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WEYL SPACE FORMS AND THEIR SUBMANIFOLDS

BY

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Dedicated to Professor Takao Takahashi on his sixtieth birthday

Abstract. We study the geometric structure of a Gauduchon manifold of constant curvature. We give a necessary and sufficient condition for a Gauduchon manifold to be a Gauduchon manifold of constant curvature, and we classify the Gauduchon manifolds of constant curvature. Next, we investigate Weyl submanifolds of such manifolds.

1. Introduction. In this paper we assume that all manifolds are connected and smooth and have dimension $n \geq 3$. Let M be a manifold with a conformal structure $[g]$ and a torsion-free affine connection D. A manifold $(M, [g], D)$ is called a *Weyl manifold* if $Dg = \omega \otimes g$, for a 1-form ω. A Weyl manifold M is said to be *Einstein–Weyl* if the symmetric part of the Ricci curvature of D is proportional to the metric g at each point of M. The Einstein–Weyl equation is conformally invariant. It is known that there is a unique metric g, up to a constant, in the conformal structure of a compact Weyl manifold with respect to which the corresponding 1-form ω is co-closed. We call g the *Gauduchon metric*. Tod showed that if g is the Gauduchon metric of a compact Einstein–Weyl manifold, then $\omega^\sharp$ is a Killing vector field with respect to g (cf. [16]). A manifold (M, g, D) with a Gauduchon metric g is called a *Gauduchon manifold*. Many examples of and general results on Einstein–Weyl manifolds have been obtained (cf. [4], [6], [10]–[13], [16]). In particular, Itoh [6] investigated Einstein–Weyl geometry over compact manifolds.

It is known that M is a manifold of constant curvature if and only if M is a Weyl conformally flat Einstein manifold. Let $S(g) = \int_M s_g \, dV_g$ be the total scalar curvature of a compact Riemannian manifold (M^n, g), where s_g and

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dV_g denote the scalar curvature and the volume element of g, respectively. For a compact Riemannian manifold (M^n, g) of volume 1, (M^n, g) is Einstein if and only if g is a critical point of $S/\mathrm{vol}^{(n-2)/n}$.

Now, we consider a Gauduchon manifold (M^n, g, D) with $\dim M^n = n$. As shown in Section 2, (M^n, g, D) is a Gauduchon manifold of constant curvature if and only if it is a Weyl conformally flat Einstein–Weyl manifold and the scalar curvature of g is constant. This last condition is equivalent to the Gauduchon metric g being a critical point of S restricted to the set $\mathrm{Conf}_0(g)$ of metrics pointwise conformal to g and having the same total volume. Moreover, from a theorem of Katagiri [8] and the three-dimensional classification of Tod [16], we know that (M^n, g, D) $(n \geq 3)$ is a Gauduchon manifold of constant curvature if and only if it is a Weyl conformally flat Einstein–Weyl manifold. Next, we classify the Gauduchon manifolds of constant curvature k with Killing dual 1-form ω. If $k > 0$ and $n \geq 4$, then $\omega = 0$ and (M^n, g) is an elliptic space form. If $k = 0$ and $\omega \neq 0$, then $\nabla\omega = 0$, the first Betti number $b_1(M^n)$ is 1 and the universal covering manifold of (M^n, g) is isometric to the Riemannian product $(S^{n-1}, h) \times \mathbb{R}^1$. If $k < 0$, then $\omega = 0$ and (M^n, g) is a hyperbolic space form. From these, we obtain the following result: Let $(M^n, [g], D)$ be a compact Einstein–Weyl manifold which is Weyl conformally flat for every $g \in [g]$. If $n \geq 4$ and $\omega \neq 0$ for every $g \in [g]$, then $(M^n, [g], D)$ is a Weyl flat manifold.

On the other hand, Pedersen, Poon and Swann [12] studied Weyl submanifolds of Weyl manifolds and gave some examples.

The Hopf manifold H^n is a locally conformal Kaehler manifold of dimension $2n$ whose Lee form $\overline{\omega}$ $(\neq 0)$ is parallel and whose Weyl curvature is zero. The dual vector field $\overline{\omega}^\sharp$ of the Lee form $\overline{\omega}$ is Killing. In [2] and [3], Dragomir investigated submanifolds of H^n.

In Section 3, we study compact Weyl totally umbilical submanifolds (M^n, g, D) of a Gauduchon manifold $(\overline{M}^m, \overline{g}, \overline{D})$ of constant curvature k.

In Section 4, we study Weyl submanifolds (M^n, g, D) of a Gauduchon flat manifold $(\overline{M}^m, \overline{g}, \overline{D})$ with Killing dual 1-form $\overline{\omega}$. Let (M^n, g, D) be a compact Weyl totally umbilical submanifold of $(\overline{M}^m, \overline{g}, \overline{D})$ which is tangent to the vector field $\overline{\omega}^\sharp$ and $\omega \neq 0$. Then M^n is a totally geodesic submanifold with Einstein–Weyl structure, the first Betti number $b_1(M^n)$ is 1 and the universal covering manifold of (M^n, g) is isometric to the Riemannian product $(S^{n-1}, h) \times \mathbb{R}^1$. If (M^n, g, D) is a Weyl hypersurface of $(\overline{M}^{n+1}, \overline{g}, \overline{D})$ which is orthogonal to the vector field $\overline{\omega}^\sharp$, then M^n is a Weyl totally umbilical and totally geodesic submanifold, and (M^n, g) is an elliptic space form.

Example 1. (I) $S^1 \times S^{n+1}$ $(n \geq 1)$ and H^n admit Gauduchon flat structures.

(II) $S^1 \times S^n$ $(n \geq 2)$ is a totally geodesic submanifold of $S^1 \times S^{n+1}$ which is tangent to the dual vector field $\omega^\sharp$ of ω on $S^1 \times S^{n+1}$ ([12]).

(III) S^{n+1} is a totally geodesic and Weyl totally umbilical submanifold of $S^1 \times S^{n+1}$ which is orthogonal to the vector field $\omega^\sharp$ on $S^1 \times S^{n+1}$ ([12]).

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2. Gauduchon manifolds. Let $(M^n, [g], D)$ be a Weyl manifold with $Dg = \omega \otimes g$ of dimension $n \geq 3$. We define a vector field B by $g(X, B) = \omega(X)$. Then

$$D_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(Y)X + \tfrac{1}{2}g(X,Y)B, \tag{1}$$

where ∇ denotes the Levi-Civita connection of g.

The curvature tensor R of ∇ is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Let R^D be the curvature tensor of D. Then

$$\begin{aligned} R^D&(X,Y)Z \\ &= R(X,Y)Z - \tfrac{1}{2}\big\{\big[(\nabla_X\omega)Z + \tfrac{1}{2}\omega(X)\omega(Z)\big]Y \\ &\quad - \big[(\nabla_Y\omega)Z + \tfrac{1}{2}\omega(Y)\omega(Z)\big]X + ((\nabla_X\omega)Y)Z - ((\nabla_Y\omega)X)Z \\ &\quad - g(Y,Z)\big(\nabla_X B + \tfrac{1}{2}\omega(X)B\big) + g(X,Z)\big(\nabla_Y B + \tfrac{1}{2}\omega(Y)B\big)\big\} \\ &\quad - \tfrac{1}{4}|\omega|^2(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{2}$$

where X, Y and Z are any vector fields on M^n (cf. [10]).

For two 2-tensors h and g the 4-tensor $h \owedge g$ is given by

$$\begin{aligned} (h \owedge g)(V,Z,X,Y) &= h(V,X)g(Z,Y) + h(Z,Y)g(V,X) \\ &\quad - h(V,Y)g(Z,X) - h(Z,X)g(V,Y). \end{aligned} \tag{3}$$

For the Riemannian manifold (M^n, g), we have the decomposition into irreducible components

$$R = W + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{s_g}{n}g\right) \owedge g + \frac{s_g}{2n(n-1)} g \owedge g, \tag{4}$$

where Ric is the Ricci tensor, s_g is the scalar curvature of ∇ and W is the Weyl conformal curvature. There is no Weyl component W when $n = 3$.

The manifold (M^n, g) is called *Weyl conformally flat* if $W = 0$.

For the Weyl manifold $(M^n, [g], D)$, we have the decomposition into irreducible components

$$R^D = W + \left[\frac{1}{n-2}\mathrm{Ric}_0 + \frac{1}{2}S_0(\nabla\omega) + \frac{1}{4}\omega\otimes_0\omega\right] \owedge g + \left[\frac{1}{2n(n-1)}s_g - \frac{n-2}{8n}|\omega|^2 - \frac{1}{2n}d^*\omega\right] g \owedge g - \left(\frac{1}{2}(d\omega \owedge g) + g\otimes d\omega\right), \tag{5}$$

where $S(\nabla\omega)(X,Y) = \frac{1}{2}((\nabla_X\omega)Y + (\nabla_Y\omega)X)$, $2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$ and $\mathrm{Ric}_0, S_0, \otimes_0$ are trace-free parts (cf. [12]).

We set

$$E(V,Z,X,Y) = R(V,Z,X,Y) + \tfrac{1}{2}(S(\nabla\omega)\owedge g)(V,Z,X,Y) + \tfrac{1}{4}((\omega\otimes\omega)\owedge g)(V,Z,X,Y) - \tfrac{1}{8}|\omega|^2(g\owedge g)(V,Z,X,Y) \tag{6}$$

and

$$F(V,Z,X,Y) = -\big(\tfrac{1}{2}(d\omega\owedge g) + g\otimes d\omega\big)(V,Z,X,Y). \tag{7}$$

Then, from (2), we obtain

$$R^D(V,Z,X,Y) = E(V,Z,X,Y) + F(V,Z,X,Y), \tag{8}$$

where $R^D(V,Z,X,Y) = g(R^D(X,Y)Z,V)$.

If $R^D = 0$ we say that the Weyl (resp. Gauduchon) manifold is *Weyl* (resp. *Gauduchon*) *flat.*

By a simple calculation, we have

LEMMA 1. (i) $R^D(V,Z,X,Y) + R^D(V,Z,Y,X) = 0$.
(ii) $R^D(V,Z,X,Y) + R^D(Z,V,X,Y) = -2d\omega(X,Y)g(Z,V)$.
(iii) $R^D(V,Z,X,Y) + R^D(V,X,Y,Z) + R^D(V,Y,Z,X) = 0$.
(iv) $E(V,Z,X,Y) + E(V,Z,Y,X) = 0$.
(v) $E(V,Z,X,Y) + E(Z,V,X,Y) = 0$.
(vi) $E(V,Z,X,Y) + E(V,X,Y,Z) + E(V,Y,Z,X) = 0$.

For each plane p in the tangent space $T_x(M^n)$, we have

$$g(R^D(X_1,X_2)X_2,X_1) = g(E(X_1,X_2)X_2,X_1), \tag{9}$$

where X_1, X_2 is an orthonormal basis for p. We set

$$K_g^D(p) = g(R^D(X_1,X_2)X_2,X_1). \tag{10}$$

Then $K_g^D(p)$ is independent of the choice of the orthonormal basis (X_1,X_2) of p.

A Gauduchon manifold (M^n,g,D) is called a *Gauduchon manifold of constant curvature* k if there is a constant k such that $K_g^D(p) = k$ for all planes p in $T_x(M^n)$ and for all points $x\in M^n$.

If (M^n, g, D) is a Gauduchon manifold of constant curvature k, then $g(E(X,Y)Y,X) = k(g(Y,Y)g(X,X) - g(X,Y)g(X,Y))$ for all tangent vectors $X, Y \in T_x(M^n)$. Thus, from Lemma 1(iv)–(vi) we obtain (cf. [9], [5])

$$E(V,Z,X,Y) = k(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)). \tag{11}$$

Hence we have the following

Lemma 2. *Let (M^n, g, D) be a Gauduchon manifold of constant curvature k. Then*

$$R^D(V,Z,X,Y) = k(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)) \\ - \big(\tfrac{1}{2}(d\omega \owedge g) + g \otimes d\omega\big)(V,Z,X,Y).$$

Let $(M^n, [g], D)$ be a Weyl manifold and ${}^D\mathrm{Ric}$ the Ricci tensor of D. Using (2), we obtain

$$\begin{aligned} {}^D\mathrm{Ric}(X,Y) &= \mathrm{Ric}(X,Y) + \tfrac{1}{2}(n-1)(\nabla_X\omega)Y - \tfrac{1}{2}(\nabla_Y\omega)X \\ &\quad + \tfrac{1}{4}(n-2)\omega(X)\omega(Y) + \big(\tfrac{1}{2}\operatorname{div} B - \tfrac{1}{4}(n-2)|\omega|^2\big)g(X,Y). \end{aligned} \tag{12}$$

The Weyl manifold $(M^n, [g], D)$ is said to have an *Einstein–Weyl structure* if there exists a function $\widetilde{\Lambda}$ on M^n such that

$${}^D\mathrm{Ric}(X,Y) + {}^D\mathrm{Ric}(Y,X) = \widetilde{\Lambda} g(X,Y). \tag{13}$$

By using (12), we can rewrite (13) as

$$\mathrm{Ric}(X,Y) + \tfrac{1}{4}(n-2)\mathcal{D}\omega(X,Y) = \Lambda g(X,Y), \tag{14}$$

where

$$\mathcal{D}\omega(X,Y) = (\nabla_X\omega)Y + (\nabla_Y\omega)X + \omega(X)\omega(Y), \\ \Lambda = \tfrac{1}{2}\widetilde{\Lambda} - \tfrac{1}{2}\big(\operatorname{div} B - \tfrac{1}{2}(n-2)|\omega|^2\big)$$

(cf. [10], [13]).

We recall the following results.

Lemma 3 ([16]). *Let $(M^n, [g], D)$ be an Einstein–Weyl manifold. If the vector field B dual to ω is a Killing vector field, then the equation* (14) *reduces to*

$$\mathrm{Ric}(X,Y) + \tfrac{1}{4}(n-2)\omega(X)\omega(Y) = \Lambda g(X,Y). \tag{15}$$

Lemma 4 ([4], [6]). *Let (M^n, g, D) be a compact Einstein–Weyl manifold with Killing dual 1-form ω. Then*

$$c := s_g - \tfrac{1}{4}(n+2)|\omega|^2 \tag{16}$$

is a constant, called the Gauduchon constant.

Now, we give a necessary and sufficient condition for a Gauduchon manifold to be a Gauduchon manifold of constant curvature.

THEOREM 1. *Let (M^n, g, D) be a Gauduchon manifold of dimension $n \geq 3$. Then (M^n, g, D) is a Gauduchon manifold of constant curvature if and only if it satisfies the following conditions*:

(I) *(M^n, g, D) is a Weyl conformally flat Einstein–Weyl manifold*;
(II) *the scalar curvature s_g of the Gauduchon metric g is constant.*

REMARK 1. (II) of Theorem 1 is equivalent to the Gauduchon metric g being a critical point of the total scalar curvature S restricted to the set $\mathrm{Conf}_0(g)$ of metrics pointwise conformal to g and having the same total volume (cf. [1], p. 121).

Proof of Theorem 1. Let (M^n, g, D) be a Gauduchon manifold of constant curvature k. From Lemma 2, we obtain

$$ {}^D\mathrm{Ric}(X,Y) + {}^D\mathrm{Ric}(Y,X) = 2k(n-1)g(X,Y). \tag{17} $$

Thus (M^n, g, D) is an Einstein–Weyl manifold. It is easily seen that a Gauduchon manifold of constant curvature is Weyl conformally flat, that is, $W = 0$. Let c be a Gauduchon constant. Since $\Lambda = \frac{1}{2}\widetilde{\Lambda} + \frac{1}{4}(n-2)|\omega|^2$, using (15) and (17), we have

$$ \begin{aligned} \mathrm{Ric}(X,Y) = \big(k(n-1) &+ \tfrac{1}{4}(n-2)|\omega|^2\big)g(X,Y) \\ &- \tfrac{1}{4}(n-2)\omega(X)\omega(Y). \end{aligned} \tag{18} $$

Thus we get $s_g = kn(n-1) + \frac{1}{4}(n-1)(n-2)|\omega|^2$. From (16), if $n \neq 4$, we have

$$ s_g = \frac{n-1}{n-4}\bigg(-(n+2)k + \frac{n-2}{n}c\bigg) = \text{constant}. $$

Suppose $n = 4$. Since (M^n, g, D) is a Weyl conformally flat Einstein–Weyl manifold, by a result of Pedersen and Swann ([14], Cor. 3.2) we obtain $\omega = 0$ or $s^D = 0$, where s^D is the scalar curvature of D. From Lemma 2 we obtain $s^D = kn(n-1)$. If $k \neq 0$, then $\omega = 0$. Thus $R^D = R$ so that (M^n, g) is a space of constant curvature. If $k = 0$, from (18), $\mathrm{Ric}(\omega) = 0$. Since the dual of ω is Killing, we have $\nabla^*\nabla\omega = \mathrm{Ric}(\omega)$ (cf. [1], p. 41). We integrate over M^n the scalar product of $\nabla^*\nabla\omega$ with ω to obtain $\int_M |\nabla\omega|^2\,dV_g = 0$, where dV_g denotes the volume element with respect to g. This implies that ω is parallel. Since $\nabla_X|\omega|^2 = 2g(\nabla_X B, B) = 0$, $|\omega| = \text{constant}$. Hence $s_g = \frac{1}{4}(n-1)(n-2)|\omega|^2 = \text{constant}$.

Conversely, suppose (I) and (II). From (5), we have

$$ \begin{aligned} R^D(V,Z,X,Y) = {} & \alpha(g \owedge g)(V,Z,X,Y) \\ & - \big(\tfrac{1}{2}(d\omega \owedge g) + g \otimes d\omega\big)(V,Z,X,Y), \end{aligned} $$

where

$$ \alpha = \frac{1}{2n(n-1)}s_g - \frac{n-2}{8n}|\omega|^2. $$

Since the scalar curvature s_g is constant and $c = s_g - \frac{1}{4}(n+2)|\omega|^2 = \text{constant}$, $|\omega|$ is constant. Thus α is constant. Therefore (M^n, g, D) is a Gauduchon manifold of constant curvature 2α. ■

We set

$$C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(n-1)}[(\nabla_X s_g)Y - (\nabla_Y s_g)X],$$

where $g(QX,Y) = \mathrm{Ric}(X,Y)$. A Riemannian manifold (M^n, g) is called *conformally flat* if $W = 0$ for $n \geq 4$ and $C = 0$ for $n = 3$.

In [8], Katagiri proved that a compact conformally flat Einstein–Weyl manifold (M^n, g, D) $(n \geq 3)$ with Gauduchon metric g has constant scalar curvature s_g.

Let (M^3, g, D) be a Gauduchon manifold with Einstein–Weyl structure. In [16] Tod gave all local forms of Gauduchon metrics in dimension three together with their corresponding one-forms and a classification. In this classification, the corresponding one-forms ω have constant length. Thus, from Lemma 4, the scalar curvature s_g of the Gauduchon metric g is constant.

Hence we have the following result.

Corollary 1. *Let (M^n, g, D) be a Gauduchon manifold of dimension $n \geq 3$. Then (M^n, g, D) is a Gauduchon manifold of constant curvature if and only if (M^n, g, D) is a Weyl conformally flat Einstein–Weyl manifold.*

Next, we classify Gauduchon manifolds of constant curvature k.

Theorem 2. *Let (M^n, g, D) be a Gauduchon manifold of constant curvature k with Killing dual 1-form ω and $n \geq 3$.*

(I) *If $k > 0$ and $n \geq 4$, then $\omega = 0$ and so (M^n, g) is an elliptic space form.*

(II) *If $k = 0$ and moreover*
- (a) *$\omega = 0$, then g is a flat metric,*
- (b) *$\omega \neq 0$, then $\nabla\omega = 0$, the first Betti number $b_1(M^n)$ is 1 and the universal covering manifold of (M^n, g) is isometric to the Riemannian product $(S^{n-1}, h) \times \mathbb{R}^1$.*

(III) *If $k < 0$, then $\omega = 0$ and so (M^n, g) is a hyperbolic space form.*

Proof. When $k > 0$ and $n \geq 4$, we assume that $\omega \neq 0$. From (II) of Theorem 1, we have $s_g = \text{constant}$. Moreover from (18) for any tangent vector X orthogonal to B at a point x of M^n,

$$\begin{aligned} &\mathrm{Ric}(B,B) = k(n-1)|\omega|^2, \qquad \mathrm{Ric}(B,X) = 0, \\ &\mathrm{Ric}(X,X) = \big(k(n-1) + \tfrac{1}{4}(n-2)|\omega|^2\big)g(X,X). \end{aligned} \tag{19}$$

Thus the Ricci curvature is positive definite. Since $W = 0$ and $n \geq 4$, (M^n, g) is conformally flat. Thus, from a theorem of Tani [15], (M^n, g) is a space of constant curvature $\widetilde{k}$. Let $X_1 = B/|B|, X_2, \dots, X_n$ be an orthonormal basis of $T_x(M)$. Since (M^n, g) is a space of constant curvature $\widetilde{k}$, $\mathrm{Ric}(X_1, X_1) = \dots = \mathrm{Ric}(X_n, X_n) = \widetilde{k}(n-1)$. On the other hand, from (18) we have $\mathrm{Ric}(X_1, X_1) = k(n-1)$ and $\mathrm{Ric}(X_i, X_i) = k(n-1) + \frac{1}{4}(n-2)|\omega|^2$ for $i = 2, 3, \dots, n$. This is a contradiction. Therefore we get $\omega = 0$. Hence (M^n, g) is an elliptic space form.

Next we assume that $k = 0$. If $\omega = 0$, then g is a flat metric. If $\omega \neq 0$, from (18) for any tangent vector X orthogonal to B at a point x of M^n,

$$\begin{aligned} &\mathrm{Ric}(B, B) = 0, \qquad \mathrm{Ric}(B, X) = 0, \\ &\mathrm{Ric}(X, X) = \tfrac{1}{4}(n-2)|\omega|^2 g(X, X). \end{aligned} \tag{20}$$

Since the dual of ω is Killing, using $\nabla^*\nabla\omega = \mathrm{Ric}(\omega)$, we get $\int_M |\nabla\omega|^2\, dV_g = 0$. This implies that ω is parallel and hence harmonic. Since the Ricci curvature is nonnegative, using the Weitzenböck formula, we have $b_1(M^n) = 1$ (cf. [6], [14]).

Since $d\omega = 0$, the foliation $\mathcal{N}$ defined by $\omega = 0$ is integrable. Let N be a leaf of $\mathcal{N}$. Let $\widetilde{M}^n$ be the universal covering manifold of (M^n, g). Since $\nabla B = 0$, by the de Rham decomposition theorem, $\widetilde{M}^n$ with the lifted metric is isometric to the Riemannian product $\widetilde{N} \times \mathbb{R}^1$, where $\widetilde{N}$ is the universal covering manifold of N. Since M^n is a Gauduchon flat manifold and N is orthogonal to the vector field B, from Theorem 7 in Section 4, N is a totally geodesic submanifold with constant positive sectional curvature. Furthermore, since M^n is complete and N is totally geodesic, N is complete with respect to the induced metric. Thus $\widetilde{N}$ is isometric to the sphere S^{n-1}.

Finally, we assume that $k < 0$. Using $\nabla^*\nabla\omega = \mathrm{Ric}(\omega)$ and $\mathrm{Ric}(\omega) = k(n-1)\omega$, we have

$$\int_M |\nabla\omega|^2\, dV_g = k(n-1) \int_M |\omega|^2\, dV_g. \tag{21}$$

Thus we obtain $\omega = 0$. By using (6) and (11), we obtain $R(V, Z, X, Y) = k(g(Y, Z)g(X, V) - g(X, Z)g(Y, V))$. Thus (M^n, g) is a manifold of constant negative sectional curvature k, that is, (M^n, g) is a hyperbolic space form. ■

REMARK 2. (a) The sign of k is related to the sign of $c - \frac{1}{4}n(n-4)|\omega|^2$ in [6].

(b) In Theorem 2, we assume that $k > 0$ and $n = 3$ and $\omega \neq 0$. Let $X_1 = B/|B|, X_2, X_3$ be an orthonormal basis of $T_x(M)$ and p_{ij} be the plane spanned by X_i and X_j $(i \neq j)$. Using (18), we have $\mathrm{Ric}(X_1, X_1) = 2k$, $\mathrm{Ric}(X_2, X_2) = 2k + \frac{1}{4}|\omega|^2$ and $\mathrm{Ric}(X_3, X_3) = 2k + \frac{1}{4}|\omega|^2$. Thus we obtain $K(p_{12}) = k$, $K(p_{23}) = k + \frac{1}{4}|\omega|^2$ and $K(p_{13}) = k$, where $K(p_{ij})$ denotes the sectional curvature of g determined by the plane p_{ij}. Therefore (M, g) is not a space of constant curvature.

In the case where $k > 0$ and $n = 3$ and $\omega = 0$, (M, g) is a space of constant curvature.

THEOREM 3. *Let $(M^n, [g], D)$ be a compact Einstein–Weyl manifold which is Weyl conformally flat for every $g \in [g]$. If $n \geq 4$ and $\omega \neq 0$ for every $g \in [g]$, then $(M^n, [g], D)$ is a Weyl flat manifold.*

Proof. Let g be a Gauduchon metric. From Corollary 1, (M^n, g, D) is a Gauduchon manifold of constant curvature k. Since $n \geq 4$ and $\omega \neq 0$, by Theorem 2, we have $k = 0$ and $d\omega = 0$. Thus, from Lemma 2, we get $R^D = 0$. Since $R^D = 0$ is conformally invariant, $(M^n, [g], D)$ is a Weyl flat manifold. ∎

REMARK 3. From the following Example 2, we see that a compact Weyl conformally flat Einstein–Weyl manifold $(M^3, [g], D)$ of dimension three is not necessarily Weyl flat.

EXAMPLE 2. Let $\pi : S^{2n+1} \to P^n(\mathbb{C})$ be the Hopf fibration. Let $\widetilde{g}$ be the Fubini–Study metric on $P^n(\mathbb{C})$ and η a canonical connection whose curvature form is proportional to the Kaehler form of the Fubini–Study metric. For a real number a with $0 < a \leq 1$, we define a Riemannian metric g_a on S^{2n+1} by $g_a = \pi^*\widetilde{g} + a^2\eta \otimes \eta$. We set

$$f^2 = \frac{8(n+1)}{2n-1}a^2(1-a^2) \quad \text{and} \quad \omega_a = f\eta.$$

We define a connection D by

$$D_X Y = \nabla^a_X Y - \tfrac{1}{2}\omega_a(X)Y - \tfrac{1}{2}\omega_a(Y)X + \tfrac{1}{2}g_a(X,Y)B,$$

where ∇^a denotes the Levi-Civita connection of g_a and B is the dual vector field of ω_a. Then

$${}^D\mathrm{Ric}(X,Y) + {}^D\mathrm{Ric}(Y,X) = 4na^2 g_a(X,Y).$$

Therefore (S^{2n+1}, g_a, D) is an Einstein–Weyl manifold (cf. [11]). Moreover, the scalar curvature $s^D = 2n(2n+1)a^2 = \text{constant}$.

We assume $n = 1$. Then the Weyl conformal curvature W of (S^3, g_a) is flat. From (5), using $s^D = s_{g_a} - 2d^*\omega_a - \frac{1}{2}|\omega_a|^2$, we obtain

$$R^D = \tfrac{1}{12}s^D(g_a \owedge g_a) - \big(\tfrac{1}{2}(d\omega_a \owedge g_a) + g_a \otimes d\omega_a\big).$$

Thus, we have $K^D_{g_a}(p) = a^2$. If $0 < a < 1$, then ω_a is not closed (cf. [5]). The Weyl curvature R^D is

$$\begin{aligned} R^D(V,Z,X,Y) &= a^2(g_a(Y,Z)g_a(X,V) - g_a(X,Z)g_a(Y,V)) \\ &\quad - \big(\tfrac{1}{2}(d\omega_a \owedge g_a) + g_a \otimes d\omega_a\big)(V,Z,X,Y). \end{aligned}$$

3. Weyl submanifolds of Gauduchon manifolds. Let $(\overline{M}, [\overline{g}], \overline{D})$ be a Weyl manifold with $\overline{D}\overline{g} = \overline{\omega} \otimes \overline{g}$ and $i : M \to \overline{M}$ an immersed submanifold. We pull back the conformal structure from $\overline{M}$ to M. A torsion-free connection D on M is given by $D_X Y = \pi(\overline{D}_X Y)$, where π is the orthogonal projection from $i^*T\overline{M}$ to TM and X, Y are vector fields on M. Since $\overline{D}\overline{g} = \overline{\omega} \otimes \overline{g}$, we obtain $Dg = \omega \otimes g$, where $g = i^*\overline{g}$ and $\omega = i^*\overline{\omega}$. The second fundamental form β of the Weyl structure is defined by

$$\overline{D}_X Y = D_X Y + \beta(X, Y). \tag{22}$$

Let $\overline{B}$ be the vector field dual to $\overline{\omega}$. The vector field B dual to ω satisfies the decomposition

$$\overline{B} = B + B^{\perp}, \tag{23}$$

where $B^{\perp}$ is the normal component with respect to M. Let α be the second fundamental form of the isometric immersion $i : (M, g) \to (\overline{M}, \overline{g})$. From (1) and (22), we obtain $\beta = \alpha + \frac{1}{2} g \otimes B^{\perp}$. Let ξ be a normal vector field on M and X be a vector field on M. We have the Weingarten equation

$$\overline{D}_X \xi = -A^{\beta}_{\xi} X + D^N_X \xi, \tag{24}$$

where $-A^{\beta}_{\xi} X$ and $D^N_X \xi$ are the tangential and normal components of $\overline{D}_X \xi$, respectively. The mean curvature vector H^{α} of M is defined to be $H^{\alpha} = \frac{1}{n} \operatorname{tr} \alpha$, $n = \dim M$. Since $\beta = \alpha + \frac{1}{2} g \otimes B^{\perp}$, the corresponding mean curvature vectors are related by $H^{\beta} = H^{\alpha} + \frac{1}{2} B^{\perp}$.

A Weyl submanifold $(M, [g], D)$ is said to be *Weyl totally geodesic* if $\beta = 0$. For a normal section ξ on $(M, [g], D)$, if $A^{\beta}_{\xi} = \lambda I$ for some function λ, then ξ is called a *Weyl umbilical section* on $(M, [g], D)$. If the Weyl submanifold $(M, [g], D)$ is Weyl umbilical with respect to every local normal section of $(M, [g], D)$, then $(M, [g], D)$ is said to be *Weyl totally umbilical*. A Weyl submanifold $(M, [g], D)$ is said to be *Weyl minimal* if $H^{\beta} = 0$. These notions are conformally invariant.

PROPOSITION 1. *Let $(M^n, [g], D)$ be a Weyl minimal submanifold of a Weyl manifold $(\overline{M}^m, [\overline{g}], \overline{D})$. Assume that $\overline{B}$ is a Killing vector field with respect to $\overline{g}$. Then B is a Killing vector field with respect to g if and only if M^n satisfies one of the following two conditions*:

(a) *(M^n, g) is minimal*;

(b) *M^n is tangent to the vector field $\overline{B}$.*

Proof. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of $\overline{g}$ and g respectively. Since $\overline{B}$ is a Killing vector field, for any tangent vector fields X and Y on M^n, we have

$$\begin{aligned} 0 &= (\overline{\nabla}_X \overline{\omega})Y + (\overline{\nabla}_Y \overline{\omega})X \\ &= (\nabla_X \omega)Y + (\nabla_Y \omega)X - 2\overline{\omega}(\alpha(X, Y)). \end{aligned} \tag{25}$$

We assume that B is a Killing vector field. From (25), this is equivalent to $\alpha(X,Y) \perp B^{\perp}$, and, consequently, H^{α} is orthogonal to $B^{\perp}$. On the other hand, since $H^{\beta}=0$, $H^{\alpha}=-\frac{1}{2}B^{\perp}$. Hence we obtain $H^{\alpha}=0$ and $B^{\perp}=0$.

As $H^{\beta}=0$ we have $H^{\alpha}=-\frac{1}{2}B^{\perp}$ and conditions (a) and (b) are equivalent. In particular, they each imply $B^{\perp}=0$ and we have $\overline{\omega}(\alpha(X,Y)) = \overline{g}(\alpha(X,Y),\overline{B})=0$. From (25), B is a Killing vector field. ■

Let $(M^n,[g],D)$ be a Weyl submanifold of a Weyl manifold $(\overline{M}^m,[\overline{g}],\overline{D})$, and $n\geq 3$. Let $R^{\overline{D}}$ and R^D be the curvature tensors of $\overline{D}$ and D respectively. Then we have the equation of Gauss

$$(26)\qquad R^{\overline{D}}(V,Z,X,Y) = R^{D}(V,Z,X,Y) + \overline{g}(\beta(X,Z),\beta(Y,V)) - \overline{g}(\beta(Y,Z),\beta(X,V)).$$

Next, we consider Weyl totally umbilical submanifolds of a Gauduchon manifold of constant curvature.

THEOREM 4. *Let $(\overline{M}^m,\overline{g},\overline{D})$ be a Gauduchon manifold of constant curvature k and (M^n,g,D) be a Weyl totally umbilical submanifold of $(\overline{M}^m,\overline{g},\overline{D})$. Then (M^n,g,D) is an Einstein–Weyl manifold.*

Proof. Let $\{\xi_1,\dots,\xi_{m-n}\}$ be an orthonormal basis in $T_xM^{\perp}$. Since (M^n,g,D) is a Weyl totally umbilical submanifold, $A^{\beta}_{\xi_i}=\lambda_i I$. Moreover, $\beta(X,Y)=g(X,Y)\sum\lambda_i\xi_i = g(X,Y)H^{\beta}$. By using Lemma 2 and (26), we have

$$(27)\qquad {}^{D}\mathrm{Ric}(X,Y) = (n-1)(k+|H^{\beta}|^2)g(X,Y) + \tfrac{1}{2}nd\omega(X,Y).$$

Thus

$$(28)\qquad {}^{D}\mathrm{Ric}(X,Y) + {}^{D}\mathrm{Ric}(Y,X) = 2(n-1)(k+|H^{\beta}|^2)g(X,Y).$$

That is, M^n is an Einstein–Weyl manifold. ■

4. Weyl submanifolds of a Gauduchon flat manifold.

We now consider compact Einstein–Weyl hypersurfaces in a Gauduchon flat manifold.

THEOREM 5. *Let $(\overline{M}^{n+1},\overline{g},\overline{D})$ be a Gauduchon flat manifold with Killing dual 1-form $\overline{\omega}$ and (M^n,g,D) be a compact Einstein–Weyl (that is, $\mathrm{Ric} = \Lambda g - \frac{1}{4}(n-2)\mathcal{D}\omega$) hypersurface in $(\overline{M}^{n+1},\overline{g},\overline{D})$ which is tangent to the vector field $\overline{B}$.*

(I) *If $\omega=0$, then (M^n,g) is an Einstein manifold.*

(II) *If $\omega\neq 0$, then $\Lambda=\frac{1}{4}(n-2)|\omega|^2$ and Λ is constant. Moreover (M^n,g) is totally geodesic or it has two principal curvatures, one nonzero of multiplicity 1, and the other zero. In addition, the first Betti number $b_1(M^n)$ is 1 and the universal covering manifold of (M^n,g) is isometric to the Riemannian product $(S^{n-1},h)\times\mathbb{R}^1$.*

Proof. Since $(\overline{M}^{n+1}, \overline{g}, \overline{D})$ is a Gauduchon flat manifold, from (ii) of Lemma 1, $\overline{\omega}$ is closed. Thus the induced 1-form ω is also closed. Since M^n is tangent to the vector field $\overline{B}$, using (25), we see that the vector field B is a Killing vector field. Hence the 1-form ω is parallel. Since $\nabla_X|\omega|^2 = \nabla_X(g(B,B)) = 2g(\nabla_X B, B) = 0$, $|\omega| =$ constant. Since $\mathrm{Ric} = \Lambda g - \frac{1}{4}(n-2)\omega\otimes\omega$, we get $\Lambda = \frac{1}{n}\big(s_g + \frac{1}{4}(n-2)|\omega|^2\big)$. By using the Bianchi's second identity and $\nabla\omega = 0$, we obtain $(n-2)\nabla s_g = 0$. Thus s_g is constant and so Λ is constant. Since the dual of ω is Killing, we have $\nabla^*\nabla\omega = \mathrm{Ric}(\omega)$. Using $\mathrm{Ric} = \Lambda g - \frac{1}{4}(n-2)\omega\otimes\omega$, we have

$$\nabla^*\nabla\omega = \big(\Lambda - \tfrac{1}{4}(n-2)|\omega|^2\big)\omega. \tag{29}$$

From this equation, we have

$$|\nabla\omega|^2 = \big(\Lambda - \tfrac{1}{4}(n-2)|\omega|^2\big)|\omega|^2. \tag{30}$$

Since ω is parallel, we obtain $\Lambda = \frac{1}{4}(n-2)|\omega|^2$ or $\omega = 0$.

If $\omega = 0$, then (M^n, g) is an Einstein manifold.

If $\omega \neq 0$, we have $\Lambda = \frac{1}{4}(n-2)|\omega|^2$. Since M^n is tangent to the vector field $\overline{B}$, using (12) and (26), we have

$$\begin{aligned}\mathrm{Ric}(X,Y) = {} & \mathrm{tr}(A)g(AX,Y) - g(A^2X,Y)\\ & - \tfrac{1}{4}(n-2)\omega(X)\omega(Y) + \tfrac{1}{4}(n-2)|\omega|^2 g(X,Y).\end{aligned} \tag{31}$$

We take an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(M^n)$ such that each e_i is an eigenvector of A, that is, $Ae_i = \lambda_i e_i$ for $i = 1, \dots, n$. From $\mathrm{Ric} = \Lambda g - \frac{1}{4}(n-2)\omega\otimes\omega$ and (31), for each i, we have

$$\lambda_i^2 - \mathrm{tr}(A)\lambda_i + \Lambda - \tfrac{1}{4}(n-2)|\omega|^2 = 0. \tag{32}$$

Since $\Lambda = \frac{1}{4}(n-2)|\omega|^2$, each λ_i satisfies

$$t^2 - \mathrm{tr}(A)t = 0. \tag{33}$$

Thus, at any point of M^n there are at most two distinct principal curvatures, say λ, μ. From (33), we obtain $\lambda\mu = 0$. Hence either $\lambda = \mu = 0$, that is, (M^n, g) is totally geodesic, or $\lambda \neq 0$, $\mu = 0$. In the last case $l\lambda = \mathrm{tr}(A) = \lambda + \mu = \lambda$. Thus $l = 1$.

In the case where (M^n, g) is totally geodesic, for any tangent vector X orthogonal to B at a point x,

$$\begin{gathered}\mathrm{Ric}(B,B) = 0, \qquad \mathrm{Ric}(B,X) = 0,\\ \mathrm{Ric}(X,X) = \tfrac{1}{4}(n-2)|\omega|^2 g(X,X).\end{gathered} \tag{34}$$

When $\lambda \neq 0, \mu = 0$, we take an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(M^n)$ such that each e_i is an eigenvector of A, that is, $Ae_1 = \lambda e_1$, $Ae_i = 0$ for $i = 2, \dots, n$. We set $B = \sum_{i=1}^n b_i e_i$. For any tangent vector X orthogonal to B at a point x,

$$(35)\qquad \begin{aligned}&\mathrm{Ric}(B,B)=\lambda^2 b_1^2-\lambda^2 b_1^2=0,\qquad \mathrm{Ric}(B,X)=0,\\ &\mathrm{Ric}(X,X)=\tfrac{1}{4}(n-2)|\omega|^2 g(X,X).\end{aligned}$$

Since the form ω is parallel and the Ricci curvature is nonnegative, using the Weitzenböck formula, we have $b_1(M^n)=1$.

Using the equation of Gauss, we obtain $R^D=0$. By the same method as in the proof of Theorem 2, we conclude that the universal covering manifold of (M^n,g) is isometric to the Riemannian product $(S^{n-1},h)\times\mathbb{R}^1$. ∎

Next, we consider compact Weyl totally umbilical submanifolds of a Gauduchon flat manifold.

THEOREM 6. *Let $(\overline{M}^m,\overline{g},\overline{D})$ be a Gauduchon flat manifold with Killing dual 1-form $\overline{\omega}$ and (M^n,g,D) be a compact Weyl totally umbilical submanifold of $(\overline{M}^m,\overline{g},\overline{D})$. Then (M^n,g,D) is an Einstein–Weyl manifold with scalar curvature $s_g\geq 0$. In particular, assume that M^n is tangent to the vector field $\overline{B}$.*

(I) *If $\omega=0$, then (M^n,g) is an Einstein manifold.*

(II) *If $\omega\neq 0$, then (M^n,g) is a totally geodesic submanifold. In addition, the first Betti number $b_1(M^n)$ is 1 and the universal covering manifold of (M^n,g) is isometric to the Riemannian product $(S^{n-1},h)\times\mathbb{R}^1$.*

Proof. From Theorem 4, (M^n,g,D) admits an Einstein–Weyl structure. By using (14) and (28), we obtain

$$(36)\qquad \begin{aligned}\mathrm{Ric}(X,Y)=&-\tfrac{1}{4}(n-2)((\nabla_X\omega)Y+(\nabla_Y\omega)X+\omega(X)\omega(Y))\\ &+\big\{(n-1)|H^\beta|^2-\tfrac{1}{2}\big(\mathrm{div}\,B-\tfrac{1}{2}(n-2)|\omega|^2\big)\big\}g(X,Y).\end{aligned}$$

Since (M^n,g,D) is a Weyl totally umbilical submanifold, we have $\beta(X,Y)=g(X,Y)\sum\lambda_i\xi_i=g(X,Y)H^\beta$. We set $\mu_i=\lambda_i-\frac{1}{2}\overline{g}(B^\perp,\xi_i)$. Since $\beta=\alpha+\frac{1}{2}g\otimes B^\perp$, we have $\alpha(X,Y)=g(X,Y)\sum\mu_i\xi_i=g(X,Y)H^\alpha$.

Since $\overline{B}$ is a Killing vector field, using (25) we have

$$(37)\qquad \begin{aligned}&(\nabla_X\omega)Y+(\nabla_Y\omega)X-2\overline{g}(H^\alpha,B^\perp)g(X,Y)=0,\\ &\mathrm{div}\,B-n\overline{g}(H^\alpha,B^\perp)=0.\end{aligned}$$

It follows from (36) that

$$(38)\qquad \begin{aligned}&\mathrm{Ric}(X,Y)\\ &\quad=-\tfrac{1}{4}(n-2)\omega(X)\omega(Y)\\ &\qquad+\big\{\tfrac{1}{4}(n-2)|\omega|^2+(n-1)|H^\alpha|^2+\tfrac{1}{4}(n-1)|B^\perp|^2\big\}g(X,Y).\end{aligned}$$

The scalar curvature s_g of (M^n,g) is

$$(39)\qquad s_g=\tfrac{1}{4}(n-1)\{(n-2)|\omega|^2+n|B^\perp|^2\}+n(n-1)|H^\alpha|^2\geq 0.$$

Assume, that M^n is tangent to the vector field $\overline{B}$. From (38), we obtain

$$\mathrm{Ric}(X,Y) = \Big(\sum \mu_i^2(n-1) + \tfrac{1}{4}(n-2)|\omega|^2\Big) g(X,Y) - \tfrac{1}{4}(n-2)\omega(X)\omega(Y). \tag{40}$$

We set $\Lambda = \sum \mu_i^2(n-1) + \frac{1}{4}(n-2)|\omega|^2$. Then $\mathrm{Ric} = \Lambda g - \frac{1}{4}(n-2)\omega \otimes \omega$. As in the proof of Theorem 5, we obtain $\Lambda = \frac{1}{4}(n-2)|\omega|^2$ or $\omega = 0$.

If $\omega = 0$, then (M^n, g) is an Einstein manifold.

If $\omega \neq 0$, then $\Lambda = \frac{1}{4}(n-2)|\omega|^2$ and $\mu_i = 0$ for $i = 1, \dots, m-n$, that is, (M^n, g) is totally geodesic. Since $B^\perp = 0$, (M^n, g, D) is Weyl totally geodesic. Thus $R^D = 0$. Moreover we obtain the equation (20). The proof of the remaining statement may be given as in the proof of Theorem 2. ∎

Finally, we consider Weyl hypersurfaces M orthogonal to the vector field $\overline{B}$.

THEOREM 7. *Let $(\overline{M}^{n+1}, \overline{g}, \overline{D})$ be a Gauduchon flat manifold with Killing dual 1-form $\overline{\omega} \neq 0$ and (M^n, g, D) be a Weyl hypersurface of $(\overline{M}^{n+1}, \overline{g}, \overline{D})$ which is orthogonal to the vector field $\overline{B}$. Then (M^n, g, D) is Weyl totally umbilical and (M^n, g) is a totally geodesic submanifold with constant positive sectional curvature, that is, (M^n, g) is an elliptic space form.*

Proof. Since $\overline{B}$ is normal to M^n, we have $\overline{B} = B^\perp$. Since $\overline{B}$ is parallel, we obtain $\overline{g}(\overline{\nabla}_X Y, B^\perp) = -\overline{g}(Y, \overline{\nabla}_X \overline{B}) = 0$ for any tangent vector fields X and Y on M^n. Thus (M^n, g) is totally geodesic. Then we have $\beta(X,Y) = \frac{1}{2} g(X,Y)\overline{B}$. For a unit normal vector field $\xi = \overline{B}/|\overline{B}|$, we have $g(A^\beta_\xi X, Y) = \overline{g}(\beta(X,Y), \xi) = \frac{1}{2}|\overline{\omega}| g(X,Y)$. This implies $A^\beta_\xi = \frac{1}{2}|\overline{\omega}| I$, that is, (M^n, g, D) is Weyl totally umbilical. By using (26), $R^D = R$ and $\beta(X,Y) = \frac{1}{2} g(X,Y)\overline{B}$, we obtain $R(V,Z,X,Y) = \frac{1}{4}|\overline{\omega}|^2 (g(Y,Z)g(X,V) - g(X,Z)g(Y,V))$. Since $\overline{B}$ is parallel, $|\overline{\omega}|$ is constant. ∎

NOTE. (a) We have recently learned of the following result from a letter of M. Itoh (Tsukuba Univ.): Let M be a compact, connected oriented Riemannian manifold with $\dim M = 2n+1$. Let $\widetilde{M}$ be the universal covering manifold of M. If $\widetilde{M}$ is isometric to $S^{2n} \times \mathbb{R}^1$, then M is an S^{2n}-bundle over S^1.

(b) We have recently learned of the existence of the paper [7] which slightly overlaps the first part of this paper and in particular gives a classification of conformally flat Einstein–Weyl manifolds.

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