COLLOQUIUM MATHEMATICUM

VOL. 89 2001 NO. 1

ON THE COMPLEXITY OF HAMEL BASES OF INFINITE-DIMENSIONAL BANACH SPACES

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Abstract. We call a subset S of a topological vector space V *linearly Borel* if for every finite number n, the set of all linear combinations of S of length n is a Borel subset of V. It is shown that a Hamel basis of an infinite-dimensional Banach space can never be linearly Borel. This answers a question of Anatoliĭ Plichko.

Throughout, let X be any infinite-dimensional Banach space. A subset S of X is called *linearly Borel* (with respect to X) if for every positive integer n, the set of all linear combinations with n vectors of S is a Borel subset of X. Since X is a complete metric space, X is a *Baire space*, i.e., a space in which non-empty open sets are not meager (cf. [1, Section 3.9]). Moreover, all Borel subsets of X have the *Baire property*, i.e., for each Borel set S, there is an open set $\mathcal{O}$ such that $\mathcal{O} \triangle S$ is meager, where $\mathcal{O} \triangle S = (\mathcal{O} \setminus S) \cup (S \setminus \mathcal{O})$.

This is already enough to prove the following.

THEOREM. *If X is an infinite-dimension Banach space and H is a Hamel basis of X, then H is not linearly Borel* (*with respect to X*).

Proof. Let X be any infinite-dimensional Banach space over the field $\mathbb{F}$ and let H be any Hamel basis of X. For a positive integer n, let $[H]^n$ be the set of all n-element subsets of H and let

$$H_n := \Big\{ \sum_{i=1}^{n} \alpha_i h_i : \alpha_1, \ldots, \alpha_n \in \mathbb{F} \setminus \{0\} \text{ and } \{h_1, \ldots, h_n\} \in [H]^n \Big\}.$$

Assume towards a contradiction that H is linearly Borel. Then, by definition, for each positive integer n, H_n is Borel, and hence, by the facts mentioned above, H_n has the Baire property. Since H is a Hamel basis of X, we get

$$B = \bigcup_{n=1}^{\infty} H_n ,$$

2000 *Mathematics Subject Classification*: Primary 46B20; Secondary 54E52.

Key words and phrases: Hamel bases, Banach spaces, Borel sets.

I would like to thank the *Swiss National Science Foundation* for their support during the period in which the research for this paper was done.

and because X is a Baire space, there must be a least positive integer m such that H_m is not meager. Because H_m has the Baire property and is not meager, there is a non-empty open set $\mathcal{O}$ such that $\mathcal{O} \triangle H_m$ is meager. Since H is a Hamel basis, $\mathcal{O} \setminus H_m$ cannot be empty, and therefore, $\mathcal{O} \setminus H_m$ is a non-empty meager set. Let $B_{v,r}$ denote the open ball with center $v \in X$ and radius r. Let $x \in H_m \cap \mathcal{O}$ and let ε be such that $B_{x,2\varepsilon} \subseteq \mathcal{O}$. Take any $y \in H_{3m+1}$ with $\|y\| < \varepsilon$. Then $B_{x+y,\varepsilon} \subseteq \mathcal{O}$ and the following map is a homeomorphism from $B_{x,2\varepsilon}$ into $B_{x+y,\varepsilon}$:

$$\Phi : B_{x,2\varepsilon} \to B_{x+y,\varepsilon}, \qquad z \mapsto x + y + \tfrac{1}{2}(z - x).$$

Since $\mathcal{O} \setminus H_m$ is meager, both sets, $B_{x,2\varepsilon} \setminus H_m$ as well as $B_{x+y,\varepsilon} \setminus H_m$, are meager, and further, by the definition of Φ, also $B_{x+y,\varepsilon} \setminus \Phi[H_m]$ is meager, where $\Phi[H_m] := \{\Phi(z) : z \in H_m \cap B_{x,2\varepsilon}\}$. Now, because we have chosen $y \in H_{3m+1}$, $\Phi[H_m] \cap H_m = \emptyset$, and hence,

$$B_{x+y,\varepsilon} = (B_{x+y,\varepsilon} \setminus H_m) \cup (B_{x+y,\varepsilon} \setminus \Phi[H_m]),$$

which implies that the open set $B_{x+y,\varepsilon}$, as the union of two meager sets, is meager. But this is a contradiction to the fact that X is a Baire space. ∎

REFERENCES

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Received 2 November 2000 (3991)