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ON THE CONDITION OF Λ-CONVEXITY IN SOME PROBLEMS OF WEAK CONTINUITY AND WEAK LOWER SEMICONTINUITY

BY

AGNIESZKA KAŁAMAJSKA (Warszawa)

Abstract. We study the functional $I_f(u) = \int_\Omega f(u(x))\,dx$, where $u = (u_1, \ldots, u_m)$ and each u_j is constant along some subspace W_j of $\mathbb{R}^n$. We show that if intersections of the W_j's satisfy a certain condition then I_f is weakly lower semicontinuous if and only if f is Λ-convex (see Definition 1.1 and Theorem 1.1). We also give a necessary and sufficient condition on $\{W_j\}_{j=1,\ldots,m}$ to have the equivalence: I_f is weakly continuous if and only if f is Λ-affine.

1. Introduction and statement of results. Assume that $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $u : \Omega \to \mathbb{R}^m$, $u = (u_1, \ldots, u_m)$, $u_i \in L^1_{\mathrm{loc}}(\Omega)$, $P = (P_1, \ldots, P_N)$ is a first order vector-valued differential operator with constant coefficients,

$$P_k u = \sum_{j=1}^{m} \sum_{i=1}^{n} a^k_{i,j} \frac{\partial u_j}{\partial x_i} \quad \text{for } k = 1, \ldots, N, \tag{1}$$

and $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Let $\{u^\nu\}_{\nu \in \mathbb{N}}$ be a bounded sequence in $L^\infty(\Omega, \mathbb{R}^m)$ such that $Pu^\nu = 0$ in the sense of distributions. The basic question of the compensated compactness theory is the following: what can we say about weak limits of $f(u^\nu)$ as $\nu \to \infty$? By weak limits we understand limits in $L^\infty(\Omega)$ with respect to weak $*$ convergence denoted by $\overset{*}{\rightharpoonup}$.

This problem has been recognized as being of crucial importance in many areas of mathematics, for example in the study of systems of conservation laws [9, 10, 13, 14, 30, 32–36], nonlinear elasticity [1, 3, 7, 13, 16, 23, 27, 31, 38], micromagnetics [8, 17, 23, 26], nonlinear geometric optics [18, 19], Skyrme's model for meson fields [12], and fluid mechanics [11].

The problem is related to the study of sequential weak lower semicontinuity and sequential weak continuity of the functional

$$I_f(u) = \int_\Omega f(u(x))\,dx, \quad u \in \operatorname{Ker} P, \tag{2}$$

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in the weak $*$ topology of $L^\infty(\Omega,\mathbb{R}^m)$. Let us recall that I_f is *sequentially weakly lower semicontinuous* if for every sequence $u^\nu \in \operatorname{Ker} P \cap L^\infty(\Omega,\mathbb{R}^m)$ such that $u^\nu \overset{*}{\rightharpoonup} u$ we have $\liminf I_f(u^\nu) \geq I_f(u)$, and I_f is *sequentially weakly continuous* if $\lim I_f(u^\nu) = I_f(u)$ as $\nu \to \infty$.

In particular, when $P = \operatorname{curl}$ is applied to each coordinate of $u = (u^1,\dots,u^m)$ ($u^i \in \mathbb{R}^n$) in a simply connected domain, we have to do with the classical functional of the calculus of variations.

The so-called Λ-convexity condition is crucial in this approach. Here by Λ we will usually denote a cone in $\mathbb{R}^m$, that is, an arbitrary set invariant under dilation: if $\lambda \in \Lambda$ and $t \in \mathbb{R}$ then $t\lambda \in \Lambda$.

DEFINITION 1.1. Let $f : \mathbb{R}^m \to \mathbb{R}$ and assume that $\Lambda \subseteq \mathbb{R}^m$ is a cone. We say that f is Λ-*convex* if for each $A \in \mathbb{R}^m$ and $\lambda \in \Lambda$ the function

$$\mathbb{R} \ni t \mapsto f(A + t\lambda) \tag{3}$$

is convex. The mapping f is called Λ-*affine* if for each $A \in \mathbb{R}^m$ and $\lambda \in \Lambda$ the function (3) is affine.

The following result was established by Murat and Tartar (see e.g. [25, Theorem 2.1], [7, Theorem 3.1], [27, Theorem 10.1], [33, Corollary 9]).

THEOREM 1.1. *Define*

$$V = \Big\{(\xi,\lambda) : \xi \in \mathbb{R}^n,\ \xi \neq 0,\ \lambda \in \mathbb{R}^m,\ \sum_{i,j} a^k_{i,j}\xi_i\lambda_j = 0 \textit{ for } k = 0,\dots,N\Big\},$$

$$\Lambda = \{\lambda \in \mathbb{R}^m : \textit{there exists } \xi \in \mathbb{R}^n,\ \xi \neq 0,\ \textit{such that } (\xi,\lambda) \in V\}.$$

If I_f given by (2) *is lower semicontinuous with respect to L^∞-weak $*$ convergence*, *then f is Λ-convex. If I_f is continuous with respect to L^∞-weak $*$ convergence*, *then f is Λ-affine.*

If f is a quadratic form, then the lower semicontinuity of I_f is equivalent to the convexity of f in the directions of Λ (see e.g. [25, Section 3], [32, Theorem 11]), while for general f there is no equivalence in the above theorem. A relevant example is well known ([7, p. 26], [32], [25]). Let $u := (u_1,u_2,u_3)$, $n = 2$, $\frac{\partial}{\partial x}u_1(x,y) = 0$, $\frac{\partial}{\partial y}u_2(x,y) = 0$ and $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})u_3(x,y) = 0$. Here $\Lambda = (\mathbb{R}\times\{0\}\times\{0\}) \cup (\{0\}\times\mathbb{R}\times\{0\}) \cup (\{0\}\times\{0\}\times\mathbb{R})$ and the function $f(x,y,z) = xyz$ is Λ-affine, in particular f and $-f$ are Λ-convex, but $I_f(u) = \int_\Omega f(u)\,dx$ is not weakly continuous. This shows that I_f and $-I_f$ cannot be lower semicontinuous.

Our goal is the following. We restrict our attention to the special case when each P_k is of the form $\partial u_{j(k)}/\partial v_k$. In particular, every coordinate function u_j is constant along some subspace W_j of $\mathbb{R}^n$. There are two problems we are concerned with.

PROBLEM 1. Describe the set $\mathcal{F}$ of all m-tuples $\{W_j\}_{j=1,\dots,m}$ of subspaces of $\mathbb{R}^n$ such that if $f : \mathbb{R}^m \to \mathbb{R}$ is continuous then the following conditions are equivalent:

(1) The functional $I_f(u)$ is continuous with respect to the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbb{R}^m) \cap \operatorname{Ker} P$.
(2) f is Λ-affine.

PROBLEM 2. Describe the set $\mathcal{G}$ of all m-tuples $\{W_j\}_{j=1,\dots,m}$ of subspaces of $\mathbb{R}^n$ such that if $f : \mathbb{R}^m \to \mathbb{R}$ is continuous then the following conditions are equivalent:

(1) The functional $I_f(u)$ is lower semicontinuous with respect to the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbb{R}^m) \cap \operatorname{Ker} P$.
(2) f is Λ-convex.

We have succeeded in solving Problem 1 (see Theorem 3.3). We show that the set $\mathcal{F}$ consists of all m-tuples $\{W_i\}_{i=1,\dots,m}$ which satisfy a condition of transversality (see Definition 2.3). Unfortunately, we have not been able to solve Problem 2 completely. In Theorem 3.2 we give a sufficient condition for $\{W_i\}_{i=1,\dots,m} \in \mathcal{G}$. We call it the parallelness condition (see Definition 2.2) and discuss it in Section 4. Also in Example 5.1 we show that the set $\mathcal{G}$ is essentially larger than the set of m-tuples which satisfy the parallelness condition. Note that we always have $\mathcal{G} \subseteq \mathcal{F}$. It may be that $\mathcal{G} = \mathcal{F}$; this hypothesis is motivated by Example 5.1, but I have not been able to prove it.

Let us mention that in the proof of Theorems 3.2 and 3.3 we apply the powerful theory of Young measure.

Although our model looks rather simple at first glance, necessary and sufficient conditions for lower semicontinuity of I_f are not known in this case. Some examples representing this model appear in the literature (see e.g. [25, Section 7.3], [28, 29], [32, Examples 5 and 6 and Propositions 15–17] and [37], see also the recent deep result of Müller [24]); a similar model appears in geometric optics [18, 19].

I believe that a further investigation of the model will bring some new geometrically transparent necessary conditions for lower semicontinuity of the functional I_f in the general setting.

2. Notation and some preliminaries.

Let $m \in \mathbb{N}$. We recall the standard order in $\{0,1\}^m$: for $I, J \in \{0,1\}^m$ we have $I > J$ if either $i_1 > j_1$, or $i_1 = j_1$ and $i_2 > j_2, \dots,$ or $i_s = j_s$ for $s = 1, \dots, l$, $l < m$, and $i_{l+1} > j_{l+1}$. For $I \in \{0,1\}^m$ we set

$$D(I) = \{r \in \{1, \dots, m\} : I \text{ has 1 on the } r\text{th place}\}, \tag{4}$$

$$D^*(I) = \{r \in \{1, \dots, m\} : I \text{ has 0 on the } r\text{th place}\}. \tag{5}$$

If $D(I) = \{i\}$ we will write $I = \delta_i$ for simplicity.

Given $I \in \{0,1\}^m$, we denote by I^* the element of $\{0,1\}^m$ such that $D(I^*) = D^*(I)$. Consequently, if $\mathcal{A} \subseteq \{0,1\}^m$ then we define $\mathcal{A}^* = \{I^* : I \in \mathcal{A}\}$. If A is a finite set then $\#A$ denotes the number of its elements.

Let W be a linear subspace of $\mathbb{R}^n$ equipped with a scalar product $\langle\langle\cdot,\cdot\rangle\rangle$. By $W^{\pm}$ we denote the subspace perpendicular to W with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. The symbol $\langle\cdot,\cdot\rangle$ will stand for the standard scalar product and $W^{\perp}$ for the space orthogonal to W with respect to the standard scalar product. The standard basis will be denoted by $\{e_1,\ldots,e_n\}$.

We denote the m-product of the sum of Grassmannians in $\mathbb{R}^n$ by

$$(6)\qquad \widetilde{\mathcal{W}}(n,m) = \{W = (W_1,\ldots,W_m) : W_i \text{ are linear subspaces of } \mathbb{R}^n\},$$

and its special subset by

$$(7)\qquad \mathcal{W}(n,m) = \{W = (W_1,\ldots,W_m) \in \widetilde{\mathcal{W}}(n,m) : W_1 + \ldots + W_m = \mathbb{R}^n\},$$

where $W_1 + \ldots + W_m = \operatorname{span}\{W_i\}_{i=1,\ldots,m}$ is the algebraic sum of the W_i.

If $W \in \widetilde{\mathcal{W}}(n,m)$ and $W = (W_1,\ldots,W_m)$, we set $W^{\pm} = (W_1^{\pm},\ldots,W_m^{\pm})$.

If $I \in \{0,1\}^m$, we define $W^I = \bigcap_{i\in D(I)} W_i$ if $D(I) \neq \emptyset$, $W^0 = \mathbb{R}^n$ (to abbreviate we write simply 0 instead of $(0,\ldots,0)$). For example, when $m = 3$ we have $W^{(1,1,0)} = W_1 \cap W_2$.

Given $m \in \mathbb{N}$, $W = (W_1,\ldots,W_m) \in \widetilde{\mathcal{W}}(n,m)$, and the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$, we introduce subsets of $\{0,1\}^m$: $\mathcal{A}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \{I \in \{0,1\}^m : (W^{\pm})^I \neq \{0\}\}$, and $\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \{I \in \mathcal{A}(\mathcal{W},\langle\langle\cdot,\cdot\rangle\rangle) : \text{if } J \in \{0,1\}^m, D(J) \supseteq D(I), \text{ and } J \neq I \text{ then } (W^{\pm})^J = \{0\}\}$.

By $\mathbb{R}^{\times I}$ we denote $\mathbb{R}^{\times i_1} \times \ldots \times \mathbb{R}^{\times i_m}$, where $\mathbb{R}^{\times 0} = \{0\}$, $\mathbb{R}^{\times 1} = \mathbb{R}$. For example $\mathbb{R}^{\times(1,0,1)} = \mathbb{R} \times \{0\} \times \mathbb{R}$.

If $W \in \widetilde{\mathcal{W}}(n,m)$, and $\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \{I_1,\ldots,I_k\}$, where $I_1 < \ldots < I_k$, we set

$$(8)\qquad W^*_{\langle\langle\cdot,\cdot\rangle\rangle} = \{W^{I^*}\}_{I\in\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle)} = \{(W^{I_1^*},\ldots,W^{I_k^*})\} \in \widetilde{\mathcal{W}}(n,k).$$

If $\langle\langle\cdot,\cdot\rangle\rangle$ is the standard scalar product $\langle\cdot,\cdot\rangle$, we write simply $\mathcal{A}(W)$, $\mathcal{B}(W)$ and W^*.

We will need the following definitions.

Definition 2.1. We say that $W \in \widetilde{\mathcal{W}}(n,m)$ is *decomposable into a direct sum* if there exist $k \in \mathbb{N}$ and subspaces $A_1,\ldots,A_k$ of $\mathbb{R}^n$ such that each W_i is of the form $W_i = A_{k_1^i} \oplus \ldots \oplus A_{k_{l_i}^i}$ for some $k_1^i,\ldots,k_{l_i}^i \in \{1,\ldots,k\}$, where $A_1 \oplus \ldots \oplus A_l$ stands for the direct sum.

Example 2.1. The collection of spaces $W_1 = \operatorname{span}\{e_1\}$, $W_2 = \operatorname{span}\{e_2\}$, and $W_3 = \operatorname{span}\{e_1 + e_2\}$, $n = 2$, is not decomposable into a direct sum.

Remark 2.1. Note that $W \in \widetilde{\mathcal{W}}(n,m)$ is decomposable into a direct sum if and only if there is a basis $w_1,\ldots,w_n$ in $\mathbb{R}^n$ such that each W_i can be represented in this basis as $W_i = \operatorname{span}\{w_j\}_{j\in\{k_1^i,\ldots,k_{l_i}^i\}}$ for some $k_1^i,\ldots,k_{l_i}^i \in$

$\{1, \dots, n\}$. In such a case, we will say that the W_i are *decomposable along the basis* $\{w_1, \dots, w_n\}$.

REMARK 2.2. If $W \in \widetilde{\mathcal{W}}(n,m)$ is decomposable into a direct sum, then $W^{\pm}$ does not need to be decomposable into a direct sum. For example take $m = n = 3$, $W_1 = \operatorname{span}\{e_1 - e_2\}$, $W_2 = \operatorname{span}\{e_1 - 2e_2\}$, $W_3 = \operatorname{span}\{e_1 - 3e_3\}$, and the standard scalar product.

To abbreviate, we will say that subpaces $\{A_i\}_{i=1,\dots,k}$ of $\mathbb{R}^n$ which satisfy the condition $A_i \cap (A_1 + \dots + A_{i-1} + A_{i+1} + \dots + A_k) = \{0\}$ for every $i \in \{1, \dots, k\}$ are *independent*.

We introduce the following condition:

DEFINITION 2.2. Let $W \in \widetilde{\mathcal{W}}(n,m)$. We say that W satisfies the *parallelness condition* if the spaces $\{W^{I^*}\}_{I \in \mathcal{B}(W)}$ span the whole $\mathbb{R}^n$.

Note that in particular W must also span $\mathbb{R}^n$. In our notation (see (7) and (8)), the parallelness condition reads as

$$W \in \mathcal{W}(n,m) \quad \text{and} \quad W^* \in \mathcal{W}(n,k), \quad \text{where } k = \#\mathcal{B}(W). \tag{9}$$

REMARK 2.3. If $I_1, I_2 \in \{0,1\}^m$ are such that $I_1 < I_2$ then $I_1^* > I_2^*$ and $W^{I_1^*} \subseteq W^{I_2^*}$. This implies that $\operatorname{span}\{W^{I^*}\}_{I \in \mathcal{B}(W)} = \operatorname{span}\{W^{I^*}\}_{I \in \mathcal{A}(W)}$.

EXAMPLE 2.2. The collection of spaces $W_1 = \operatorname{span}\{e_1\}$, $W_2 = \operatorname{span}\{e_2\}$, $W_3 = \operatorname{span}\{e_1 + e_2\}$, $n = 2$, $m = 3$, does not satisfy the parallelness condition.

We refer to Section 4 for a detailed discussion of the parallelness condition. In particular Theorem 4.2 there can be used to construct examples.

DEFINITION 2.3. Let $W \in \widetilde{\mathcal{W}}(n,m)$. We say that W satisfies the *condition of transversality* if for each $A \subseteq \{1, \dots, m\}$ the following condition is satisfied: if for each $i, j \in A$ we have $W_i \cap W_j = \{0\}$ then all the spaces $\{W_i\}_{i \in A}$ are independent.

EXAMPLE 2.3. $W = (W_1, W_2, W_3)$ in Example 2.2 does not satisfy the condition of transversality.

EXAMPLE 2.4. Let $m = n = 3$ and $W_1 = \operatorname{span}\{e_1, e_3\}$, $W_2 = \operatorname{span}\{e_2, e_3\}$, $W_1 = \operatorname{span}\{e_1 + e_2, e_3\}$. Then there is no subset of $\{W_1, W_2, W_3\}$ which consists of pairwise independent subspaces. Since an implication with false predecessor is always true, the collection (W_1, W_2, W_3) does satisfy the condition of transversality.

As usual, $C(\Omega)$ denotes the space of continuous functions on Ω, $C_0(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity, while $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω. If $f \in C(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, then

(f,μ) will stand for $\int_\Omega f(\lambda)\,\mu(d\lambda)$. By $\fint_A f\,dx$ we mean $|A|^{-1}\int_A f\,dx$. We denote by $\rightarrow$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ the strong, weak and weak $*$ convergence respectively.

For $W \in \widetilde{\mathcal{W}}(n,m)$, $p \in [1,\infty]$ we set

$$\mathcal{K}(\Omega, W, p) = \{u \in L^p(\Omega, \mathbb{R}^m) : \partial_w u_j = 0 \text{ for each } w \in W_j,\ j = 1,\ldots,m\},$$

equipped with the topology of weak sequential convergence in $L^p(\Omega,\mathbb{R}^m)$ and weak $*$ convergence if $p=\infty$. More generally, for P given by (1), we set $\mathcal{K}(\Omega,P,p) = \{u \in L^p(\Omega,\mathbb{R}^m) : P_j u = 0 \text{ for } j = 1,\ldots,N\}$.

We will need the following lemma (see e.g. [6, Theorem 13], [21], [22] for its classical variant related to the operator $Pu = (\operatorname{curl} u^1, \ldots, \operatorname{curl} u^m)$, $m \in \mathbb{N}$, $u^i \in \mathbb{R}^n$, and $u \in \mathcal{K}(\Omega,P,p)$).

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $1 \le p < \infty$. Then for every $u \in \mathcal{K}(\Omega,W,p)$ and every $\lambda > 0$ there exists a closed set $F_\lambda \subset \Omega$ and a mapping $u^\lambda \in \mathcal{K}(\Omega,W,\infty)$ such that*

(i) $\lambda^p|\Omega\setminus F_\lambda| \to 0$ *as* $\lambda\to\infty$,
(ii) $u = u^\lambda$ *for almost every* $x \in F_\lambda$,
(iii) $|u^\lambda(x)| \le \lambda$ *for almost every* $x\in\Omega$,
(iv) $\|u-u^\lambda\|_{L^p(\Omega,\mathbb{R}^m)} \to 0$ *as* $\lambda\to\infty$.

Proof. Let $v_\lambda : \mathbb{R}\to\mathbb{R}$ be the Lipschitz function defined by $v_\lambda(y) = y$ if $|y|\le\lambda$ and $v_\lambda(y) = \lambda y/|y|$ if $|y| > \lambda$. An easy computation shows that the function $u^\lambda(x) = (v_\lambda(u^1(x)),\ldots,v_\lambda(u^m(x)))$ satisfies the assertions of the lemma with $F_\lambda = \{x : |u(x)| \le \lambda\}$. ∎

We recall the fundamental theorem of Young (see [2]).

THEOREM 2.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded measurable set. Assume that $u^j : \Omega\to\mathbb{R}^m$, $j = 1,2,\ldots,$ is a sequence of measurable functions satisfying the tightness condition*

$$\sup_j |\{x\in\Omega : |u^j(x)| \ge k\}| \overset{k\to\infty}{\longrightarrow} 0.$$

Then there exists a subsequence $\{u^k\}$ and a family $\{\nu_x\}_{x\in\Omega}$ of probability measures $\nu_x \in \mathcal{M}(\mathbb{R}^m)$ such that

(i) *for every $f\in C_0(\mathbb{R}^m)$ the function $x\mapsto(f,\nu_x)$ is measurable*,

(ii) *if $K\subseteq\mathbb{R}^n$ is a closed set, and $u^j(x)\in K$ for every j and almost every x, then* $\operatorname{supp}\nu_x \subseteq K$ *for almost every x*,

(iii) *if $A\subseteq\Omega$ is measurable, $f:\Omega\times\mathbb{R}^m\to\mathbb{R}$ is a Carathéodory function and the sequence $\{f(x,u^k(x))\}$ is sequentially weakly relatively compact in $L^1(A)$, then $\{f(x,u^k(x))\}$ converges weakly in $L^1(A)$ to $\overline{f}$ given by*

$$\overline{f}(x) = \int_{\mathbb{R}^m} f(x,\lambda)\,\nu_x(d\lambda).$$

DEFINITION 2.4. We say that the sequence $\{u^j\}_{j\in\mathbb{N}}$ *generates the Young measure* $\{\nu_x\}_{x\in\Omega}$ if $\{\nu_x\}_{x\in\Omega}$ satisfies (i) of Theorem 2.1 and $f(u^j) \overset{*}{\rightharpoonup} \overline{f} = (f,\nu_x)$ in $L^\infty(\Omega)$ for every $f \in C_0(\mathbb{R}^m)$.

Applying the same techniques as in [20, Lemma 3.1], one can easily obtain the following.

LEMMA 2.2. *Suppose that* $p \in [1,\infty)$, $u^\nu \in \mathcal{K}(\Omega, W, p)$ *for each* $\nu \in \mathbb{N}$, $\{u^\nu\}_{\nu\in\mathbb{N}}$ *is weakly convergent in* $L^p(\Omega,\mathbb{R}^m)$ *and generates the Young measure* $\{\nu_x\}_{x\in\Omega}$. *Let* $u^{\nu,k} \in \mathcal{K}(\Omega, W, \infty)$, $k,\nu \in \mathbb{N}$, *be the function of Lemma* 2.1 *with* $u = u^\nu$ *and* $\lambda = k$. *Passing to a subsequence, we may assume that* $\{u^{\nu,k}\}_{\nu\in\mathbb{N}}$ *generates the Young measure* $\{\nu_x^k\}_{x\in\Omega}$, *for every* $k \in \mathbb{N}$. *Let* $f \in C(\mathbb{R}^m)$ *satisfy* $|f(\lambda)| \le C(1+|\lambda|^p)$. *Then for every* $\varepsilon > 0$ *there exists a set* $E \subseteq \Omega$ *such that* $|E| < \varepsilon$ *and* $(f,\nu_x^k) \to (f,\nu_x)$ *in* $L^1(\Omega \setminus E)$ *as* $k \to \infty$.

3. The main results. Consider the case when (1) has the simple form

$$P_k u = \frac{\partial}{\partial v_k} u_{j(k)} \quad \text{with } j(k) \in \{1,\dots,m\},\ k = 1,\dots,N,\ v_k \in \mathbb{R}^n. \tag{10}$$

The space of solutions of the system $Pu = 0$ is the space of functions $u = (u_1,\dots,u_m)$ such that u_i is constant along

$$W_i = \operatorname{span}\{v_k : j(k) = i\} \quad (i = 1,\dots,m). \tag{11}$$

Note that every u_i can be written in the form

$$u_i(x) = v_i(\pi_i(x)), \tag{12}$$

where $\pi_i : \mathbb{R}^n \to W_i^{\pm_i}$ is the orthogonal projection with respect to an arbitrary scalar product $\langle\langle\cdot,\cdot\rangle\rangle_i$. In particular, we can assume that $\langle\langle\cdot,\cdot\rangle\rangle_i = \langle\cdot,\cdot\rangle$ for all $i \in \{1,\dots,m\}$. We will also assume that $W_i \ne \mathbb{R}^n$ for each i.

We have the following characterization of the characteristic cone Λ and the manifold V, associated with the functional I (see Theorem 1.1).

THEOREM 3.1. *Consider the system* (10) *with* $W = (W_1,\dots,W_m)$ *and* W_i *defined by* (11). *Then the manifold* V *and the characteristic cone* Λ *associated with* (10) *are given by* $V = \bigcup_{I\in\mathcal{A}(W)} (W^\perp)^I \times \mathbb{R}^{\times I}$ *and* $\Lambda = \bigcup_{I\in\mathcal{B}(W)} \mathbb{R}^{\times I}$.

The proof of the above theorem is left to the reader. Note that the equation $\langle\xi, w\rangle\lambda_i = 0$ for each $w \in W_i$ is satisfied if either $\xi \perp W_i$, or $\lambda_i = 0$.

Consider the functional

$$I_f(u) = \int_\Omega f(u(x))\,dx, \tag{13}$$

where $u \in \mathcal{K}(\Omega, W, p)$, and $f : \mathbb{R}^m \to \mathbb{R}$ is continuous.

The following property is similar to quasiconvexity and A-quasiconvexity (see e.g. [7, p. 13], [4], [15]).

DEFINITION 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $f : \mathbb{R}^m \to \mathbb{R}$ be continuous, and $p \in [1,\infty]$. We say that f is *integrally convex* on $\mathcal{K}(\Omega, W, p)$ if for every $u \in \mathcal{K}(\Omega, W, p)$,

$$\fint_{\Omega} f(u(x))\,dx \ge f\Big(\fint_{\Omega} u(x)\,dx\Big). \tag{14}$$

We will prove the following theorem.

THEOREM 3.2. *Suppose that $W \in \mathcal{W}(n,m)$ satisfies the parallelness condition and $\Lambda = \bigcup_{I\in\mathcal{B}(W)} \mathbb{R}^{\times I}$. Then the following conditions for f and I_f are equivalent.*

(i) *I_f is lower semicontinuous on $\mathcal{K}(\Omega, W, \infty)$.*

(ii) *f is Λ-convex.*

(iii) *f is integrally convex in $\mathcal{K}(Q, W, \infty)$ for every parallelepiped $Q \subset \mathbb{R}^n$ whose sides are parallel to the basis $\{w_i\}_{i=1,\dots,n}$ with $w_i \in \bigcup_{I\in\mathcal{B}(W)} W^{I^*}$ for every i.*

(iv) *If $p \in [1,\infty]$, $\{\nu_x\}_{x\in\Omega}$ is an arbitrary Young measure generated by a sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ such that $u^\nu \in \mathcal{K}(\Omega, W, p)$ and u^ν is weakly convergent in $L^p(\Omega, \mathbb{R}^m)$, then for every $f \in C(\mathbb{R}^m)$ such that $|f(\lambda)| \le C(1+|\lambda|^p)$ if $p<\infty$ and for almost all $x \in \Omega$, we have*

$$(f, \nu_x) \ge f((\lambda, \nu_x)). \tag{15}$$

Proof. (i)⇒(ii) follows from Theorem 1.1.

The implication (ii)⇒(iii) is a consequence of the following lemma.

LEMMA 3.1. *Let $W \in \mathcal{W}(n,m)$, $\Omega \subset \mathbb{R}^n$, $E = +_{I\in\mathcal{B}(W)} W^{I^*}$, $Q \subset E$ be an arbitrary parallelepiped whose sides are parallel to the spaces W^{I^*} for $I \in \mathcal{B}(W)$, and let $\Lambda = \bigcup_{I\in\mathcal{B}(W)} \mathbb{R}^{\times I}$. Then for every $u \in \mathcal{K}(\Omega, W, 1)$ and every Λ-convex continuous function $f : \mathbb{R}^m \to \mathbb{R}$, we have*

$$\fint_{Q} f(u(x))\,dx \ge f\Big(\fint_{Q} u(x)\,dx\Big). \tag{16}$$

Proof. Let $w_1, \dots, w_k$ be a basis in E such that for each l we can find $I_l \in \mathcal{B}(W)$ with $w_l \in W^{I_l^*}$. Choose the parallelepiped $Q = \sum_{i=1}^k t_i w_i + y$ with $t_i \in (0,1)$ for $i = 1,\dots,k$ and $y \in E^\perp$. Since $w_1 \in W_i$ for all $i \in D^*(I_1)$, and u_i are constant along W_i, we see that the image of the mapping

$$\mathbb{R} \ni t_1 \mapsto \phi_1(t_1,\dots,t_k) = u\Big(\sum_{i=1}^k t_i w_i + y\Big) \in \mathbb{R}^m$$

is a subset of $B + \mathbb{R}^{\times I_1}$ with $B = u(\sum_{i=2}^{k} t_i w_i + y)$. By assumption f is convex in the direction of $\mathbb{R}^{\times I_1}$, hence

$$\int_0^1 f(\phi_1(t_1,\ldots,t_k))\,dt_1 \geq f\Big(\int_0^1 \phi_1(t_1,\ldots,t_k)\,dt_1\Big).$$

Proceeding in the same way with variables t_i for $i = 2,\ldots,k$, and vector-valued functions

$$\mathbb{R} \ni t_i \mapsto \phi_i(t_i,\ldots,t_k) = \int_0^1 \ldots \int_0^1 u\Big(\sum_{i=1}^{k} t_i w_i + y\Big)\,dt_1 \ldots dt_{i-1}$$

we see that

$$\begin{aligned}\fint_Q f(u(x))\,dx &= \int_0^1 \ldots \int_0^1 f\Big(u\Big(\sum_{i=1}^{k} t_i w_i + y\Big)\Big)\,dt_1 \ldots dt_k \\ &\geq f\Big(\int_0^1 \ldots \int_0^1 u\Big(\sum_{i=1}^{k} t_i w_i + y\Big)\,dt_1 \ldots dt_k\Big) = f\Big(\fint_Q u(x)\,dx\Big). \quad \blacksquare\end{aligned}$$

(iii)⇒(iv). Assume that $\{u^\nu\}_{\nu\in\mathbb{N}}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$. First we consider the case $u^\nu \in \mathcal{K}(\Omega, W, \infty)$ and $u^\nu \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega,\mathbb{R}^m)$, and then the general case.

Case 1. Take a parellelepiped $Q \subseteq \Omega$ as in (iii). By assumption we have $\fint_Q f(u^\nu(x))\,dx \geq f(\fint_Q u^\nu(x)\,dx)$ for each $\nu \in \mathbb{N}$. Letting $\nu \to \infty$ and using Theorem 2.1 we obtain $\fint_Q \int_{\mathbb{R}^m} f(\lambda)\,\nu_x(d\lambda) \geq f(\fint_Q \int_{\mathbb{R}^m} \lambda \nu_x(d\lambda))$. Now (15) follows from Lebesgue's Differentiation Theorem.

Case 2. We will modify the sequence slightly, proceeding in a similar way to the proof of Theorem 1.2 in [20]. Let $k \in \mathbb{N}$ and $\{u^{\nu,k}\}_{\nu\in\mathbb{N}}$ be the sequence defined in Lemma 2.1 with $\lambda = k$ and $u = u^\nu$. Using the diagonal procedure and passing to a subsequence we can assume that each sequence $\{u^{\nu,k}\}_{\nu\in\mathbb{N}}$ generates the Young measure $\{\nu_x^k\}_{x\in\Omega}$. Since by Case 1, for each x in a set $\Omega(k)$ of full measure we have

$$(f,\nu_x^k) \geq f((\lambda,\nu_x^k)), \tag{17}$$

it follows that (17) is satisfied on the set $\Omega_0 = \bigcap_k \Omega(k)$, also of full measure. Now it suffices to apply Lemma 2.2.

(iv)⇒(i). This part is standard (see e.g. [20, proof of Theorem 1.1]). Let $u^\nu \in \mathcal{K}(\Omega, W, \infty)$, $u^\nu \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega,\mathbb{R}^m)$, and $\alpha = \liminf_{\nu\to\infty} I_f(u^\nu)$. According to Theorem 2.1 we find a subsequence $\{u^l\}$ with the properties: 1) $I_f(u^l) \to \alpha$ as $l \to \infty$, 2) the sequence $\{u^l\}_{l\in\mathbb{N}}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$, 3) for almost every $x\in\Omega$ we have $(\lambda,\nu_x)=u(x)$. Then

$$\alpha=\lim_{l\to\infty}\int_{\Omega} f(u^l)\,dx=\int_{\Omega}\int_{\mathbb{R}^m} f(\lambda)\nu_x(d\lambda)\,dx$$
$$\geq\int_{\Omega} f\Big(\int_{\mathbb{R}^m}\lambda\nu_x(d\lambda)\Big)dx=\int_{\Omega} f(u(x))\,dx. \ \blacksquare$$

We will prove the following theorem, which solves Problem 1.

THEOREM 3.3. *Let $m,n\in\mathbb{N}$. Assume that the manifold V and the characteristic cone Λ are associated with the system* (10) *for $W=(W_1,\ldots,W_m)$* (*see Theorem* 3.1) *such that $W_i\neq\mathbb{R}^n$ for every i and $W_1^\perp+\ldots+W_m^\perp=\mathbb{R}^n$. The following conditions are equivalent*:

(i) *$W^\perp$ satisfies the condition of transversality.*

(ii) *A continuous function $f:\mathbb{R}^m\to\mathbb{R}$ is Λ-affine if and only if f has the following property*:

(18) *For all $r\in\mathbb{N}$ and $(\xi_1,\lambda_1),\ldots,(\xi_r,\lambda_r)\in V$ such that* $\operatorname{rank}\{\xi_1,\ldots,\xi_r\}\leq r-1$, *and all $s\in\mathbb{R}^m$ we have*

$$f^{(r)}(s)(\lambda_1,\ldots,\lambda_r)=0.$$

(iii) *A continuous function $f:\mathbb{R}^m\to\mathbb{R}$ is Λ-affine if and only if I_f is weakly continuous in $\mathcal{K}(\Omega,W,\infty)$.*

(iv) *A continuous function $f:\mathbb{R}^m\to\mathbb{R}$ is Λ-affine if and only if for every Young measure $\{\nu_x\}_{x\in\Omega}$ generated by a sequence from $\mathcal{K}(\Omega,W,\infty)$ and for almost every $x\in\Omega$, we have $(f,\nu_x)=f((\lambda,\nu_x))$.*

We start by recalling the following result due to Murat and Tartar (see e.g. [32], [7, p. 27]).

LEMMA 3.2. *Assume that $\Omega\subset\mathbb{R}^n$, P, V, Λ, f and I_f are given by* (1), *Theorem* 1.1 *and* (2). *If I_f is weakly $*$ continuous in $\mathcal{K}(\Omega,P,\infty)$ then f satisfies* (18).

We also state the following lemma. Its proof is left to the reader.

LEMMA 3.3. *Assume that $W\in\mathcal{W}(n,m)$ and $\Lambda=\bigcup_{I\in\mathcal{B}(W)}\mathbb{R}^{\times I}$ spans all of $\mathbb{R}^m$. The space of all Λ-affine functions is spanned by all monomials $\{\lambda^\alpha\}_{\alpha\in\{0,1\}^m}$ such that for each $I\in\mathcal{B}(W)$ the set $D(I)\cap D(\alpha)$ has at most one element.*

Proof of Theorem 3.3. (iii)⇒(ii). Let f be a Λ-affine function. Then I_f is weakly continuous in $\mathcal{K}(\Omega,W,\infty)$. By Lemma 3.2, f satisfies (18). The reverse implication in (ii) follows from (18) by taking $r=2$ and $\xi_1=\xi_2$, $\lambda_1=\lambda_2$.

(ii)⇒(i). According to Lemma 3.3 define

(19) $$\mathcal{C}=\{\alpha\in\{0,1\}^m:\#(D(I)\cap D(\alpha))\leq 1 \text{ for each } I\in\mathcal{B}(W)\}.$$

Take $r \in \mathbb{N}$ and $W_{i_1}, \dots W_{i_r}$ such that $\{W_j^\perp\}_{j\in\{i_1,\dots,i_r\}}$ are pairwise independent. We will see that $W_{i_1}^\perp, \dots, W_{i_r}^\perp$ must be independent. Take $\alpha \in \{0,1\}^m$ such that $D(\alpha) = \{i_1, \dots, i_r\}$, so that $\alpha \in \mathcal{C}$ and $f(\alpha) = \lambda_{i_1} \dots \lambda_{i_r}$ is Λ-affine, hence it satisfies (18). If there are $\xi_1 \in W_{i_1}^\perp, \dots, \xi_r \in W_{i_r}^\perp$, $\xi_i \neq 0$ such that $\operatorname{rank}\{\xi_1, \dots, \xi_r\} \le r-1$, by Theorem 3.1 we see that $(\xi_k, \delta_{i_k}) \in V$ for $k = 1, \dots, r$, and $f^{(r)}(p)(\delta_{i_1}, \dots, \delta_{i_r}) = 0$ by Lemma 3.2. That leads to a contradiction since $f^{(r)}(p)(\delta_{i_1}, \dots, \delta_{i_r}) = 1$. Therefore $\operatorname{rank}\{\xi_1, \dots, \xi_r\} = r$.

(i)⇒(iii). The implication ⇐ in (iii) is always satisfied (Theorem 1.1). To see that ⇒ in (iii) is also true it suffices to consider all monomials $f(\lambda) = \lambda^\alpha$ where $\alpha \in \{0,1\}^m$ is as in Lemma 3.3 (note that if only $W_i \neq \mathbb{R}^n$ for each i then Λ spans all of $\mathbb{R}^m$).

Take $\alpha \in \mathcal{C}$ with $\mathcal{C}$ given by (19), and $D(\alpha) = \{i_1, \dots, i_r\}$. Since $W_{i_1}^\perp, \dots$ $\dots, W_{i_r}^\perp$ are pairwise independent, by an easy calculation we see that I_f is weakly continuous.

(iii)⇐(iv). This is an immediate consequence of the Young Theorem and the Lebesgue Differentiation Theorem. ■

4. The parallelness condition. We start with the following characterization showing that the parallelness condition can be expressed without the use of the scalar product.

THEOREM 4.1. *Let $W \in \widetilde{W}(n,m)$. The following are equivalent.*

(i) *W satisfies the parallelness condition.*

(ii) *There is $k \in \mathbb{N}$ and k pairs $(C(i), D(i))$, $i = 1, \dots, k$, of complementary subsets of $\{1, \dots, m\}$ such that*

$$\dim \operatorname{span}\{W_i\}_{i\in D(j)} \le n-1 \quad \text{for each } j, \tag{20}$$

$$\dim \operatorname{span}\Big\{\bigcap_{i\in C(j)} W_i\Big\}_{j=1,\dots,k} = n. \tag{21}$$

(iii) *If $\langle\langle\cdot,\cdot\rangle\rangle$ is an arbitrary scalar product then the spaces $\{W^{I^*}\}_{I\in\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle)}$ span $\mathbb{R}^n$.*

Proof. (i)⇒(ii). Assume that $W \in \mathcal{W}(m,n)$ satisfies the condition of Definition 2.2 and $\mathcal{B}(W) = \{I_1, \dots, I_k\}$. Then (21) is always satisfied with $C(j) = D^*(I_j)$. Hence it suffices to prove that (20) is also satisfied with $D(j) = D(I_j)$.

If $\mathcal{B}(W) = \{(1, \dots, 1)\}$ then (20) holds. Since it is not possible that $\mathcal{B}(W) = \{0\}$, we can assume that $k > 1$ and each $I \in \mathcal{B}(W)$ has some 0 and 1.

By definition if $I \in \mathcal{B}(W)$ then $\bigcap_{i\in D(I)} W_i^\perp \neq \{0\}$. Hence, we can find some $w \in \mathbb{R}^n$, $w \neq 0$, such that $w \in W_i^\perp$ for each $i \in D(I)$. In particular, for each $i \in D(I)$, we have $W_i \subseteq \{\operatorname{span} w\}^\perp$, and (20) is also satisfied.

(ii)⇒(i). Assume that $W \in \mathcal{W}(n,m)$ satisfies (20) and (21). Let $j \in \{1,\dots,k\}$ and define $I_j \in \{0,1\}^m$ in such a way that $D(I_j) = D(j)$, $D^*(I_j) = C(j)$. We show that $\mathcal{A}(W) \supseteq \{I_1,\dots,I_k\} = \mathcal{E}$.

Take $I \in \mathcal{E}$. Since $\dim \operatorname{span}\{W_i\}_{i\in D(I)} \le n-1$, we find $w \in \mathbb{R}^n$, $w \neq 0$, such that $W_i \subseteq \{\operatorname{span} w\}^\perp$ for every $i \in D(I)$. This implies that $\bigcap_{i\in D(I)} W_i^\perp \ni w \neq 0$. Hence $I \in \mathcal{A}(W)$ and the parallelness condition is satisfied.

(ii)⇔(iii). We proceed in the same way as in the proof of (ii)⇔(i), but with the standard scalar product replaced with $\langle\langle\cdot,\cdot\rangle\rangle$. ∎

Remark 4.1. If $W^\pm$ is a collection of independent subspaces of $\mathbb{R}^n$ which span $\mathbb{R}^n$, then the parallelness condition is satisfied. Indeed, we can find a basis $\{w_1,\dots,w_n\}$ in $\mathbb{R}^n$ such that each $W_i^\pm$ is spanned by some vectors from this basis: $W_i^\pm = \operatorname{span}\{w_j\}_{j\in C(i)}$ where $C(i) \subseteq \{1,\dots,m\}$ are pairwise disjoint subsets and $\bigcup_i C(i) = \{1,\dots,m\}$. Since $\mathcal{A}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \{\delta_i : i = 1,\dots,m\}$, it follows that $(\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle))^* = \{(1,\dots,1)-\delta_i : i = 1,\dots,m\}$ and $W^{(1,\dots,1)-\delta_i} = (\operatorname{span}\{\{w_1,\dots,w_n\} \setminus \{w_j\}_{j\in C(i)}\})^\pm$. Now it suffices to apply the following.

Lemma 4.1. *Assume that $\{w_j\}_{j=1,\dots,n}$ is a basis in $\mathbb{R}^n$, and $C(1),\dots$ $\dots,C(k) \subseteq \{1,\dots,n\}$ are disjoint subsets such that $\bigcup_{l=1}^k C(l) = \{1,\dots,n\}$. Then the spaces*

$$R_l = (\operatorname{span}\{\{w_1,\dots,w_n\} \setminus \{w_i\}_{i\in C(l)}\})^\pm, \qquad l = 1,\dots,k,$$

are linearly independent, of respective dimensions $\#C(l)$, and they span $\mathbb{R}^n$.

Proof. Since R_l is defined as the set of solutions of $n-\#C(l)$ independent equations

$$\langle\langle w_i, x\rangle\rangle = 0 \quad \text{for } i \in \{1,\dots,n\} \setminus C(l), \tag{22}$$

the dimension of R_l is $\#C(l)$. On the other hand $\sum_{l=1}^k \#C(l) = n$. Hence, it suffices to show that $R_1,\dots,R_k$ are linearly independent. Assume by contradiction that there are coefficients $\alpha_1,\dots,\alpha_k$, not all zero, and vectors $0 \neq v_l \in R_l$ such that $\sum_{l=1}^k \alpha_l v_l = 0$. With no loss of generality we can assume that $0 \neq v_k = \sum_{l=1}^{k-1} \alpha_l v_l$. Since $v_k \in R_k$ is nonzero, we can find w_s with some $s \in C(k)$ such that $\langle\langle w_s, v_k\rangle\rangle \neq 0$. On the other hand, $C(k) \subseteq \{1,\dots,n\} \setminus C(l)$ for each $l \in \{1,\dots,k-1\}$, and we see from (22) that $\langle\langle w_s, v_l\rangle\rangle = 0$ for $l = 1,\dots,k-1$. That leads to a contradiction, since $0 \neq \langle\langle w_s, v_k\rangle\rangle = \sum_{l=1}^{k-1} \alpha_l \langle\langle w_s, v_l\rangle\rangle = 0$. ∎

The situation when the system is of the form (10) and $W^\pm$ is a collection of independent subspaces of $\mathbb{R}^n$ has been investigated in [7, 28, 29, 32, 37].

Remark 4.1 can be generalized as follows.

Theorem 4.2. *If $\langle\langle\cdot,\cdot\rangle\rangle$ is a scalar product in $\mathbb{R}^n$, $W \in \widetilde{\mathcal{W}}(n,m)$ is such that $W^{\pm} \in \mathcal{W}(n,m)$ and $W^{\pm}$ is decomposable into a direct sum then W satisfies the parallelness condition.*

Proof. Assume that $\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle) = \{I_1,\ldots,I_k\}$ and find a basis $\{w_1,\ldots$ $\ldots,w_n\}$ in $\mathbb{R}^n$ such that $W^{\pm}$ is decomposable along $\{w_1,\ldots,w_n\}$. In particular $(W^{\pm})^{I_l} = \operatorname{span}\{w_i\}_{i\in C(l)}$, where $l = 1,\ldots,k$ and $C(l) \subseteq \{1,\ldots,n\}$ are disjoint subsets. Define $C_0 = \{1,\ldots,n\} \setminus \bigcup_{l=1}^k C(l)$. Note that if $r \in D^*(I_l)$ then $w_j \notin W_r^{\pm}$ for any $j \in C(l)$. In particular, for each $l \in \{1,\ldots,k\}$, $w_j \notin W_r^{\pm}$ for any $j \in C(l)$ and any $r \in D^*(I_l)$. This implies that $W_r \supseteq \{\operatorname{span}\{\{w_1,\ldots,w_n\} \setminus \{w_j\}_{j\in C(l)}\}\}^{\pm}$ for each $r \in D^*(I_l)$. Hence

$$W^{I_l^*} \supseteq \{\operatorname{span}\{\{w_1,\ldots,w_n\} \setminus \{w_j\}_{j\in C(l)}\}\}^{\pm} = R_l. \tag{23}$$

We consider two cases: $C_0 = \emptyset$ and $C_0 \neq \emptyset$. In the first case we apply Lemma 4.1. In the second case, take $i \in C_0$ and define $E(i) = \{r \in \{1,\ldots,m\} : w_i \in W_r^{\pm}\}$. By assumption $E(i) \neq \emptyset$ for each i; moreover, if we define $J_i \in \{0,1\}^m$ to satisfy $D(J_i) = E(i)$, then $(W^{\pm})^{J_i} \ni w_i \neq \{0\}$, and there is $I \in \mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle)$ such that $D(J_i) \subseteq D(I)$. Since $w_i \notin (W^{\pm})^I$ for any $I \in \mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle)$, we must have $J_i \neq I$.

If $I = (1,\ldots,1)$ $(k=1)$, then $W^{I^*} = \mathbb{R}^n$ and the assertion is satisfied. Hence we may assume that I has some zeros. This means that $D^*(I) \neq \emptyset$ and for each $r \in D^*(I)$,

$$w_i \notin W_r^{\pm}. \tag{24}$$

Define the function $l : C_0 \to \{1,\ldots,k\}$ by $l(i) = l$ if $J_i < I_l$ $(I_l \in \mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle))$. Note that $l(i)$ may not be uniquely defined. According to (24), we obtain $w_i \notin W_r^{\pm}$ for each $r \in D(I_{l(i)})$, and by the same arguments as for (23),

$$W^{I^*_{l(i)}} \supseteq \{\operatorname{span}\{\{w_1,\ldots,w_n\} \setminus \{w_i\}\}\}^{\pm} = S_i. \tag{25}$$

Taking into account (23) and (25) we see that $\operatorname{span}\{W^{I^*}\}_{I\in\mathcal{B}(W,\langle\langle\cdot,\cdot\rangle\rangle)}$ contains the spaces $\{R_i\}_{i=1,\ldots,l}$ and $\{S_i\}_{i\in C_0}$. Now the assertion follows from Lemma 4.1. ∎

Remark 4.2. One may ask if it is possible that if $W \in \mathcal{W}(n,m)$ satisfies the parallelness condition then there is a scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ in $\mathbb{R}^n$ such that $W^{\pm}$ is decomposable into a direct sum. The answer is "no". Take $n=5$, $m=3$, and $W = (W_1,W_2,W_3)$ where $W_1 = \operatorname{span}\{e_1,e_5\}$, $W_2 = \operatorname{span}\{e_2,e_4\}$, $W_3 = \operatorname{span}\{e_3,e_4-e_5\}$. Then W satisfies the parallelness condition. Assume that there is a scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ such that $W^{\pm}$ is decomposable into a direct sum. According to Remark 2.1 we can find a basis $\{w_1,\ldots,w_5\}$ such that $W^{\pm}$ is decomposable along $\{w_1,\ldots,w_5\}$. Note that if $i \neq j$, where $i,j \in \{1,2,3\}$, then $W_i^{\pm} \cap W_j^{\pm}$ is one-dimensional (four independent linear equations must be satisfied). On the other hand $W_1^{\pm} \cap W_2^{\pm} \cap W_3^{\pm} = \{0\}$. Assuming that $W_1^{\pm} = \mathrm{span}\{w_1, w_2, w_3\}$, $W_2^{\pm} = \mathrm{span}\{w_3, w_4, w_5\}$, we see that $W_3^{\pm}$ is spanned by one w_i with $i \in \{1, 2\}$, one w_i with $i \in \{4, 5\}$, and one more w_i with $i \neq 3$. In all cases one of $W_1^{\pm} \cap W_3^{\pm}$, $W_2^{\pm} \cap W_3^{\pm}$ is two-dimensional.

5. Examples, questions and remarks

REMARK 5.1. It follows from the Chacon Biting Lemma (see e.g. [5]) and standard techniques of Young measures (see e.g. [20, the proof of Theorem 1.1]) that if $W \in \mathcal{W}(n, m)$ satisfies the parallelness condition, $p \in [1, \infty]$, f is Λ-convex and is nonnegative then I_f is lower semicontinuous on $\mathcal{K}(\Omega, W, p)$.

REMARK 5.2. It is proved in Theorem 3.2 that if W satisfies the parallelness condition, and Λ associated with W is given by Theorem 3.1, then the Λ-convexity of f is equivalent to integral convexity in $\mathcal{K}(Q, W, p)$ for some specific parallelepiped Q. The condition of integral convexity is similar to quasiconvexity, and to the more general condition of P-quasiconvexity (see e.g. [7, p. 13], [4], [15]) in the case when P has the constant rank property. The P-quasiconvexity condition reads: for every cube $Q \subset \mathbb{R}^n$ and $\phi \in C^\infty(\overline{Q}, \mathbb{R}^m) \cap \mathrm{Ker}\, P$, periodic with periodicity cell Q, we have $\fint_Q f(\phi(x))\, dx \geq f(\fint_Q \phi(x)\, dx)$. In the case when P has the constant rank property the cube Q can be taken arbitrary. This is not our case where the sides of Q are parallel to particular subspaces in $\mathbb{R}^n$. To the best of our knowledge such P-quasiconvexity conditions are missing in the literature.

REMARK 5.3. Let $\mathcal{F}$ and $\mathcal{G}$ be the subsets of $\widetilde{\mathcal{W}}(n, m)$ described in Problems 1 and 2 in the introduction. Obviously, we have $\mathcal{G} \subseteq \mathcal{F}$. On the other hand it is easy to find $W \in \mathcal{F}$ which does not satisfy the parallelness condition, e.g. $m = n = 3$, $W_1 = \mathrm{span}\{e_2\}$, $W_2 = \mathrm{span}\{e_1\}$, $W_3 = \mathrm{span}\{e_1 - e_2, e_3\}$. We think it is possible that $\mathcal{G} = \mathcal{F}$. This conjecture is motivated by the following example showing that the class of spaces which satisfy the parallelness condition is essentially smaller than $\mathcal{G}$.

EXAMPLE 5.1. Let $m = n = 3$, $W = (W_1, W_2, W_3) \in \mathcal{F}$, $W_1 = \mathrm{span}\{e_2\}$, $W_2 = \mathrm{span}\{e_1\}$, $W_3 = \mathrm{span}\{e_1 - e_2, e_3\}$ so that $W_1^\perp = \mathrm{span}\{e_1, e_3\}$, $W_2^\perp = \mathrm{span}\{e_2, e_3\}$, $W_3^\perp = \mathrm{span}\{e_1 + e_2\}$, $\mathcal{B}(W) = \{(1, 1, 0), (0, 0, 1)\}$. Hence $u \in \mathrm{Ker}\, P$ if $u = (u_1, u_2, u_3)$ with $u_1 = u_1(x, z)$, $u_2 = u_2(y, z)$, $u_3 = u_3(x + y)$, and $\Lambda = (\mathbb{R} \times \mathbb{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbb{R})$. According to Lemma 3.1 we have $E = \mathrm{span}\{e_1 - e_2, e_3\}$. Integrating in directions of E and $v = e_1 + e_2$, using Lemma 3.1 and techniques similar to the proof of Theorem 3.2, one can prove that the functional $I_f(u) = \int_\Omega f(u(x))\, dx$ is sequentially lower semicontinuous on $\mathcal{K}(\Omega, W, \infty)$ if and only if f is Λ-convex.

REMARK 5.4. It is easy to check that if $f : \mathbb{R}^m \to \mathbb{R}$ is continuous and satisfies (18) with V given by Theorem 3.1, then I_f is weakly continuous.

REMARK 5.5. The assumption $W_i \neq \mathbb{R}^n$ for every i in (11) is purely technical. If $W_i = \mathbb{R}^n$ then u_i is constant and the weak $*$ convergence $u_i \overset{*}{\rightharpoonup} u$ is the convergence of constants.

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Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: kalamajs@mimuw.edu.pl

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