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ON ASSOCIATED AND ATTACHED PRIME IDEALS OF CERTAIN MODULES

BY

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Abstract. Primary and secondary functors have been introduced in [2] and applied to extend some results concerning asymptotic prime ideals. In this paper, the theory of primary and secondary functors is developed and examples of non-exact primary and non-exact secondary functors are presented. Also, as an application, the sets of associated and of attached prime ideals of certain modules are determined.

1. Introduction and preliminaries. The theory of asymptotic prime ideals is one of the most widely used theories in commutative algebra and it has many applications in local cohomology and algebraic geometry. The subject has started by a question raised by L. J. Ratliff. For an ideal I of a Noetherian ring R, he conjectured that the two sequences of sets $\mathrm{Ass}_R(R/I^n)$ and $\mathrm{Ass}_R(I^n/I^{n+1})$ are ultimately constant. M. Brodmann proved this conjecture. Since then a lot of research has been done in this field, especially by Ratliff, S. McAdam, D. Katz and P. Schenzel. For a survey of asymptotic results and their applications, we refer the reader to [6]. Motivation for the present work comes from [2], where far-reaching generalizations of some of the previously known asymptotic results are proved. This has been done by introducing two classes of linear functors, primary and secondary functors. We now recall their definition.

Let R be a commutative ring and let $\mathscr{C}_R$ denote the category of all R-modules and all R-homomorphisms.

1.1. Definition. Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and $T : \mathscr{C} \to \mathscr{C}_R$ be a linear functor. The functor T is called *primary* on $\mathscr{C}$ if, for each prime ideal $\mathfrak{p}$ of R with $R/\mathfrak{p} \in \mathscr{C}$, the following conditions are equivalent:

(i) $T(N) \neq 0$ for any $\mathfrak{p}$-coprimary R-module N in $\mathscr{C}$.
(ii) $T(N) \neq 0$ for some $\mathfrak{p}$-coprimary R-module N in $\mathscr{C}$.
(iii) $T(R/\mathfrak{p}) \neq 0$.

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The theory of secondary representation and attached prime ideals is dual to that of primary decomposition and associated primes. For a complete treatment of this theory, we refer the reader to [3, Appendix to Section 6].

1.2. Definition. Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$. A linear functor $T : \mathscr{C} \to \mathscr{C}_R$ is called *secondary* on $\mathscr{C}$ if, for each prime ideal $\mathfrak{p}$ of R, the following conditions are equivalent:

(i) $T(S) \neq 0$ for any $\mathfrak{p}$-secondary R-module S in $\mathscr{C}$.
(ii) $T(S) \neq 0$ for some $\mathfrak{p}$-secondary R-module S in $\mathscr{C}$.

Let $\mathscr{C}_{\mathcal{N}}$ and $\mathscr{C}_{\mathcal{A}}$ denote the subcategories of Noetherian and Artinian R-modules respectively. The following result determines a large class of primary and secondary functors.

1.3. Theorem. (i) ([2, Corollary 1.5]). *Let $T : \mathscr{C}_{\mathcal{N}} \to \mathscr{C}_R$ be an exact functor. Then T is primary on $\mathscr{C}_{\mathcal{N}}$.*

(ii) ([2, Lemma 2.7(i)]). *Let R be a complete semi-local Noetherian ring and let $T : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$ be an exact functor. Then T is secondary on $\mathscr{C}_{\mathcal{A}}$.*

In [2], we did not present any example of non-exact primary or of non-exact secondary functors. The aim of this article is to study the theory of primary and secondary functors systematically and to present such examples. As an application, we describe the sets of associated and of attached prime ideals of certain modules which are obtained by applying primary (resp. secondary) functors to some modules.

Throughout R will denote a commutative ring with identity. All functors considered are assumed to be linear.

2. Non-exact primary functors. The following result plays an essential role in this section.

2.1. Theorem (see [7, 4.3]). *If M and N are R-modules with N finitely generated, then $M \otimes_R N = 0$ if and only if $M = (\mathrm{Ann}_R N)M$.*

2.2. Theorem. *Let M be an R-module. Then the functor $M \otimes_R (\cdot)$ is primary on $\mathscr{C}_{\mathcal{N}}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal and N be a $\mathfrak{p}$-coprimary Noetherian R-module. Then $\sqrt{\mathrm{Ann}_R N} = \mathfrak{p}$. It is enough to show that $M \otimes_R N = 0$ if and only if $M = \mathfrak{p}M$.

Suppose that $M \otimes_R N = 0$ .Then by 2.1, $M = (\mathrm{Ann}_R N)M$, and so $M = \mathfrak{p}M$. Conversely, assume that $M = \mathfrak{p}M$. Let $x_1, \ldots, x_t$ be a set of generators for N. Define $f : R/\mathrm{Ann}_R N \to \bigoplus_{i=1}^t Rx_i$ by $f(r+\mathrm{Ann}_R N) = (rx_1, \ldots, rx_t)$ for all $r \in R$. Clearly, f is an injective R-homomorphism. Since $\bigoplus_{i=1}^t Rx_i$ is a Noetherian R-module, it follows that $R/\mathrm{Ann}_R N$ is Noetherian as an

R-module and so Noetherian as a ring. Hence there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n \subseteq \mathrm{Ann}_R N$. Now, we have

$$M = \mathfrak{p}M = \mathfrak{p}^n M \subseteq (\mathrm{Ann}_R N)M \subseteq M.$$

Thus $M = (\mathrm{Ann}_R N)M$ and so $M \otimes_R N = 0$ by 2.1, as required. ■

We shall use the following result.

2.3. THEOREM (see [7, Theorem 2.1]). *If M and N are R-modules such that N is finitely generated, then the module* $\mathrm{Hom}_R(N, M)$ *is zero if and only if* $(0 :_M \mathrm{Ann}_R N) = 0$.

As an immediate consequence of 2.3, we deduce the following.

2.4. COROLLARY. *Let R be a Noetherian ring and N be a finitely generated R-module. Then the functor* $\mathrm{Hom}_R(N, \cdot)$ *is primary on* $\mathscr{C}_R$.

Proof. For an R-module M, it follows easily from 2.3 that $\mathrm{Hom}_R(N, M) \neq 0$ if and only if $\mathrm{Ann}_R N \subset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R M$. This finishes the proof. ■

The following example shows that the "finitely generated" assumption on N is necessary in 2.4.

2.5. EXAMPLE. Let R be a Noetherian domain which is not a field and let K denote the quotient field of R. Then the functor $\mathrm{Hom}_R(K, \cdot)$ is not primary on $\mathscr{C}_R$. To see this, let $N_1 = R$ and $N_2 = K$. Then N_1 and N_2 are both 0-coprimary. It is easy to see that $\mathrm{Hom}_R(K, N_1) = 0$, while $\mathrm{Hom}_R(K, N_2) \neq 0$.

2.6. THEOREM. *Let M be an R-module. Then the functor* $\mathrm{Hom}_R(\cdot, M)$ *is primary on* $\mathscr{C}_{\mathcal{N}}$.

Proof. For a $\mathfrak{p}$-coprimary Noetherian R-module N, we show that $\mathrm{Hom}_R(N, M) = 0$ if and only if $(0 :_M \mathfrak{p}) = 0$. This will complete the proof. As we have seen in the proof of 2.2, there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n \subseteq \mathrm{Ann}_R N$. Hence

$$(0 :_M \mathfrak{p}) \subseteq (0 :_M \mathrm{Ann}_R N) \subseteq (0 :_M \mathfrak{p}^n).$$

But it is easy to see that $(0 :_M \mathfrak{p}) = 0$ if and only if $(0 :_M \mathfrak{p}^n) = 0$. Therefore $(0 :_M \mathfrak{p}) = 0$ if and only if $(0 :_M \mathrm{Ann}_R N) = 0$. Now, by 2.3, the conclusion follows. ■

The following corollary extends [2, Corollary 1.5] (see 1.3(i)).

2.7. COROLLARY. *Let $T : \mathscr{C}_{\mathcal{N}} \to \mathscr{C}_R$ be a functor. Suppose that T is either covariant right exact or contravariant left exact. Then T is primary on $\mathscr{C}_{\mathcal{N}}$.*

Proof. First, we prove the assertion for R Noetherian. In that case, if T is covariant right exact (resp. contravariant left exact), then there is a natural equivalence of functors $v : (\cdot) \otimes_R T(R) \to T$ (resp. $w : T \to \mathrm{Hom}_R(\cdot, T(R))$) from $\mathscr{C}_{\mathcal{N}}$ to $\mathscr{C}_R$ (see e.g. [2, Lemma 1.1]). Hence, by 2.2 and 2.6 the result follows.

Now, we prove the general case. To this end, let $\mathfrak{p}$ be a prime ideal of R and N_1 and N_2 be two Noetherian $\mathfrak{p}$-coprimary R-modules. The ring $R' = R/\mathrm{Ann}_R N_1 \cap \mathrm{Ann}_R N_2$ is Noetherian and both N_1 and N_2 are Noetherian R'-modules. Hence, by the first part of the proof, $T(N_1) = 0$ if and only if $T(N_2) = 0$. Note that T may be considered as a functor from the subcategory of Noetherian R'-modules to the category of all R'-modules. ■

We now set some notations. Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and $M \in \mathscr{C}$. Let X be a submodule of M and let $i : X \to M$ and $\pi : M \to M/X$ be the natural maps.

(i) If $T : \mathscr{C} \to \mathscr{C}_R$ is a covariant left exact functor, then we identify the module $T(i)T(X)$ with $T(X)$.

(ii) If $T : \mathscr{C} \to \mathscr{C}_R$ is a contravariant left exact functor, then we identify the module $T(\pi)T(M/X)$ with $T(M/X)$.

2.8. Lemma. *Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and $T : \mathscr{C} \to \mathscr{C}_R$ be a functor. Let $M \in \mathscr{C}$ and X_1, X_2 be submodules of M.*

(i) *If T is covariant left exact, then $T(X_1 \cap X_2) = T(X_1) \cap T(X_2)$.*

(ii) *If T is contravariant right exact, then $T(\pi)T(M/X_1 \cap X_2) = T(\pi_1)T(M/X_1) + T(\pi_2)T(M/X_2)$, where $\pi_i : M \to M/X_i$, $i = 1, 2$, and $\pi : M \to M/X_1 \cap X_2$ are the natural epimorphisms.*

Proof. (i) Let

$$0 \to X_1 \cap X_2 \xrightarrow{i} M \xrightarrow{\pi} M/X_1 \cap X_2 \to 0,$$
$$0 \to X_j \xrightarrow{i_j} M \xrightarrow{\pi_j} M/X_j \to 0, \quad j = 1, 2,$$

be the canonical exact sequences, and let

$$\widetilde{T(\pi)} : T(M)/T(i)T(X_1 \cap X_2) \to T(M/X_1 \cap X_2),$$
$$\widetilde{T(\pi_j)} : T(M)/T(i_j)T(X_j) \to T(M/X_j), \quad j = 1, 2,$$

be the induced monomorphisms. Let $l_j : M/X_j \to M/X_1 \oplus M/X_2$, $j = 1, 2$, be the natural injections. Then the map

$$\lambda : T(M)/T(i_1)T(X_1) \oplus T(M)/T(i_2)T(X_2) \to T(M/X_1 \oplus M/X_2),$$

defined by $\lambda(x_1 + T(i_1)T(X_1), x_2 + T(i_2)T(X_2)) = \sum_{j=1}^{2} T(l_j)T(\pi_j)(x_j)$ for all $x_1, x_2 \in T(M)$ is a monomorphism. Let $g : M/X_1 \cap X_2 \to M/X_1 \oplus M/X_2$

be the natural monomorphism. From the commutative diagram

$$\begin{array}{ccc} T(M)/T(i)T(X_1 \cap X_2) & \xrightarrow{\theta} & T(M)/T(i_1)T(X_1) \oplus T(M)/T(i_2)T(X_2) \\ \downarrow{\scriptstyle \widetilde{T(\pi)}} & & \downarrow{\scriptstyle \lambda} \\ T(M/X_1 \cap X_2) & \xrightarrow{T(g)} & T(M/X_1 \oplus M/X_2) \end{array}$$

in which θ is the natural map, we deduce that θ is a monomorphism. Hence

$$T(i)T(X_1 \cap X_2) = T(i_1)T(X_1) \cap T(i_2)T(X_2)$$

as required.

(ii) Let $\bigoplus_{i=1}^2 M/X_i \xrightarrow{p_i} M/X_i,\ i = 1, 2$, be the natural projections. Then the map

$$\varphi : \bigoplus_{i=1}^{2} T(M/X_i) \to T\Big(\bigoplus_{i=1}^{2} M/X_i\Big)$$

defined by $\varphi((x_1, x_2)) = \sum_{i=1}^2 T(p_i)(x_i)$ for all $x_1 \in T(M/X_1)$ and $x_2 \in T(M/X_2)$ is an isomorphism. Set $\psi = T(g\pi)\varphi$. Since $T(g)$ is an epimorphism, it follows that $\operatorname{Im}\psi = T(\pi)T(M/X_1 \cap X_2)$. On the other hand, it is easy to see that $\operatorname{Im}\psi = \sum_{i=1}^2 T(\pi_i)T(M/X_i)$. Now, the result follows. ∎

2.9. Lemma. *Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and let $T : \mathscr{C} \to \mathscr{C}_R$ be a covariant left exact functor. If $\mathfrak{p} \in \operatorname{Spec} R$ and M is a $\mathfrak{p}$-coprimary R-module such that $T(M) \neq 0$, then $T(M)$ is also $\mathfrak{p}$-coprimary.*

Proof. Let $a \in \mathfrak{p}$. There is a positive integer n such that $a^n \operatorname{id}_M = 0$. Since T is linear, it follows that

$$a^n \operatorname{id}_{T(M)} = a^n T(\operatorname{id}_M) = T(a^n \operatorname{id}_M) = 0.$$

Hence $a^n T(M) = 0$. Now, let $a \notin \mathfrak{p}$. Then the map $M \xrightarrow{a} M$ is injective. This yields that the map $T(M) \xrightarrow{a} T(M)$ is also injective, because T is linear and left exact. ∎

2.10. Theorem. *Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and $T : \mathscr{C} \to \mathscr{C}_R$ be a primary functor on $\mathscr{C}$. Let $M \in \mathscr{C}$ and let $0 = \bigcap_{i=1}^k Q_i$ be a minimal primary decomposition of the zero submodule of M, where M/Q_i is $\mathfrak{p}_i$-coprimary. Suppose that $R/\mathfrak{p}_i \in \mathscr{C}$ for $i = 1, \dots, k$ and that $T(R/\mathfrak{p}_i) \neq 0$ for $i = 1, \dots, r$, while this does not hold for $i = r + 1, \dots, k$.*

(i) *If T is covariant left exact, then $0 = \bigcap_{i=1}^r T(Q_i)$ is a minimal primary decomposition of the zero submodule of $T(M)$ and the quotient modules $T(M)/T(Q_i)$ are $\mathfrak{p}_i$-coprimary for $i = 1, \dots, r$.*

(ii) *If T is contravariant right exact, then $T(M) = \sum_{i=1}^r T(\pi_i)T(M/Q_i)$ is a minimal secondary representation of $T(M)$, where $\pi_i : M \to M/Q_i$, $i = 1, \dots, r$, are the natural epimorphisms.*

Proof. (i) By applying the functor T to the exact sequence

$$0 \to Q_i \to M \to M/Q_i \to 0,$$

we get the exact sequence

$$0 \to T(Q_i) \to T(M) \to T(M/Q_i).$$

Then by 2.9, $T(Q_i)$ is a $\mathfrak{p}_i$-primary submodule of $T(M)$ for $i = 1, \ldots, r$. Also,

$$T(Q_i) = T(M) \quad \text{for } i = r+1, \ldots, k.$$

It follows that $0 = \bigcap_{i=1}^{r} T(Q_i)$ is a primary decomposition of the zero submodule of $T(M)$. Suppose that

$$\bigcap_{\substack{i=1 \\ i \neq j}}^{r} T(Q_i) \subseteq T(Q_j)$$

for some $1 \leq j \leq r$. Set $\overline{Q}_j = \bigcap_{i=1,\, i \neq j}^{k} Q_i$. Then $\overline{Q}_j$ is a $\mathfrak{p}_j$-coprimary module as can be easily seen. Now, we have

$$T(\overline{Q}_j) = \bigcap_{\substack{i=1 \\ i \neq j}}^{r} T(Q_i) = \bigcap_{i=1}^{r} T(Q_i) = 0.$$

This contradicts the choice of $\mathfrak{p}_j$, and hence this decomposition is minimal.

(ii) $T(M/Q_i)$ is $\mathfrak{p}_i$-secondary for $i = 1, \ldots, r$ and $T(M/Q_i) = 0$ for $i = r+1, \ldots, k$. Hence $T(\pi_i)T(M/Q_i)$ is a $\mathfrak{p}_i$-secondary submodule of $T(M)$ for $i = 1, \ldots, r$. Let $S_i = T(\pi_i)T(M/Q_i)$. It follows from 2.8(ii) that $T(M) = \sum_{i=1}^{r} S_i$ is a secondary representation of $T(M)$. Now, we show that this representation is minimal. Suppose $S_j \subseteq \sum_{i=1,\, i \neq j}^{r} S_i$ for some $1 \leq j \leq r$. By 2.8(ii),

$$T\Big(M \big/ \bigcap_{\substack{i=1 \\ i \neq j}}^{k} Q_i\Big) = \sum_{\substack{i=1 \\ i \neq j}}^{r} S_i = T(M).$$

Therefore, it follows from the canonical exact sequence

$$0 \to \bigcap_{\substack{i=1 \\ i \neq j}}^{k} Q_i \to M \to M \big/ \bigcap_{\substack{i=1 \\ i \neq j}}^{k} Q_i \to 0$$

that $T(\bigcap_{i=1,\, i \neq j}^{k} Q_i) = 0$. This yields a contradiction because the module $\bigcap_{i=1,\, i \neq j}^{k} Q_i$ is $\mathfrak{p}_j$-coprimary and T is primary. ■

Suppose that $T : \mathscr{C}_{\mathcal{N}} \to \mathscr{C}_R$ (resp. $T : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$) is a covariant (resp. contravariant) left exact functor. It follows from [2, Remarks 1.3(i) and 2.4] that for a Noetherian (resp. Artinian) R-module N (resp. A), the set of prime ideals which appears in a minimal primary decomposition of the zero submodule of $T(N)$ (resp. $T(A)$) coincides with $\mathrm{Ass}_R(T(N))$ (resp. $\mathrm{Ass}_R(T(A))$). Hence we deduce the following, which improves Theorem 1.4 of [2].

2.11. COROLLARY. *Let $T : \mathscr{C}_{\mathcal{N}} \to \mathscr{C}_R$ be a primary functor on $\mathscr{C}_{\mathcal{N}}$. Let N be a Noetherian R-module.*

(i) *If T is covariant left exact, then* $\mathrm{Ass}_R(T(N)) = \{\mathfrak{p} \in \mathrm{Ass}_R N : T(R/\mathfrak{p}) \neq 0\}$.

(ii) *If T is contravariant right exact, then* $\mathrm{Att}_R(T(N)) = \{\mathfrak{p} \in \mathrm{Ass}_R N : T(R/\mathfrak{p}) \neq 0\}$.

In view of the proof of 2.4, we deduce the following result.

2.12. COROLLARY. *Let R be a Noetherian ring and M be an R-module. Let N be a finitely generated R-module. Suppose that $0 = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition of the zero submodule of M. Then $\bigcap_{\mathrm{Ann}_R N \subseteq \mathfrak{p}_i} \mathrm{Hom}_R(N, Q_i)$ is a minimal primary decomposition of the zero submodule of* $\mathrm{Hom}_R(N, M)$.

As an application, we establish [1, p. 267, Proposition 10].

2.13. COROLLARY. *Let R be a Noetherian ring and let N be a finitely generated R-module. Then for any R-module M,*

$$\mathrm{Ass}_R(\mathrm{Hom}_R(N, M)) = \mathrm{Supp}\, N \cap \mathrm{Ass}_R M.$$

Proof. We may express M as $M = \bigcup_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ is the set of finitely generated submodules of M. As N is finitely generated, the image of any homomorphism $f : N \to M$ is finitely generated. It therefore follows easily that

$$\mathrm{Hom}_R(N, M) = \bigcup_{i \in I} \mathrm{Hom}_R(N, M_i).$$

Now, $\mathrm{Ass}_R M = \bigcup_{i \in I} \mathrm{Ass}_R M_i$ and

$$\mathrm{Ass}_R(\mathrm{Hom}_R(N, M)) = \bigcup_{i \in I} \mathrm{Ass}_R(\mathrm{Hom}_R(N, M_i)).$$

Hence the claim follows from 2.12. ■

The following result may be considered as a tool for determining primary functors.

2.14. PROPOSITION. *Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and let T, U and V be functors from $\mathscr{C}$ to $\mathscr{C}_R$. Let $\varphi : T \to U$ and $\psi : U \to V$ be natural transformations of functors. Suppose that T and V are primary on $\mathscr{C}$ and that, for any $\mathfrak{p}$-coprimary R-module N in $\mathscr{C}$ with $R/\mathfrak{p} \in \mathscr{C}$, the sequence*

$$0 \to T(N) \xrightarrow{\varphi_N} U(N) \xrightarrow{\psi_N} V(N) \to 0$$

is exact. Then U is also primary on $\mathscr{C}$.

Proof. Suppose the contrary is true. Then there exists a prime ideal $\mathfrak{p}$ with $R/\mathfrak{p} \in \mathscr{C}$ and $\mathfrak{p}$-coprimary R-modules $N_1, N_2 \in \mathscr{C}$ such that $U(N_1) = 0$ and $U(N_2) \neq 0$. From the exact sequence

$$0 \to T(N_2) \to U(N_2) \to V(N_2) \to 0,$$

we deduce that $T(N_2) \neq 0$ or $V(N_2) \neq 0$. But T and V are primary, so that $T(N_1) \neq 0$ or $V(N_1) \neq 0$. Now, it follows from the exact sequence

$$0 \to T(N_1) \to U(N_1) \to V(N_1) \to 0$$

that in any case $U(N_1) \neq 0$. This yields a contradiction. ∎

Let $T : \mathscr{C} \to \mathscr{C}$ be a covariant (resp. contravariant) functor. We say that T *commutes with direct limits* if for any direct system $\{M_i\}_{i \in I}$ of R-modules,

$$T(\varinjlim_{i \in I} M_i) \cong \varinjlim_{i \in I} T(M_i)$$

(resp.

$$T(\varinjlim_{i \in I} M_i) \cong \varprojlim_{i \in I} T(M_i)).$$

2.15. Theorem. *Let R be a Noetherian ring and $T : \mathscr{C}_R \to \mathscr{C}_R$ be either a covariant left exact or a contravariant right exact functor which commutes with direct limits. Suppose T is primary on $\mathscr{C}_{\mathcal{N}}$. Then T is also primary on $\mathscr{C}_R$.*

Proof. Suppose the contrary is true. Then there are a prime ideal $\mathfrak{p}$ and $\mathfrak{p}$-coprimary modules M and M' such that $T(M) \neq 0$, while $T(M') = 0$. We may write $M = \varinjlim_{i \in I} M_i$, where M_i*'s* are finitely generated submodules of M. Hence $T(M) \cong \varinjlim_{i \in I} T(M_i)$ or $T(M) \cong \varprojlim_{i \in I} T(M_i)$ according as T is covariant or contravariant. Therefore there exists $i \in I$ such that $T(M_i) \neq 0$. Since T is primary on $\mathscr{C}_{\mathcal{N}}$, it follows that $T(R/\mathfrak{p}) \neq 0$. As M' is $\mathfrak{p}$-coprimary, it has a submodule isomorphic to $R/\mathfrak{p}$. Thus there exists an exact sequence

$$0 \to R/\mathfrak{p} \to M'.$$

Now, by applying T to this sequence, we get $T(M') \neq 0$, which is a contradiction. ∎

2.16. Example. Let R be a Noetherian ring. Let E and F be an injective and a flat R-module respectively. The functors $\mathrm{Hom}_R(\cdot, E)$ and $F \otimes_R (\cdot)$ are primary on $\mathscr{C}_R$. Note that these functors are primary on $\mathscr{C}_{\mathcal{N}}$, by 2.7.

3. Non-exact secondary functors. All examples of secondary functors which we considered up to now are exact. [2, Example 2.2] provides examples of exact functors which are not secondary. In this section, we give some examples of non-exact secondary functors.

One may prove the following extended version of [2, Lemma 2.3], by the same proof.

3.1. Lemma. *Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{C}_R$ and $T : \mathscr{C} \to \mathscr{C}_R$ be a functor. Let $M \in \mathscr{C}$ and X_1, X_2 be submodules of M.*

(i) *If T is covariant right exact, then $T(i)T(X_1 + X_2) = T(i_1)T(X_1) + T(i_2)T(X_2)$, where $i_j : X_j \to M$, $j = 1, 2$, and $i : X_1 + X_2 \to M$ are the natural monomorphisms.*

(ii) *If T is contravariant left exact, then $T(M/X_1 + X_2) = T(M/X_1) \cap T(M/X_2)$.*

With the comment preceding 2.11 in mind, one may deduce the following improvement of [2, Theorem 2.5], by employing arguments similar to the proof of 2.10 and applying 3.1.

3.2. Theorem. *Let $T : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$ be a secondary functor on $\mathscr{C}_{\mathcal{A}}$. Let $A = \sum_{i=1}^{k} S_i$ be a minimal secondary representation of the Artinian R-module A, where S_i is $\mathfrak{p}_i$-secondary. Suppose that $T(S_i) \neq 0$ for $i = 1, \ldots, r$, while $T(S_i) = 0$ for $i = r + 1, \ldots, k$.*

(i) *If T is covariant right exact, then $T(A) = \sum_{j=1}^{r} T(i_j)T(S_j)$, where $i_j : S_j \to A$, $j = 1, \ldots, k$, are inclusion maps, is a minimal secondary representation of $T(A)$ and so*

$$\mathrm{Att}_R(T(A)) = \{\mathfrak{p} \in \mathrm{Att}_R A : T(S) \neq 0 \textit{ for some } (\textit{equivalently all})\ \mathfrak{p}\textit{-secondary Artinian } R\textit{-modules } S\}.$$

(ii) *If T is contravariant left exact, then $0 = \bigcap_{i=1}^{r} T(A/S_i)$ is a minimal primary decomposition of the zero submodule of $T(A)$ and so*

$$\mathrm{Ass}_R(T(A)) = \{\mathfrak{p} \in \mathrm{Att}_R A : T(S) \neq 0 \textit{ for some } (\textit{equivalently all})\ \mathfrak{p}\textit{-secondary Artinian } R\textit{-modules } S\}.$$

The next result may be regarded as a slight generalization of [4, Proposition 5.2].

3.3. Corollary. *Let N be a finitely generated R-module. Then the functor $N \otimes_R (\cdot)$ is secondary on $\mathscr{C}_{\mathcal{A}}$ and hence, for any Artinian R-module A,*

$$\mathrm{Att}_R(N \otimes_R A) = \mathrm{Att}_R A \cap \mathrm{Supp}\, N.$$

Proof. First, we will show that for a $\mathfrak{p}$-secondary Artinian R-module S, $N \otimes_R S \neq 0$ if and only if $\mathrm{Ann}_R N \subseteq \mathfrak{p}$. This will prove the first assertion. By 2.1, $N \otimes_R S \neq 0$ if and only if $S \neq (\mathrm{Ann}_R N)S$. It follows from [5, Proposition 3.4] that $S = (\mathrm{Ann}_R N)S$ if and only if $S = xS$ for some $x \in \mathrm{Ann}_R N$. (Note that in the proof of [5, Proposition 3.4], there is no need for R to be Noetherian.) Hence $N \otimes_R S \neq 0$ if and only if $\mathrm{Ann}_R N \subseteq \mathfrak{p}$. The second assertion follows from the first one and 3.2(i). ∎

By using the same method as in the proof of 2.14, one may prove the following result.

3.4. PROPOSITION. *Let* $\mathscr{C}$ *be a Serre subcategory of* $\mathscr{C}_R$ *and let* T, U *and* V *be functors from* $\mathscr{C}$ *to* $\mathscr{C}_R$. *Let* $\varphi : T \to U$ *and* $\psi : U \to V$ *be natural transformations of functors. Suppose that* T *and* V *are secondary on* $\mathscr{C}$ *and that, for any* $\mathfrak{p}$*-secondary* R*-module* S *in* $\mathscr{C}$, *the sequence*

$$0 \to T(S) \xrightarrow{\varphi_S} U(S) \xrightarrow{\psi_S} V(S) \to 0$$

is exact. Then U *is also secondary on* $\mathscr{C}$.

In the remainder of this section, we assume that R is a semi-local Noetherian complete ring and that E is the minimal injective cogenerator of R. Put $\mathrm{D}(\cdot) = \mathrm{Hom}_R(\cdot, E)$.

3.5. PROPOSITION. *Let* R *be a semi-local Noetherian complete ring. Suppose that* $T : \mathscr{C}_{\mathcal{N}} \to \mathscr{C}_R$ *is a primary functor on* $\mathscr{C}_{\mathcal{N}}$. *Then both functors* $U = T(\mathrm{D}(\cdot)) : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$ *and* $V = \mathrm{D}(T(\mathrm{D}(\cdot))) : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$ *are secondary on* $\mathscr{C}_{\mathcal{A}}$.

Proof. Let S_1 and S_2 be Artinian $\mathfrak{p}$-secondary R-modules. Then it follows from [5, Theorem 1.6] that $N_i = \mathrm{D}(S_i)$, $i = 1, 2$, are Noetherian $\mathfrak{p}$-coprimary R-modules. Thus $T(N_1) = 0$ if and only if $T(N_2) = 0$. Therefore $U(S_1) = 0$ if and only if $U(S_2) = 0$. Similarly, $V(S_1) = 0$ if and only if $V(S_2) = 0$. Note that for an R-module M, $\mathrm{D}(M) = 0$ if and only if $M = 0$. ∎

3.6. EXAMPLE. Let R be a semi-local Noetherian complete ring. Let M be an R-module. Then the functor $\mathrm{Hom}_R(M, \cdot)$ is secondary on $\mathscr{C}_{\mathcal{A}}$. To see this, note that $A = \mathrm{D}(\mathrm{D}(A))$ for any Artinian R-module A.

The next result generalizes [2, Lemma 2.7(i)] (see 1.3(ii)).

3.7. PROPOSITION. *Let the situation be as in* 3.5. *Let* $T : \mathscr{C}_{\mathcal{A}} \to \mathscr{C}_R$ *be a functor which is either contravariant right exact or covariant left exact. Then* T *is secondary on* $\mathscr{C}_{\mathcal{A}}$.

Proof. Put $U = T(\mathrm{D}(\cdot))$. Then U is a functor from $\mathscr{C}_{\mathcal{N}}$ to $\mathscr{C}_R$ and it is either covariant right exact or contravariant left exact. In each case U is primary on $\mathscr{C}_{\mathcal{N}}$, by 2.7. Let $\mathfrak{p}$ be a prime ideal of R and S_1 and S_2 be two Artinian $\mathfrak{p}$-secondary R-modules. Then $N_i = \mathrm{D}(S_i)$, $i = 1, 2$, are Noetherian $\mathfrak{p}$-coprimary R-modules, and so $U(N_1) = 0$ if and only if $U(N_2) = 0$. But $U(N_i) = T(S_i), i = 1, 2$ (see [5, Theorem 1.6]). Therefore $T(S_1) = 0$ if and only if $T(S_2) = 0$. ∎

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