mathbf{lat} \;\middle|\; H, L \in \mathcal{H}_u,\ H \subseteq L^+ \right\}.$$

Proof. Clearly, σ_u maps $[\mathcal{H}'_u]$ into $\big[\binom{I}{P}\big]$, whence $\widetilde{\sigma}_u$ is well defined. Conversely, suppose that a morphism $h : \binom{F}{G} \to \binom{F'}{G'}$ in Λ'-$\mathbf{lat}$ has the property that $\sigma_u h$ factors through $\binom{I^s}{P^s}$ for some $s \in \mathbb{N}$. Then h is a composition $g \circ f$ with $f \in \mathrm{Hom}_\Lambda(F, I^s)$ and $g \in \mathrm{Hom}_\Lambda(P^s, G')$. Hence, h factors through $\binom{H}{L} \in \mathcal{H}'_u$ with $H := g^{-1}(F') \cap I^s$ and $L := f(G) + P^s$. This proves that $\widetilde{\sigma}_u$ is a faithful functor which is dense by virtue of (35). ■

In general, however, $\widetilde{\sigma}_u$ is not full, and for that reason, there is no way to replace Λ'-$\mathbf{lat}^s/\big[\binom{I}{P}\big]$ in Theorem 1 by Λ'-$\mathbf{lat}/[\mathcal{H}'_u]$. In fact, there may be indecomposable Λ'-lattices neither in Λ'-$\mathbf{lat}^s$ nor in $\mathcal{H}'_u$ (see Examples 3, 4 in §7).

As in [19], Proposition 13, we usually can replace $\Lambda' = \partial_u\Lambda$ by a Morita equivalent R-order with less indecomposable projectives. Retaining assumption (C), let

$$\Lambda = Q \oplus Q_0 \tag{36}$$

be a decomposition of Λ-lattices such that $\mathrm{Hom}_\Lambda(Q', I/P) \neq 0$ for each indecomposable direct summand Q' of Q, and $\mathrm{Hom}_\Lambda(Q_0, I/P) = 0$. We define the *reduced u-derivative* of Λ by

$$\partial'_u\Lambda := \begin{pmatrix} \mathrm{Hom}_\Lambda(Q, Q^+) & \mathrm{Hom}_\Lambda(Q, \Lambda^{+-} + \Lambda^{-+}) \\ Q_- & \Lambda^- \end{pmatrix}. \tag{37}$$

PROPOSITION 16. *If* (C) *is valid, then the reduced u-derivative $\partial'_u\Lambda$ is Morita equivalent to $\partial_u\Lambda$.*

Proof. Since $(Q_0)_- = Q_0$, Lemma 3 implies $(\Lambda^{+-}+\Lambda^{-+})Q_0 = \Lambda^{-+}Q_0 = \Lambda^+Q_0$. Hence $\partial_u Q_0$ is a simultaneous direct summand of $\partial_u({}_\Lambda\Lambda)$ and $Q' := \binom{\Lambda^{+-}+\Lambda^{-+}}{\Lambda^-}$, and $\partial_u Q \oplus Q'$ is a progenerator of $\partial_u\Lambda$. By Proposition 2, the decomposition $\Lambda_- = Q_- \oplus Q_0 = \Lambda_- Q \oplus \Lambda_- Q_0$ yields $Q_- = \Lambda_- Q \subseteq \Lambda_- Q^+ \subseteq Q_-$. Similarly, $Q^+ = \Lambda^+ Q$, and thus

$$\mathrm{End}_{\partial_u\Lambda}(\partial_u Q) = \mathrm{Hom}_\Lambda(Q, Q^+),$$
$$\mathrm{Hom}_{\partial_u\Lambda}(\partial_u Q, Q') = \mathrm{Hom}_\Lambda(Q, \Lambda^{+-} + \Lambda^{-+}).$$

Consequently, the progenerator $\partial_u Q \oplus Q'$ leads to the Morita equivalent R-order (37). ∎

5. Splitting over-orders. Recall that a generalized over-order Γ of Λ is given by a ring homomorphism $f : \Lambda \to \Gamma$ with R-torsion cokernel. Equivalently, Γ is given by its inverse image $\Omega = f^{-1}(\Gamma)$ in A, which is an *overring* of Λ, i.e. an R-subalgebra Ω of A with $\Omega \supset \Lambda$. If Ω is given, then $\Gamma \cong \Omega/\Omega_\infty$, where $\Omega_\infty := \{a \in A \mid Ka \subseteq \Omega\} \lhd A$. In this way, we have a one-to-one correspondence between generalized over-orders Γ and overrings Ω of Λ. For a Λ-lattice E, define $\Gamma E := \Gamma \odot_\Lambda E$, where "$\odot$" denotes the tensor product modulo R-torsion. Hence ΓE can be identified with the set of finite sums $\sum a_i x_i$ in $K\Gamma \otimes_A KE$ with $a_i \in \Gamma$, $x_i \in E$. The same is true for right Λ-lattices. In particular, if Λ_1 and Λ_2 are generalized over-orders of Λ, then $\Lambda_1\Lambda_2$ and $\Lambda_2\Lambda_1$ are full R-lattices in $K\Lambda_1 \otimes_A K\Lambda_2 = K\Lambda_2 \otimes_A K\Lambda_1$, the largest common factor algebra of $K\Lambda_1$ and $K\Lambda_2$. Moreover, the intersection of the overrings belonging to Λ_1 and Λ_2 corresponds to a generalized over-order $\Lambda_1 \cap \Lambda_2$ of Λ which we also call the *intersection* of Λ_1 and Λ_2 (cf. [3], §1).

Let us define a *splitting* of Λ as a pair of generalized over-orders Λ_1, Λ_2 such that $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ is an order, and each indecomposable Λ-lattice is a Λ_i-lattice for some $i \in \{1, 2\}$. (In general, of course, $\Lambda_1\Lambda_2$ and $\Lambda_2\Lambda_1$ need not be equal!) In particular, the indecomposable projectives can be arranged in two classes, which gives rise to a decomposition

$$\Lambda = P_1 \oplus P_2 \tag{38}$$

with $P_i \in \Lambda_i$-**lat**. Therefore, $\Lambda_1 = P_1 \oplus \Lambda_1 P_2$ and $\Lambda_2 = \Lambda_2 P_1 \oplus P_2$, whence

$$\Lambda_1 = P_1 \oplus \Gamma P_2, \qquad \Lambda_2 = \Gamma P_1 \oplus P_2 \tag{39}$$

with $\Gamma := \Lambda_1 \Lambda_2$, and

$$\Lambda = \Lambda_1 \cap \Lambda_2. \tag{40}$$

The splitting will be called *proper* if Λ does not coincide with Λ_1 or Λ_2. If Γ is hereditary, we shall speak of a *hereditary* splitting.

NOTE. For a hereditary R-order Γ, the algebra $K\Gamma$ is necessarily semisimple ([4], Theorem 1.7.1). In fact, for each indecomposable projective $K\Gamma$-module S, the full Γ-lattices in S form a chain. Hence S must be simple.

For example, if

$$\Lambda_{mn} := \begin{pmatrix} \Delta & \Pi^n \\ \Pi^m & \Delta \end{pmatrix} \subseteq \mathrm{M}_2(D)$$

with Δ the maximal order in a skew field D (finite-dimensional over K), and $\Pi := \operatorname{Rad} \Delta$, then the pairs Λ_{30}, Λ_{03} and Λ_{31}, Λ_{03} are hereditary splittings of Λ_{33}.

PROPOSITION 17. *Let Λ_1, Λ_2 be generalized over-orders of Λ, and Γ a generalized over-order of Λ_1 and Λ_2. The bifunctor $(E_1, E_2) \mapsto E_1 \oplus E_2$ induces a faithful functor between additive categories*

$$\Lambda_1\text{-}\mathbf{lat}/[\Gamma] \times \Lambda_2\text{-}\mathbf{lat}/[\Gamma] \to \Lambda\text{-}\mathbf{lat}/[\Gamma]. \tag{41}$$

The following are equivalent:

(a) *Λ_1, Λ_2 form a splitting of Λ, with $\Gamma = \Lambda_1 \Lambda_2$.*
(b) *The functor* (41) *is an equivalence.*

Proof. It is easily seen that (41) is always faithful. The property that (41) is full signifies that for Λ_i-lattices E_i, $i \in \{1, 2\}$, each Λ-linear map between E_1 and E_2 (in either direction) lies in $[\Gamma]$. This means that each $E_1 \to E_2$ factors through ΓE_1, and each $E_2 \to E_1$ factors through ΓE_2. Hence $\Gamma = \Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ implies that (41) is full. Conversely, if (41) is full, we deduce that the natural maps $\Lambda_1 \to \Lambda_2 \Lambda_1$ and $\Lambda_2 \to \Lambda_1 \Lambda_2$ factor through Γ. Hence, $\Gamma = \Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$. Finally, the density of (41) states that each indecomposable Λ-lattice is a Λ_i-lattice for some $i \in \{1, 2\}$. ■

By the preceding proposition, the usefulness of splitting pairs of generalized over-orders becomes apparent, especially in the case of a hereditary splitting. As an application of Theorem 1, we shall see below that a special class of pre-hereditary monomorphisms gives rise to a hereditary splitting of Λ. Here the projectivity condition (P) is not assumed, but another restriction on Λ has to be imposed which forces Λ to be subhereditary if the algebra $A = K\Lambda$ is simple. In that case, we obtain an equivalent version of D. Simson's splitting theorem ([24], Theorem 17.53) for vector space categories.

Let us first consider an important special class of splitting. For a decomposition (38) of Λ, and a hereditary generalized over-order Γ of Λ, define

$$\Omega_i := (\mathrm{End}_\Lambda P_i)^{\mathrm{op}}, \qquad \Gamma_i := (\mathrm{End}_\Gamma \Gamma P_i)^{\mathrm{op}} \tag{42}$$

for $i \in \{1, 2\}$. Then there are functors

$$\Omega_1\text{-}\mathbf{lat} \times \Omega_2\text{-}\mathbf{lat} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \Lambda\text{-}\mathbf{lat} \tag{43}$$

with

$$\mathcal{F}(F_1, F_2) := (P_1 \odot_{\Omega_1} F_1) \oplus (P_2 \odot_{\Omega_2} F_2),$$
$$\mathcal{G}E := (\mathrm{Hom}_\Lambda(P_1, E),\ \mathrm{Hom}_\Lambda(P_2, E)),$$

and in accordance with (38), Λ and Γ can be written in the form

$$\Lambda = \begin{pmatrix} \Omega_1 & \Omega_{12} \\ \Omega_{21} & \Omega_2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{21} & \Gamma_2 \end{pmatrix}, \tag{44}$$

where $\Omega_{ij} := \mathrm{Hom}_\Lambda(P_i,\ P_j)$ and $\Gamma_{ij} := \mathrm{Hom}_\Gamma(\Gamma P_i,\ \Gamma P_j)$. We shall call (38) a *complete splitting* of Λ into Ω_1 and Ω_2 if ΓP_1 and ΓP_2 have no indecomposable direct summand in common, and $\Omega_{12} = \Gamma_{12}$, $\Omega_{21} = \Gamma_{21}$, i.e. the natural maps $\Omega_{ij} \to \Gamma_{ij}$ are isomorphisms for $i \neq j$.

Define the *multiplier* of a Λ-lattice E as the generalized over-order $\mathrm{O}(E)$ of Λ corresponding to the overring $\{a \in A \mid aE \subseteq E\}$. Then for a complete splitting, the generalized over-orders $\Lambda_i := \Gamma \cap \mathrm{O}(P_i)$ are

$$\Lambda_1 = \begin{pmatrix} \Omega_1 & \Gamma_{12} \\ \Gamma_{21} & \Gamma_2 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{21} & \Omega_2 \end{pmatrix}, \tag{45}$$

and thus $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = \Gamma$. Moreover, they form a splitting by the following

THEOREM 3. *Let Γ be a hereditary generalized over-order of Λ, and $\Lambda = P_1 \oplus P_2$ a decomposition of Λ-lattices such that ΓP_1 and ΓP_2 have no indecomposable direct summand in common. Then this gives a complete splitting if and only if the functors* (43) *induce a pair of mutually inverse equivalences*

$$\Omega_1\text{-}\mathbf{lat}/[\Gamma_1] \times \Omega_2\text{-}\mathbf{lat}/[\Gamma_2] \underset{\mathcal{G}'}{\overset{\mathcal{F}'}{\rightleftarrows}} \Lambda\text{-}\mathbf{lat}/[\Gamma].$$

In this case, (45) *is a hereditary splitting of Λ.*

Proof. Since Γ_1 and Γ_2 are hereditary, the functor $\mathcal{G}'$ is always well defined, whereas $\mathcal{F}'$ is defined if and only if $P_i\Gamma_i = P_i \odot_{\Omega_i} \Gamma_i$ are Γ-lattices for

$i \in \{1,2\}$, i.e. if the natural homomorphism $P_i\Gamma_i \to \Gamma \odot_\Lambda P_i\Gamma_i$ is bijective. Now $\Gamma \odot_\Lambda P_i\Gamma_i = (\Gamma P_i)\Gamma_i = \Gamma P_i$. Hence

$$\mathcal{F}' \text{ well defined } \Leftrightarrow (\Omega_{21}\Gamma_1 = \Gamma_{21},\ \Omega_{12}\Gamma_2 = \Gamma_{12}). \tag{46}$$

For an Ω_1-lattice F_1, there is an exact sequence

$$\mathrm{T}(P_1 \otimes_{\Omega_1} F_1) \hookrightarrow P_1 \otimes_{\Omega_1} F_1 \twoheadrightarrow P_1 \odot_{\Omega_1} F_1 \tag{47}$$

where "T" denotes the R-torsion part. Applying $\mathrm{Hom}_\Lambda(P_2,-)$ gives a short exact sequence

$$\mathrm{Hom}_\Lambda(P_2, \mathrm{T}(P_1 \otimes_{\Omega_1} F_1)) \hookrightarrow \mathrm{Hom}_\Lambda(P_2, P_1 \otimes_{\Omega_1} F_1) \twoheadrightarrow \mathrm{Hom}_\Lambda(P_2, P_1 \odot_{\Omega_1} F_1)$$

where the left-hand term is an R-torsion module, and the right-hand term is torsion-free. Thus $\mathrm{Hom}_\Lambda(P_2,\, P_1 \odot_{\Omega_1} F_1) = \Omega_{21} \odot_{\Omega_1} F_1$. Similarly, if we apply $\mathrm{Hom}_\Lambda(P_1,-)$ to (47), we get $\mathrm{Hom}_\Lambda(P_1,\, P_1 \odot_{\Omega_1} F_1) = \Omega_1 \odot_{\Omega_1} F_1 = F_1$, whence by symmetry,

$$\mathcal{G}\mathcal{F}(F_1,F_2) = (F_1,F_2) \oplus (\Omega_{12} \odot_{\Omega_2} F_2, \Omega_{21} \odot_{\Omega_1} F_1).$$

Consequently,

$$\mathcal{G}'\mathcal{F}' \cong 1 \ \Leftrightarrow\ (\Gamma_1\Omega_{12} = \Omega_{12},\ \Gamma_2\Omega_{21} = \Omega_{21}). \tag{48}$$

For the rest of the proof, let us assume that $\mathcal{F}'$ is well defined, and $\mathcal{G}'\mathcal{F}' \cong 1$. Then by (46) and (48) it remains to show that

$$\mathcal{F}'\mathcal{G}' \cong 1 \ \Leftrightarrow\ (\Omega_{12} = \Gamma_{12},\ \Omega_{21} = \Gamma_{21}). \tag{49}$$

Suppose first that $\mathcal{F}'\mathcal{G}' \cong 1$. Let Ω_1 be mapped onto the order Ω_1' by the natural map $K\Omega_1 \twoheadrightarrow K\Omega_1/\mathrm{Rad}\, K\Omega_1$. Then Ω_{21} is a right Ω_1'-lattice since $\Omega_{21} \in \Gamma_2$-**lat** and $K\Gamma_2$ is semisimple. Hence, Λ has a generalized over-order

$$\Lambda' := \begin{pmatrix} \Omega_1' & \Gamma_{12} \\ \Omega_{21} & \Gamma_2 \end{pmatrix}$$

such that each Λ'-lattice $E = \binom{E_1}{E_2}$ is a direct summand of $\mathcal{F}\mathcal{G}E \oplus \Gamma^s$ for some $s \in \mathbb{N}$. Thus if E_1 has no direct summand in common with Γ_1, then E is a direct summand of $\binom{E_1}{\Omega_{21}\odot_{\Omega_1'} E_1}$. The kernel of $\Lambda' \to \Gamma$ is of the form $\begin{pmatrix} N_1 & 0 \\ N_{21} & 0 \end{pmatrix}$, and by (46), we have $KN_{21} = K\Omega_{21}N_1$. Since $K\Omega_1'$ is semisimple, the ideal KN_1 is idempotent, and N_1 has no Γ_1-lattice $\neq 0$ as a direct summand. Hence, $\Gamma_{12}N_{21} \subseteq K\Gamma_{12}\Omega_{21}N_1 \subseteq KN_1$ and $N_1\Gamma_{12} = 0$ implies $\Gamma_{12}N_{21} = 0$. Therefore, $\binom{0}{N_{21}}$ is a Λ'-sublattice of $\binom{N_1}{N_{21}}$, and by the above, $\binom{N_1}{N_{21}}/\binom{0}{N_{21}}$ must be a direct summand of $\binom{N_1}{\Omega_{21}N_1}$. Consequently, $\Omega_{21}N_1 = 0$ and thus $N_{21} = 0$, i.e. $\Omega_{21} \subseteq \Gamma_{21}$. In order to prove $\Omega_{21} = \Gamma_{21}$, it now suffices to show $\Omega_{21}F_1 = \Gamma_{21}F_1$ for every Ω_1'/N_1-lattice F_1. Since ΓP_1 and ΓP_2 have no common direct summand, we have $\Gamma_{12}\Gamma_{21} \subseteq \mathrm{Rad}\,\Gamma_1$, and there exists an integer $i \in \mathbb{N}$ with $(\Gamma_{12}\Gamma_{21})^i F_1 \subseteq F_1$. We choose i minimal. By (46), we may assume that F_1 has no Γ_1-lattice $\neq 0$ as a direct summand,

and thus $i > 0$. Since $F_1' := F_1 + (\Gamma_{12}\Gamma_{21})^{i-1}F_1$ satisfies $(\Gamma_{12}\Gamma_{21})^{i-1}F_1' \subseteq F_1'$, assume $\Omega_{21}F_1' = \Gamma_{21}F_1'$ by induction. Then $\Gamma_{12}\Gamma_{21}F_1 = \Gamma_{12}\Gamma_{21}F_1' = \Gamma_{12}\Omega_{21}F_1' \subseteq F_1$, and thus $E := \binom{F_1}{\Gamma_{21}F_1}$ is a Λ'-lattice. Hence, E is a direct summand of $\binom{F_1}{\Omega_{21}F_1}$, and our claim $\Omega_{21}F_1 = \Gamma_{21}F_1$ is proved. By symmetry, the implication "⇒" in (49) follows.

Conversely, suppose $\Omega_{12} = \Gamma_{12}$, $\Omega_{21} = \Gamma_{21}$, and let $E = \binom{E_1}{E_2}$ be a Λ-lattice. Then ΓE has a decomposition $\Gamma E = H_1 \oplus H_2$ with epimorphic images H_i of ΓP_i. Moreover, $\mathfrak{F}\mathfrak{G}E = (P_1 \odot_{\Omega_1} E_1) \oplus (P_2 \odot_{\Omega_2} E_2)$, and we have an exact sequence

$$JE \hookrightarrow \mathfrak{F}\mathfrak{G}E \overset{c}{\twoheadrightarrow} E, \tag{50}$$

where c is defined by the natural homomorphisms $P_i \otimes_{\Omega_i} \mathrm{Hom}_\Lambda(P_i, E) \to E$, and J denotes the following ideal of Λ:

$$J := \begin{pmatrix} \Gamma_{12}\Gamma_{21} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{21}\Gamma_{12} \end{pmatrix} \lhd \begin{pmatrix} \Omega_1 & \Gamma_{12} \\ \Gamma_{21} & \Omega_2 \end{pmatrix} = \Lambda.$$

Clearly, the map $r : P_1 \odot_{\Omega_1} E_1 \to E \to \Gamma E \twoheadrightarrow H_2$ has its image in JH_2. Hence, r yields a retraction of the embedding $JH_2 \hookrightarrow P_1 \odot_{\Omega_1} E_1$. Similarly, $JH_1 \hookrightarrow P_2 \odot_{\Omega_2} E_2$ has a retraction. Therefore, the exact sequence (50) splits. Thus $\mathfrak{F}'\mathfrak{G}' \cong 1$, and our proof of (49) is complete. Finally, we infer that (45) is a hereditary splitting of Λ. ∎

There is a particular case of a complete splitting of R-orders which has some analogy with one-point extensions of algebras ([13], §2.5). Let Λ be an R-order in $A = A_0 \times A_1$ with A_0 simple, and I a tame irreducible (see §1) Λ-lattice with $S := KI \in A_0$-**mod**, $\Delta := (\mathrm{End}_\Lambda I)^{\mathrm{op}}$, and $\Pi := \mathrm{Rad}\,\Delta$. Suppose $I\Pi I^* \subseteq \Lambda$, where $I^* = \mathrm{Hom}_R(I \otimes_\Delta \Delta, R) = \mathrm{Hom}_\Delta(I, \Delta^*)$ is identified with $\mathrm{Hom}_\Delta(I, \Delta)$. Then we call

$$\Lambda' := \begin{pmatrix} \Delta & I^* \\ I\Pi & \Lambda \end{pmatrix} \tag{51}$$

the *trivial extension* of Λ with respect to I. If $A_0 = \mathrm{M}_n(D)$ with $D := (\mathrm{End}_A S)^{\mathrm{op}}$, then (51) is an order in $\mathrm{M}_{n+1}(D) \times A_1$. Clearly, the columns in (51) yield a complete splitting with respect to any hereditary generalized over-order of the form

$$\Gamma' := \begin{pmatrix} \Delta & I^* \\ I\Pi & \Gamma \end{pmatrix},$$

where Γ is a hereditary generalized over-order of Λ such that I is a Γ-lattice. Therefore, Theorem 3 yields an equivalence

$$\Lambda\text{-}\mathbf{lat}/[\Gamma] \xrightarrow{\sim} \Lambda'\text{-}\mathbf{lat}/[\Gamma']. \tag{52}$$

Other instances of complete splittings are given in §7, Example 5.

For the remainder of this section, let P and I be Λ-lattices in a simple A-module S. Assume that $\Delta := (\mathrm{End}_\Lambda I)^{\mathrm{op}} = (\mathrm{End}_\Lambda P)^{\mathrm{op}}$ is the maximal order in $D := (\mathrm{End}_A S)^{\mathrm{op}}$ with $\Pi := \mathrm{Rad}\,\Delta$. We call a pre-hereditary monomorphism $u : P \hookrightarrow I$ *splitting* if the inclusion $\mathrm{Hom}_\Delta(I, P\Pi) \hookrightarrow \mathrm{End}_\Delta(I)$ lifts along the natural ring homomorphism $\Lambda \to \mathrm{End}_\Delta(I)$ to a (Λ,Λ)-bimodule homomorphism $\mathrm{Hom}_\Delta(I, P\Pi) \to \Lambda$. Clearly, this implies that $A = A_0 \times A_1$ with $A_0 := \mathrm{End}_D(S)$. If, as above, I^* is identified with $\mathrm{Hom}_\Delta(I,\Delta)$, the map $\mathrm{Hom}_\Delta(I, P\Pi) \to \Lambda$ gives an inclusion

$$P\Pi I^* \subseteq \Lambda. \tag{53}$$

Our splitting theorem will be a consequence of

PROPOSITION 18. *Let $u : P \hookrightarrow I$ be splitting pre-hereditary. Then the maximal order Γ_0 in $\mathrm{M}_2(A_0)$ with $\binom{I}{P}$ as indecomposable representation is a generalized over-order of $\partial_u\Lambda$ with $\mathrm{Rad}\,\Gamma_0 \subseteq \partial_u\Lambda$.*

NOTE. For $A = A_0$, the proposition implies that $\partial_u\Lambda$ is subhereditary:

$$\mathrm{Rad}\,\Gamma_0 \subseteq \partial_u\Lambda \subseteq \Gamma_0. \tag{54}$$

However, this is no longer true for $\delta_u\Lambda$ (see §7, Example 6).

Proof of Proposition 18. Explicitly, we have

$$\Gamma_0 = \begin{pmatrix} II^* & IP^* \\ PI^* & PP^* \end{pmatrix} \supseteq \mathrm{Rad}\,\Gamma_0 = \begin{pmatrix} I\Pi I^* & I\Pi P^* \\ P\Pi I^* & P\Pi P^* \end{pmatrix}.$$

By virtue of (53), the elements of ΠI^* can be regarded as homomorphisms $P \to \Lambda$. Therefore, $P_- = P$ is mapped into Λ_-, whence $P\Pi I^* \subseteq \Lambda_-$. Moreover, $I\Pi I^* \subseteq \Lambda^+$, and dually, $P\Pi P^* \subseteq \Lambda^-$. Hence, $I\Pi P^* \subseteq \Lambda^{-+}$ and thus $\mathrm{Rad}\,\Gamma_0 \subseteq \partial_u\Lambda$. Finally, since $\binom{I}{P}$ is a $\partial_u\Lambda$-lattice, the natural epimorphism $\mathrm{M}_2(A) \twoheadrightarrow \mathrm{M}_2(A_0)$ maps $\partial_u\Lambda$ into the maximal order Γ_0. ∎

Before we proceed further, let us analyse the splitting condition (53) in the case of a tiled order Λ. Define

$$\mathfrak{S}_\Lambda := \{E \in \Lambda\text{-}\mathbf{lat} \mid KE = S\}. \tag{55}$$

PROPOSITION 19. *Let $\Lambda = (\Pi^{e_{ij}})$ be a tiled order in $A = \mathrm{M}_n(D)$, and $u : P \hookrightarrow I$ a pre-hereditary monomorphism between Λ-lattices $P, I \in \mathfrak{S}_\Lambda$. Then u is splitting if and only if $E \subseteq I$ or $E \supseteq P$ holds for each $E \in \mathfrak{S}_\Lambda$.*

Proof. The splitting condition (53) is tantamount to $P\Pi I^* E \subseteq E$ for each $E \in \mathfrak{S}_\Lambda$. Furthermore, there is no restriction if E is subject to the condition $I^*E = \Delta$, i.e. $E \subseteq I$ and $E \not\subseteq I\Pi$. For these E, (53) reduces to $P\Pi \subseteq E$, which yields the desired result. ∎

REMARK. For a tiled order Λ and a splitting pre-hereditary monomorphism $u : P \hookrightarrow I$, it can be shown that apart from indecomposables $\binom{H}{L}$ with $P \subseteq L \subseteq H \subseteq I$, each indecomposable $\partial_u\Lambda$-lattice E' can be obtained by ∂_u, i.e. there exists an indecomposable Λ-lattice E with $\partial_u E = E' \oplus \binom{I}{P}^s$

for some $s \in \mathbb{N}$. This fact is no longer true if Λ is not tiled, as Example 7 in §7 will show.

Now we shall derive our general splitting theorem:

THEOREM 4. *For an R-order Λ in $A = A_0 \times A_1 \times A_2$ with A_0 simple, let $u : P \hookrightarrow I$ be splitting pre-hereditary and H a tame irreducible Λ-lattice with $\Delta := (\mathrm{End}_\Lambda H)^{\mathrm{op}}$, $\Pi := \mathrm{Rad}\,\Delta$, and $H\Pi \subseteq P \subseteq I \subseteq H$. Assume that $S := KH$ is the simple A_0-module, and $\mathrm{Rad}(\mathrm{End}_\Delta H) \subseteq \Lambda$. Moreover, suppose ${}_\Lambda\Lambda$ has a decomposition $\Lambda = P_0 \oplus P_1 \oplus P_2$ with $P_i \subseteq A_0 + A_i$, and for $U_0 := I/P$, $U_1 := H/I$, and $U_2 := P/H\Pi$, suppose $\mathrm{Hom}_\Lambda(P_i, U_j) = 0$ whenever $i \neq j$. Under these assumptions, if $p_i : A \twoheadrightarrow A_0 \times A_i$ denotes the natural projection for $i \in \{1, 2\}$, then $\Lambda_1 := p_1(\Lambda) + \mathrm{Hom}_\Delta(H, P)$ and $\Lambda_2 := p_2(\Lambda) + \mathrm{Hom}_\Delta(I, H\Pi)$ constitute a hereditary splitting of Λ.*

REMARK. If $A = A_0$, then $\mathrm{Rad}(\mathrm{End}_\Delta H) \subseteq \Lambda$ implies that Λ is subhereditary. In this case, the theorem can be interpreted as a statement on vector space categories, and then it coincides with D. Simson's splitting theorem ([24], §17.53). In fact, Simson [24] defines a splitting decomposition $\mathbb{K}_F = \mathbb{K}''_F + \mathbb{L}_F + \mathbb{K}'_F$ of a vector space category $\mathbb{K}_F$ by three conditions (i)–(iii) related to the assumptions of Theorem 4 as follows: His first condition (i) that $\mathbb{L}_F$ is of chain type corresponds to the property that $u : P \hookrightarrow I$ is pre-hereditary. The second one (ii) says that there are no morphisms from $\mathbb{K}'_F$ to $\mathbb{L}_F$ or $\mathbb{K}''_F$, and none from $\mathbb{L}_F$ to $\mathbb{K}''_F$. This is equivalent to our disjointness assumption $\mathrm{Hom}_\Lambda(P_i, U_j) = 0$. Thirdly, Simson's dimension property (iii) is tantamount to our splitting condition (53).

Proof of Theorem 4. Let Ω be the hereditary order in A_0 with H, I, P as indecomposables, and Ω_0 the hereditary suborder which has, in addition, all the Λ-lattices between I and P as indecomposables. The splitting condition (53) and the assumption $\mathrm{Rad}(\mathrm{End}_\Delta H) \subseteq \Lambda$ imply $\mathrm{Hom}_\Delta(H, P) \cdot \mathrm{Hom}_\Delta(I, H\Pi) \subseteq \mathrm{Hom}_\Delta(I, P\Pi) \subseteq \Lambda$ and $\mathrm{Hom}_\Delta(I, H\Pi) \cdot \mathrm{Hom}_\Delta(H, P) \subseteq \mathrm{Hom}_\Delta(H, H\Pi) \subseteq \Lambda$. Hence, if $p_0 : A \twoheadrightarrow A_0$ denotes the natural projection, then

$$\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = p_0(\Lambda) + \mathrm{Hom}_\Delta(H, P) + \mathrm{Hom}_\Delta(I, H\Pi) \subseteq \Omega_0.$$

Now $\Omega P_1 = H^k$, $\Omega P_2 = P^l$, and $\Omega P_0 = I^m$ for some $k, l, m \in \mathbb{N}$. Then $\Lambda_2 P_1 = H^k$, $\Lambda_1 P_2 = P^l$, and $\Lambda_1 P_0 = P_0 + P^m \in \Omega_0$-**lat**. Hence

$$\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = \Omega_0. \tag{56}$$

If $P = H\Pi$, then $\Lambda_1 = p_1(\Lambda) = \Lambda$. Similarly, $I = H$ implies $\Lambda_2 = \Lambda$. Therefore, we may exclude these trivial cases. Then $H^+ = H = H_-$, and the maximal order $\Theta := \mathrm{End}_\Delta(H)$ is a generalized over-order of Λ^+ and Λ^-. By Lemma 3, we infer $(\Lambda^{+-} + \Lambda^{-+})H \subseteq H$, and thus $\mathrm{M}_2(\Theta)$ is a generalized over-order of $\partial_u\Lambda$. Moreover, $\mathrm{Rad}\,\Theta \subseteq \Lambda$ and $(\mathrm{Rad}\,\Theta)I \subseteq H\Pi \subseteq$

P implies $\operatorname{Rad}\Theta \subseteq \Lambda_-$ and thus $\operatorname{Rad}\mathrm{M}_2(\Theta) \subseteq \partial_u\Lambda$. By Proposition 18, the maximal order Γ_0 in $\mathrm{M}_2(A_0)$ with the indecomposable representation $\binom{I}{P}$ is a generalized over-order of $\partial_u\Lambda$ with $\operatorname{Rad}\Gamma_0 \subseteq \partial_u\Lambda$. Consequently, the inclusions $\binom{H\Pi}{H\Pi} \subseteq \binom{I}{P} \subseteq \binom{H}{H}$ imply that

$$\Gamma := \mathrm{M}_2(\Theta) \cap \Gamma_0$$

is a hereditary order in $\mathrm{M}_2(A_0)$, and a generalized over-order of $\partial_u\Lambda$ with

$$\operatorname{Rad}\Gamma = \operatorname{Rad}\mathrm{M}_2(\Theta) + \operatorname{Rad}\Gamma_0 \subseteq \partial_u\Lambda. \tag{57}$$

Now we have a decomposition of $\partial_u\Lambda$-lattices

$$\begin{aligned}\partial_u\Lambda &= \begin{pmatrix} P_0^+ \\ (P_0)_- \end{pmatrix} \oplus \begin{pmatrix} P_1^+ \\ P_1 \end{pmatrix} \oplus \begin{pmatrix} P_2^+ \\ P_2 \end{pmatrix} \oplus \begin{pmatrix} (\Lambda^{+-} + \Lambda^{-+})P_0 \\ \Lambda^- P_0 \end{pmatrix} \oplus \begin{pmatrix} P_1^+ \\ P_1 \end{pmatrix} \oplus \begin{pmatrix} P_2^+ \\ P_2 \end{pmatrix} \\ &= Q_1 \oplus Q_2\end{aligned}$$

with

$$Q_1 := \begin{pmatrix} (\Lambda^{+-} + \Lambda^{-+})P_0 \\ \Lambda^- P_0 \end{pmatrix} \oplus \begin{pmatrix} P_1^+ \\ P_1 \end{pmatrix}^2, \quad Q_2 := \begin{pmatrix} P_0^+ \\ (P_0)_- \end{pmatrix} \oplus \begin{pmatrix} P_2^+ \\ P_2 \end{pmatrix}^2$$

such that

$$\Gamma Q_1 = \begin{pmatrix} H \\ H \end{pmatrix}^{n_1}, \quad \Gamma Q_2 = \begin{pmatrix} I \\ P \end{pmatrix}^{n_2}$$

for suitable integers n_1, n_2. In order to show by Theorem 3 that $\partial_u\Lambda = Q_1 \oplus Q_2$ is a complete splitting with respect to the hereditary order Γ, we have to verify for $\{i, j\} = \{1, 2\}$ that the natural homomorphism

$$\operatorname{Hom}_{\partial_u\Lambda}(Q_i, Q_j) \to \operatorname{Hom}_\Gamma(\Gamma Q_i, \Gamma Q_j) \tag{58}$$

is an isomorphism. Note that $\operatorname{Hom}_\Gamma(\Gamma Q_i, \Gamma Q_j) = \operatorname{Hom}_{\partial_u\Lambda}(Q_i, \Gamma Q_j)$. Then the injectivity of (58) follows since $Q_i \subseteq \mathrm{M}_2(A_0) \oplus \mathrm{M}_2(A_i)$; the surjectivity follows by (57) since each homomorphism $Q_i \to \Gamma Q_j$ has its image in $(\operatorname{Rad}\Gamma)Q_j \subseteq Q_j$. Hence Theorem 3 applies, and by (45), there is a pair of splitting generalized over-orders Λ_1', Λ_2' of $\partial_u\Lambda$. If $p_i' : \mathrm{M}_2(A) \twoheadrightarrow \mathrm{M}_2(A_0 \times A_i)$ denotes the natural projection for $i \in \{1, 2\}$, then

$$\Lambda_i' = p_i'(\partial_u\Lambda) + J_i \tag{59}$$

with

$$J_1 = \left\{ a \in \Gamma \;\middle|\; a\begin{pmatrix} H \\ H \end{pmatrix} \subseteq \begin{pmatrix} I \\ P \end{pmatrix} \right\}, \quad J_2 = \left\{ a \in \Gamma \;\middle|\; a\begin{pmatrix} I \\ P \end{pmatrix} \subseteq \begin{pmatrix} H\Pi \\ H\Pi \end{pmatrix} \right\}.$$

Now for each indecomposable Λ-lattice E, we have $\partial_u E = E' \oplus E''$ with E' indecomposable and $E'' \in \Gamma$-**lat**. Therefore, our proof will be completed by the equivalence

$$\Lambda_i E = E \;\Leftrightarrow\; \Lambda_i'(\partial_u E) = \partial_u E$$

for $i \in \{1,2\}$ and $E \in \Lambda$-**lat**. Since $E \in p_i(\Lambda)$-**lat** $\Leftrightarrow \partial_u E \in p'_i(\partial_u \Lambda)$-**lat**, it remains to show that for each Λ-lattice E, the equivalences

$$\begin{aligned} PH^* \cdot E \subseteq E \;&\Leftrightarrow\; J_1(\partial_u E) \subseteq \partial_u E, \\ H\Pi I^* \cdot E \subseteq E \;&\Leftrightarrow\; J_2(\partial_u E) \subseteq \partial_u E \end{aligned} \tag{60}$$

are satisfied. Since $\Theta E = \Theta(E^+)$ and $H^* \in \Theta^{\mathrm{op}}$-**lat**, the inclusion $PH^*E \subseteq E$ implies $PH^*E^+ \subseteq E$ and thus $PH^*E^+ = P_-H^*E^+ \subseteq E_-$. By duality, we also have $H\Pi I^*E \subseteq E \Leftrightarrow H\Pi I^*E^+ \subseteq E_-$. Therefore, (60) follows by the implication $PH^*E^+ \subseteq E_- \Rightarrow PH^*E^+ \subseteq E \Rightarrow IH^*E^+ \subseteq E^+$ and its dual $H\Pi I^*E^+ \subseteq E_- \Rightarrow H\Pi P^*E_- \subseteq E_-$. ∎

6. An extended derivative. In [19], Proposition 14, we characterized hereditary monomorphisms $u : P \hookrightarrow I$ between tame irreducible Λ-lattices P, I. If the projectivity condition (P) is dropped, this gives a characterization of pre-hereditary u. In particular, we have $P \not\cong I$ for $u : P \hookrightarrow I$ pre-hereditary. In the present section, we shall prove that the categorial equivalence in Theorem 1 extends to a case (Proposition 20 below) where the assumption $P \not\cong I$ does not hold. The weak minimality condition (M°) is satisfied, and we get an equivalence $\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\mathcal{H}_u] \xrightarrow{\sim} \partial_u\Lambda\text{-}\mathbf{lat}/\left[\binom{I}{P}\right]$, where the quotient category $\partial_u\Lambda\text{-}\mathbf{lat}/\left[\binom{I}{P}\right]$ coincides with a category Λ'-**lat** for some order Λ' in a factor algebra of $\mathrm{M}_2(A)$ (see Examples 1 and 2 of §7). Moreover, $\mathcal{H}_u$ consists of the Λ-lattices belonging to some rational component of A. There is a close relationship between the functors $\widetilde{\partial}_u$ in Theorem 1 and Proposition 20 on the one hand, and the two cases occurring in the proof of the rejection lemma ([19], Proposition 7) on the other hand.

PROPOSITION 20. *Let Λ be an R-order in $A = A_0 \times A_1$ with A_0 simple such that the natural projection $A \twoheadrightarrow A_0$ maps Λ onto the hereditary order Λ_0. Let S denote the simple A_0-module, and Δ the unique maximal order in $D := (\mathrm{End}_A S)^{\mathrm{op}}$ with $\Pi := \mathrm{Rad}\,\Delta$. For an indecomposable Λ_0-lattice I which is neither projective nor injective as a Λ-lattice, with $P := I\Pi$, suppose $\mathrm{Hom}_\Delta(I, P) \subseteq \Lambda$. Then the u-differentiation* (8) *induces an equivalence*

$$\widetilde{\partial}_u : \Lambda\text{-}\mathbf{lat}/[\Lambda_0] \xrightarrow{\sim} \begin{pmatrix} \Lambda_1 & \Lambda_1 \\ N_1 & \Lambda_1 \end{pmatrix}\text{-}\mathbf{lat}, \tag{61}$$

where $\Lambda_1 := (\Lambda + A_0) \cap A_1$ and $N_1 := \Lambda \cap A_1$.

NOTE. Equivalently, the assumption of the theorem says that Λ is a subdirect product $\Lambda \subseteq \Lambda_0 \times \Lambda_1$ with Λ_0 hereditary and $K\Lambda_0$ simple, and that Λ_0 has a maximal over-order Θ such that $\mathrm{Rad}\,\Theta = \{a \in \Lambda \mid \Theta a \subseteq \Lambda\} = \{a \in \Lambda \mid a\Theta \subseteq \Lambda\}$.

Proof of Proposition 20. There is a natural epimorphism of R-orders

$$\Gamma := \begin{pmatrix} \Delta & I^* \\ P & \Lambda \end{pmatrix} \twoheadrightarrow \Gamma_0 := \begin{pmatrix} \Delta & I^* \\ P & \Lambda_0 \end{pmatrix},$$

where Γ is a trivial extension of Λ. Hence (52) gives an equivalence

$$\mathcal{F}' : \Lambda\text{-}\mathbf{lat}/[\Lambda_0] \xrightarrow{\sim} \Gamma\text{-}\mathbf{lat}/[\Gamma_0]$$

induced by the functor $\mathcal{F} : \Lambda\text{-}\mathbf{lat} \to \Gamma\text{-}\mathbf{lat}$ with $\mathcal{F}(E) = \binom{I^* \odot_\Lambda E}{E}$. By [19], Proposition 14, we have a pre-hereditary monomorphism $v : \binom{\Delta}{P} \hookrightarrow \binom{\Delta}{I}$ in Γ-**lat** with $\binom{\Delta}{P}$ projective and $\binom{\Delta}{I}$ injective. Since $\binom{I^*}{\Lambda_0} = \mathcal{F}(\Lambda_0)$, a Γ-lattice $\binom{H}{E}$ is of the form $\mathcal{F}(E)$ if and only if it does not have $\binom{\Delta}{P}$ as a direct summand. For these Γ-lattices, $\mathrm{Hom}_\Gamma\big(\binom{H}{E}, \binom{\Delta}{I}\big) = \mathrm{Hom}_\Lambda(E, I)$, and therefore

$$\begin{pmatrix} H \\ E \end{pmatrix}_- = \begin{pmatrix} H \\ E_- \end{pmatrix}.$$

Dually, the same argument holds for $\binom{H}{E}^* = (H^* \;\; E^*)$, and thus

$$\begin{pmatrix} H \\ E \end{pmatrix}^+ = \begin{pmatrix} H \\ E^+ \end{pmatrix}$$

if $\binom{H}{E}$ does not have $\binom{\Delta}{I}$ as a direct summand. Since ${}_\Lambda I$ is neither projective nor injective, we obtain

$$\Gamma^+ = \begin{pmatrix} \Delta & I^* \\ I & \Lambda^+ \end{pmatrix} = \begin{pmatrix} \Delta & I^* \\ I & II^* \end{pmatrix} \times \Lambda_1,$$
$$\Gamma^- = \begin{pmatrix} \Delta & P^* \\ P & \Lambda^- \end{pmatrix} = \begin{pmatrix} \Delta & P^* \\ P & PP^* \end{pmatrix} \times \Lambda_1,$$
$$\Gamma_- = \begin{pmatrix} \Delta & I^* \\ P & \Lambda_- \end{pmatrix} = \begin{pmatrix} \Delta & I^* \\ P & PI^* \end{pmatrix} \times N_1,$$
$$\Gamma^{+-} = \Gamma^{-+} = \begin{pmatrix} \Delta & P^* \\ I & IP^* \end{pmatrix} \times \Lambda_1.$$

Consequently, we have

$$\partial_v \Gamma = \Gamma_0' \times \begin{pmatrix} \Lambda_1 & \Lambda_1 \\ N_1 & \Lambda_1 \end{pmatrix},$$

where Γ_0' is the maximal order in $\mathrm{M}_2(K\Gamma_0)$ with the indecomposable representation

$$\begin{pmatrix} \Delta \\ I \\ \Delta \\ P \end{pmatrix}.$$

Hence, Theorem 1 gives an equivalence

$$\widetilde{\partial}_v : \Gamma\text{-}\mathbf{lat}/[\Gamma_0] \xrightarrow{\sim} \begin{pmatrix} \Lambda_1 & \Lambda_1 \\ N_1 & \Lambda_1 \end{pmatrix}\text{-}\mathbf{lat},$$

and the composition $\widetilde{\partial}_v \circ \mathcal{F}'$ coincides with $\widetilde{\partial}_u$. In fact, the preceding calculation in particular yields

$$\partial_u \Lambda = \begin{pmatrix} II^* & IP^* \\ PI^* & PP^* \end{pmatrix} \times \begin{pmatrix} \Lambda_1 & \Lambda_1 \\ N_1 & \Lambda_1 \end{pmatrix}, \tag{62}$$

where the left-hand factor is the maximal order with $\binom{I}{P}$ as indecomposable representation. ∎

REMARKS. 1. If $\partial_u\Lambda$ is replaced by $\delta_u\Lambda$, then the first factor in (62) becomes a hereditary order with an additional indecomposable representation $\binom{P}{P}$. This gives another point for our preference for $\partial_u\Lambda$.

2. If ${}_\Lambda I$ is projective or injective, then $\partial_u\Lambda$ is no longer defined. In this case, however, Λ is a trivial extension. Therefore, the equivalence (61) of the proposition remains valid, although it is only partially induced by some ∂_u.

3. Recently, O. Iyama [5] obtained a similar result where Λ_0 is not assumed to be hereditary. The right-hand order $\left(\begin{smallmatrix} \Lambda_1 & \Lambda_1 \\ N_1 & \Lambda_1 \end{smallmatrix}\right)$ in (61) is then replaced by an order which is defined in terms of the Auslander–Reiten quiver of Λ.

7. Examples. In the following examples, let $\mathfrak{p}$ denote the radical of R, and $\mathfrak{k} := R/\mathfrak{p}$. For any pair of R-orders Λ_0, Λ_1 with $\Lambda_0/\operatorname{Rad}\Lambda_0 \cong \Lambda_1/\operatorname{Rad}\Lambda_1 \cong \mathfrak{k} \times \ldots \times \mathfrak{k}$, we define by the pullback

$$\begin{array}{ccc} \Lambda_0 & \twoheadrightarrow & \mathfrak{k} \times \ldots \times \mathfrak{k} \\ \uparrow & & \uparrow \\ \Lambda_0 \diamond \Lambda_1 & \twoheadrightarrow & \Lambda_1 \end{array}$$

an R-order $\Lambda_0 \diamond \Lambda_1$ in $K\Lambda_0 \times K\Lambda_1$ which will be called the *dyad* (cf. [10]) of Λ_0 and Λ_1. Clearly, $\Lambda_0 \diamond \Lambda_1$ has the same residue algebra $\mathfrak{k} \times \ldots \times \mathfrak{k}$ as Λ_0 and Λ_1, and the operation $\diamond$ is associative and commutative. For Λ_i-lattices E_i with $E_0/\operatorname{Rad}E_0 \cong E_1/\operatorname{Rad}E_1$, a similar pullback yields a $\Lambda_0 \diamond \Lambda_1$-lattice which we denote by $E_0 \diamond E_1$ whenever it is unique up to isomorphism. Sometimes it will be convenient to write $\Lambda_0 \text{—} \Lambda_1$ instead of $\Lambda_0 \diamond \Lambda_1$.

EXAMPLE 1. In [19], Example 1, we considered the R-order $\Lambda := \Lambda_0 \diamond \Lambda_1$ in $\mathrm{M}_2(K)$ with

$$\Lambda_0 := \begin{pmatrix} R & \mathfrak{p} \\ R & R \end{pmatrix}, \quad \Lambda_1 := \begin{pmatrix} R & \mathfrak{p} \\ \mathfrak{p} & R \end{pmatrix}.$$

Λ has five irreducible representations, namely the Λ_0-lattices $H_1 := \binom{R}{R}$, $H_2 := \binom{\mathfrak{p}}{R}$, and the Λ_1-lattices $L_1 := \binom{R}{\mathfrak{p}}$, $L_2 := \binom{\mathfrak{p}}{R}$, $L_3 := \binom{R}{R}$. The remaining indecomposable Λ-lattices are the two projectives $P_1 := H_1 \diamond L_1$ and $P_2 := H_2 \diamond L_2$, the corresponding injectives $I_1 := H_1 \diamond L_3$ and $I_2 := H_2 \diamond L_3$, and an additional Λ-lattice $L := \Lambda_0 \diamond L_3$.

In [19] we already considered the hereditary monomorphism $P_1 \hookrightarrow I_1$. In order to illustrate Proposition 20, we choose $u : \mathfrak{p}H_1 \hookrightarrow H_1$. Then for each indecomposable Λ-lattice E, there exists an integer r with $\partial_u E \cong \binom{H_1}{\mathfrak{p}H_1}^r \oplus E'$, where E' is either zero or an indecomposable representation of

$$\Lambda' := \begin{pmatrix} \Lambda_1 & \Lambda_1 \\ \operatorname{Rad} \Lambda_1 & \Lambda_1 \end{pmatrix},$$

a tiled order of weight two [3]. The 8 indecomposable Λ'-lattices are therefore all irreducible. The map $E \mapsto E'$ is given by the table

E	H_1	H_2	L_1	L_2	L_3	P_1	P_2	I_1	I_2	L
E'	0	0	R	$\mathfrak{p}$	R	R	$\mathfrak{p}$	R	R	R
			$\mathfrak{p}$	R	R	$\mathfrak{p}$	R	R	R	R
			R	$\mathfrak{p}$	R	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	R	$\mathfrak{p}$
			$\mathfrak{p}$	R	R	$\mathfrak{p}$	$\mathfrak{p}$	R	$\mathfrak{p}$	$\mathfrak{p}$

Example 2. Next let us consider the local R-order $\Lambda := R \diamond \Sigma_m$ in $A = K \times K \times K$, where $m \geq 1$, and Σ_m is given by the pullback

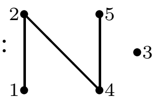
$$\begin{array}{ccc} R & \twoheadrightarrow & R/\mathfrak{p}^m \\ \uparrow & & \uparrow \\ \Sigma_m & \twoheadrightarrow & R \end{array}$$

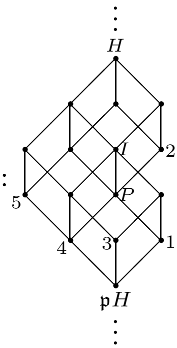
The maximal order $\Lambda_0 = R$ in the first simple component $A_0 = K$ of A is a generalized over-order of Λ with $\operatorname{Rad} \Lambda_0 \subseteq \Lambda$. Hence Proposition 20 yields an equivalence $\Lambda\textbf{-lat}/[\Lambda_0] \xrightarrow{\sim} \Lambda'\textbf{-lat}$, where

$$\Lambda' := \begin{pmatrix} \Sigma_m & \Sigma_m \\ \operatorname{Rad} \Sigma_m & \Sigma_m \end{pmatrix}$$

is an order of weight two [3]. Hence by [3], Theorem 4.9, the $4m + 3$ indecomposable Λ'-lattices can be obtained by successive application of the rejection lemma ([3], 2.9). Therefore, Λ itself has $4(m+1)$ indecomposables.

Example 3. By [19], Proposition 16, representations of a finite poset Ω can be regarded as Λ-lattices for a subhereditary tiled order Λ. For such orders, Theorem 1 becomes equivalent to Zavadskiĭ's algorithm for posets Ω if and only if (P) is satisfied. Otherwise, we obtain various almost embeddings $\mathbf{Rep}_{\mathfrak{k}}(\Omega) \to \mathbf{Rep}_{\mathfrak{k}}(\Omega')$ according to the possible pre-hereditary monomorphisms. For example:

Ω :


$\mathfrak{S}_\Lambda$:


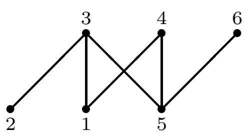
$$\Lambda = \begin{pmatrix} R & R & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & R & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & R & \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \end{pmatrix} \qquad H = \begin{pmatrix} R \\ R \\ R \\ R \\ R \end{pmatrix}$$

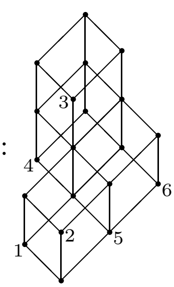
Here the poset Ω is realized by the projective Λ-lattices in $\mathfrak{S}_\Lambda$ (see (55)) between H and $\mathfrak{p}H$, and the $\leq$ relations in Ω are also expressed by the exponents $0, 1$ of $\mathfrak{p}$ in Λ. The irreducible Λ-lattices, up to isomorphism, are represented by the half-open interval $(\mathfrak{p}H, H]$ in $\mathfrak{S}_\Lambda$, whereas the closed interval $[\mathfrak{p}H, H]$ coincides with the (distributive) lattice V_Ω of one-dimensional Ω^{op}-representations.

Now let us consider the pre-hereditary monomorphism

$$u : P = \begin{pmatrix} R \\ \mathfrak{p} \\ \mathfrak{p} \\ R \\ \mathfrak{p} \end{pmatrix} \hookrightarrow I = \begin{pmatrix} R \\ \mathfrak{p} \\ R \\ R \\ \mathfrak{p} \end{pmatrix}.$$

Then the reduced u-derivative $\Lambda' = \partial'_u \Lambda$ together with the interval $V_{\Omega'}$ in $\mathfrak{S}_{\Lambda'} = \bigcup_{i \in \mathbb{Z}} \mathfrak{p}^i V_{\Omega'}$ and the corresponding poset Ω' are as follows:

Ω' :


$V_{\Omega'}$:


$$\Lambda' = \begin{pmatrix} R & \mathfrak{p} & R & R & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & R & R & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R & R & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \end{pmatrix}$$

Hence, the poset Ω' should be called the *u-derivative* of Ω, and Theorem 1 yields a map

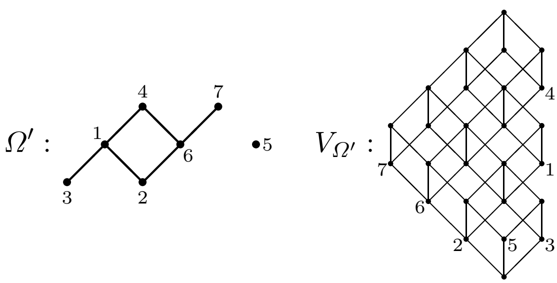
$$\operatorname{ind} \Omega \to \operatorname{ind} \Omega' \tag{63}$$

which is almost injective in the sense that only the Ω-representations corresponding to P and I are collapsed. By [17], Satz 4, the indecomposables of Ω can be read off from V_Ω, namely, there are 16 one-dimensional representations, and 5 two-dimensional indecomposables corresponding to the 3 cubes and 2 double cubes in V_Ω. For Ω' there are 20 one-dimensional and 7 two-dimensional indecomposables, according to the 4 cubes and 3 double cubes. Hence, apart from the two one-dimensional Ω'-representations associated with the Λ'-lattices $\binom{P}{P}$ and $\binom{I}{I}$, there are 5 indecomposable Ω'-representations not in the image of (63). Two of them are one-dimensional, and three two-dimensional.

Example 4. In the preceding example, consider instead of u the following pre-hereditary monomorphism:

$$v : P = \begin{pmatrix} R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \\ R & \!\!-\!\! R \\ \mathfrak{p} & R \\ \mathfrak{p} & R \end{pmatrix} \hookrightarrow I = \begin{pmatrix} R & \mathfrak{p} \\ R & \mathfrak{p} \\ R & \!\!-\!\! R \\ R & R \\ \mathfrak{p} & R \end{pmatrix}$$

between the binomial indecomposables P, I corresponding to the two double cubes in $V_\Omega = [\mathfrak{p}H, H]$. (Here R—R means the dyad $R \diamond R$.) In fact, it is easily verified that v satisfies (C$''$). In this example, $\Lambda^+ = \Lambda^- = \Lambda$, and we obtain the v-derivative



which has 26 one-dimensional, 15 two-dimensional, and 2 three-dimensional indecomposables. (If D_n denotes a chain of n elements, the 15 two-dimensional indecomposables arise from the six simple cubes D_2^3, six double cubes $\mathrm{D}_2^2 \times \mathrm{D}_3$, two treble cubes $\mathrm{D}_2^2 \times \mathrm{D}_4$, and one cube isomorphic to $\mathrm{D}_2 \times \mathrm{D}_3^2$. Moreover, $\mathrm{D}_2 \times \mathrm{D}_3^2$ itself yields a pair of three-dimensional indecomposables.) Since I/P is of length two, the image of (63) consists of $|\mathrm{ind}\,\Omega| - 2 = 19$ indecomposables. Six of the 24 remaining indecomposable Ω'-representations correspond to $\partial_v\Lambda$-lattices in the category $\mathcal{H}'_v$ of Proposition 15.

Example 5. Generalized Brauer tree orders of "defect p" type [15, 18] give rise to complete splittings. More generally, we define [18] a *cycle hypergraph* H by a surjective map $\varepsilon : C \twoheadrightarrow E$ between finite sets, together with a permutation π on C. The cycles of π are then the vertices of H, the elements of E the edges, and ε gives the rule of attachment between vertices and edges. If every edge has exactly two vertices (with multiplicities counted), then H is equivalent to a Brauer graph [15]. Now let Γ be a hereditary R-order corresponding to π, i.e. there is a bijection $P : C \xrightarrow{\sim} \operatorname{ind}\Gamma$ onto a complete system of indecomposable Γ-lattices such that $\operatorname{Rad} P_c = P_{\pi c}$ for all $c \in C$. For simplicity, suppose Γ is *totally split*, i.e. $\Gamma/\operatorname{Rad}\Gamma \cong \mathfrak{k} \times \ldots \times \mathfrak{k} = \operatorname{Map}(C, \mathfrak{k})$. Then ε induces an embedding of rings

$$\varepsilon^* : \operatorname{Map}(E, \mathfrak{k}) \hookrightarrow \operatorname{Map}(C, \mathfrak{k}), \tag{64}$$

and the R-order Λ_H associated with H is given by the pullback

$$\begin{array}{ccc} \Gamma & \twoheadrightarrow & \operatorname{Map}(C, \mathfrak{k}) \\ \uparrow & & \uparrow \varepsilon^* \\ \Lambda_H & \twoheadrightarrow & \operatorname{Map}(E, \mathfrak{k}) \end{array}$$

Hence Λ_H is a Bäckström order, i.e. $\operatorname{Rad}\Lambda_H = \operatorname{Rad}\Gamma$, and the embedding (64) shows that there is a one-to-one correspondence between the indecomposable projective Λ_H-lattices and the edges of H. In particular, Λ_H is local if and only if H has only one edge. Hence, every Λ_H allows a complete splitting into R-orders $\Lambda_{H'}$ and $\Lambda_{H''}$ with cycle hypergraphs H' and H'' such that $\Lambda_{H'}$ is local.

Example 6. Consider the following R-order Λ with a splitting prehereditary monomorphism u:

$$\Lambda = \begin{pmatrix} R & \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \end{pmatrix}, \qquad u : P = \begin{pmatrix} \mathfrak{p} \\ R \\ R \end{pmatrix} \hookrightarrow I = \begin{pmatrix} R \\ R \\ R \end{pmatrix},$$

where the dyad $R \diamond R$ is again indicated by a connecting line. Then

$$\Lambda^+ = \begin{pmatrix} R & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \end{pmatrix}, \qquad \Lambda^- = \begin{pmatrix} R & \mathfrak{p}^2 & \mathfrak{p}^2 \\ R & R & \mathfrak{p} \\ R & \mathfrak{p} & R \end{pmatrix}$$

and

$$\mathfrak{p}IP^* = \begin{pmatrix} R & \mathfrak{p} & \mathfrak{p} \\ R & \mathfrak{p} & \mathfrak{p} \\ R & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \not\subseteq \begin{pmatrix} R & \mathfrak{p} & \mathfrak{p} \\ R & R & \mathfrak{p} \\ R & \mathfrak{p} & R \end{pmatrix} = \Lambda^+\Lambda^-$$

shows that Proposition 18 is not valid for $\delta_u\Lambda$ instead of $\partial_u\Lambda$.

Example 7. The order

$$\Lambda = \begin{pmatrix} R\text{—}R & R\text{—}R \\ \mathfrak{p} \times \mathfrak{p} & R\text{—}R \end{pmatrix} \subseteq \mathrm{M}_2(K) \times \mathrm{M}_2(K)$$

has 4 irreducibles, namely $P := \binom{R}{\mathfrak{p}}$ and $I := \binom{R}{R}$ in the first rational component, and the corresponding irreducibles P' and I' in the second component. Moreover, there are 3 binomial indecomposables

$$P_1 := \begin{pmatrix} R\boldsymbol{-}R \\ \mathfrak{p}\times\mathfrak{p} \end{pmatrix}, \qquad I_2 := \begin{pmatrix} R\times R \\ R\boldsymbol{-}R \end{pmatrix}, \qquad B := \begin{pmatrix} R\boldsymbol{-}R \\ R\boldsymbol{-}R \end{pmatrix},$$

where the latter is bijective. The splitting pre-hereditary monomorphism $u : P \hookrightarrow I$ yields $\Lambda^+ = \Lambda$ and

$$\Lambda^- = \begin{pmatrix} R\boldsymbol{-}R & R\times R \\ \mathfrak{p}\times\mathfrak{p} & R\times R \end{pmatrix} = \Lambda^{-+} = \Lambda^{+-}, \qquad \Lambda_- = \begin{pmatrix} R\boldsymbol{-}R & R\boldsymbol{-}R \\ \mathfrak{p}\times\mathfrak{p} & \mathfrak{p}\times\mathfrak{p} \end{pmatrix}.$$

Hence, the reduced u-derivative is

$$\partial'_u\Lambda = \begin{pmatrix} R\boldsymbol{-}R & \mathfrak{p}\times\mathfrak{p} & R\times R \\ R\boldsymbol{-}R & R\boldsymbol{-}R & R\times R \\ \mathfrak{p}\times\mathfrak{p} & \mathfrak{p}\times\mathfrak{p} & R\times R \end{pmatrix},$$

a twofold trivial extension of the order $\left(\begin{smallmatrix} R\boldsymbol{-}R & \mathfrak{p}\times\mathfrak{p} \\ R\boldsymbol{-}R & R\boldsymbol{-}R \end{smallmatrix}\right) \cong \Lambda$. Therefore, counting indecomposables shows that apart from $\binom{I}{I}$ and $\binom{P}{P}$, there must be one more indecomposable $\partial_u\Lambda$-lattice which is not obtained by the differentiation functor. In fact, this $\partial_u\Lambda$-representation is given by the $\partial'_u\Lambda$-lattice

$$\begin{pmatrix} R\boldsymbol{-}R \\ R\times R \\ \mathfrak{p}\times\mathfrak{p} \end{pmatrix}.$$

(By the remark following Proposition 19, such $\partial_u\Lambda$-lattices are not possible if Λ is tiled.)

EXAMPLE 8. Finally, let us illustrate Theorem 4 by a simple example. To this end, let D be an unramified quadratic extension of K with maximal order Δ and $\Pi := \operatorname{Rad}\Delta$. With the R-order $\Omega := R + \Pi$ we form the dyad $\Omega \diamond R$ and consider the R-order

$$\Lambda := \begin{matrix} R\boldsymbol{-} \\ \\ \\ \end{matrix}\!\!\begin{pmatrix} \Omega & \Pi & \Pi \\ \Pi & \Delta & \Pi \\ \Delta & \Pi & \Omega \end{pmatrix}\!\!\begin{matrix} \\ \\ \boldsymbol{-}R \end{matrix}$$

in $K \times \mathrm{M}_3(D) \times K$. By [19], Proposition 14,

$$u : P = \begin{pmatrix} \Pi \\ \Pi \\ \Delta \end{pmatrix} \hookrightarrow I = \begin{pmatrix} \Pi \\ \Delta \\ \Delta \end{pmatrix}$$

is pre-hereditary, and u satisfies the splitting condition (53). For the maximal order $\Theta := \mathrm{M}_3(\Delta)$, the Θ-lattice $H := \Theta I$ satisfies $H\Pi \subseteq P \subseteq I \subseteq H$ and $\operatorname{Rad}\Theta \subseteq \Lambda$. Moreover, there is a decomposition $\Lambda = P_1 \oplus P_0 \oplus P_2$ with

$$P_1 := R\text{---}\!\begin{pmatrix}\Omega\\ \Pi\\ \Delta\end{pmatrix}, \quad P_0 := \begin{pmatrix}\Pi\\ \Delta\\ \Pi\end{pmatrix}, \quad P_2 := \begin{pmatrix}\Pi\\ \Pi\\ \Omega\end{pmatrix}\!\text{---}R$$

satisfying the assumption of Theorem 4. Hence, Λ has a pair of splitting over-orders

$$\Lambda_1 = R\text{---}\!\begin{pmatrix}\Omega & \Pi & \Pi\\ \Pi & \Delta & \Pi\\ \Delta & \Delta & \Delta\end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix}\Delta & \Pi & \Pi\\ \Delta & \Delta & \Pi\\ \Delta & \Pi & \Omega\end{pmatrix}\!\text{---}R$$

with

$$\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = \begin{pmatrix}\Delta & \Pi & \Pi\\ \Delta & \Delta & \Pi\\ \Delta & \Delta & \Delta\end{pmatrix}.$$

Furthermore, Λ_1 and Λ_2 are trivial extensions of the order

$$\Xi := \begin{pmatrix}\Delta & \Pi\\ \Pi & \Omega\end{pmatrix}\!\text{---}R$$

in $\mathrm{M}_2(D) \times K$. By Proposition 20, the indecomposable Ξ-lattices except R can be obtained from the indecomposables of an order in $\mathrm{M}_4(D)$ Morita equivalent to the order

$$\Xi_0 := \begin{pmatrix}\Omega & \Pi & \Omega\\ \Pi & \Delta & \Pi\\ \Pi & \Pi & \Omega\end{pmatrix}$$

which corresponds to a Schurian vector space category of type F_4'' listed in [7]. The 19 indecomposable Ξ_0-lattices are given (as representations of the corresponding $\mathfrak{k}$-structure) in [2], §3. Therefore, Λ_1 and Λ_2 have 21 indecomposables each, and consequently, there are $2 \cdot 21 - 3 = 39$ indecomposable Λ-lattices. Alternatively, a twofold application of Proposition 20 to Λ yields an order Morita equivalent to a subhereditary order Λ' in $\mathrm{M}_5(D)$, and Simson's splitting theorem applies to Λ'.

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Mathematisch-Geographische Fakultät
Katholische Universität Eichstätt
Ostenstr. 26-28
D-85071 Eichstätt, Germany
E-mail: wolfgang.rump@ku-eichstaett.de

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