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ON LOCAL DERIVATIONS IN THE KADISON SENSE

BY

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Abstract. Let k be a field. We prove that any polynomial ring over k is a Kadison algebra if and only if k is infinite. Moreover, we present some new examples of Kadison algebras and examples of algebras which are not Kadison algebras.

1. Introduction. Let k be a field and A a commutative k-algebra with unity. A k-linear mapping $d : A \to A$ is called a *derivation* of A if $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$. A k-linear mapping $\gamma : A \to A$ is called a *local derivation* of A if for each $a \in A$ there exists a derivation d_a of A such that $\gamma(a) = d_a(a)$.

Every derivation of A is a local derivation of A. There exist local derivations which are not derivations (see [1]). We say that a k-algebra A is a *Kadison algebra* if every local derivation of A is a derivation.

R. Kadison [1], in 1990, proved that polynomial rings over $\mathbb{C}$ are Kadison algebras. His proof of this fact is valid for any polynomial algebra over a field k of characteristic zero. Yong Ho Yon [2], in 1999, tried to prove the same in the case when k is infinite of any characteristic, but in his proof there are some gaps. He repeats Kadison's arguments which are not valid in positive characteristic. However, the assertion is indeed true. We present here a short proof of this fact.

We prove that any polynomial ring over k is a Kadison algebra if and only if k is infinite. Moreover, we present some new examples of Kadison algebras and examples of algebras which are not Kadison algebras. We prove, among other things, that if P is a prime ideal of the polynomial ring $k[x_1, \ldots, x_n]$, then the local algebra $k[x_1 \ldots, x_n]_P$ is not a Kadison algebra.

2. Results. We denote by $k[x_1, \ldots, x_n]$ the polynomial algebra over k. If $n = 1$, then we denote this algebra by $k[t]$.

THEOREM 1. *Let S be a multiplicative subset of the polynomial algebra $k[x_1, \ldots, x_n]$, where k is an infinite field. Let A be the algebra of quotients*

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$S^{-1}k[x_1,\ldots,x_n]$ *and let*

$$M = \{\lambda \in k^n : \exists_{f\in S}\ f(\lambda) = 0\}.$$

If the set $k^n \setminus M$ *is dense in the Zariski topology of* k^n, *then* A *is a Kadison algebra.*

Proof. Step 1. Assume that $\gamma : A \to A$ is a local derivation such that $\gamma(x_1) = \ldots = \gamma(x_n) = 0$. We shall show that $\gamma = 0$. To this end, observe that if $\lambda \in k^n \setminus M$, then for every $\varphi \in A$, we may define, in a natural way, the value $\varphi(\lambda)$ belonging to k.

Let $w \in A$ and let λ be an arbitrary point belonging to $k^n \setminus M$. Let

$$h = w - \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(\lambda)x_i.$$

Since γ is a local derivation of A, there exists a derivation δ of A such that $\gamma(h) = \delta(h)$. Then

$$\gamma(h) = \gamma(w) - \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(\lambda)\gamma(x_i) = \gamma(w) - \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(\lambda)0 = \gamma(w)$$

and

$$\begin{aligned}\delta(h) &= \delta(w) - \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(\lambda)\delta(x_i)\\ &= \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\delta(x_i) - \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(\lambda)\delta(x_i)\\ &= \sum_{i=1}^{n} \left(\frac{\partial w}{\partial x_i}(\lambda) - \frac{\partial w}{\partial x_i}\right)\delta(x_i),\end{aligned}$$

and so

$$\delta(h)(\lambda) = \sum_{i=1}^{n} \left(\frac{\partial w}{\partial x_i}(\lambda) - \frac{\partial w}{\partial x_i}(\lambda)\right)\delta(x_i)(\lambda) = \sum_{i=1}^{n} 0\delta(x_i)(\lambda) = 0.$$

Thus we have

$$\gamma(w)(\lambda) = \gamma(h)(\lambda) = \delta(h)(\lambda) = 0$$

for any $\lambda \in k^n \setminus M$. Since k is infinite and the set $k^n \setminus M$ is dense, $\gamma(w) = 0$. This implies that $\gamma = 0$.

Step 2. Now assume that γ is an arbitrary local derivation of A. Denote by $\varphi_1,\ldots,\varphi_n$ the elements $\gamma(x_1),\ldots,\gamma(x_n)$, respectively. There exists a unique derivation d of A such that $d(x_i) = \varphi_i$ for $i = 1,\ldots,n$. Consider the mapping $\beta = \gamma - d$. It is a local derivation of A such that $\beta(x_1) = \ldots = \beta(x_n) = 0$. Then $\beta = 0$ (by Step 1) and therefore, $\gamma = d$, that is, γ is a derivation. ∎

The above theorem implies that if k is infinite and $S = \{t^n : n \geq 0\}$, then $S^{-1}k[t]$ is a Kadison algebra. Using the above theorem in the case when S is the group of units of $k[x_1, \dots, x_n]$ we get:

PROPOSITION 2. *If k is infinite, then $k[x_1, \dots, x_n]$ is a Kadison algebra.* ■

Now we shall show that polynomial algebras over finite fields are not Kadison algebras.

PROPOSITION 3. *Let k be a finite field of cardinality q and let $A = k[x_1, \dots, x_n]$. Then the mapping $\gamma : A \to A$ defined by*

$$\gamma(f) = \frac{\partial f}{\partial x_1} - \left(\frac{\partial f}{\partial x_1}\right)^q \quad (\textit{for } f \in A)$$

is a local derivation of A which is not a derivation.

Proof. Let $p = \mathrm{char}(k)$ and let $q = p^r$ for some $r \geq 1$. Each element λ of k satisfies the equality $\lambda^q = \lambda$. Moreover, $(f + g)^q = f^q + g^q$ for all polynomials f and g. Hence, γ is a k-linear mapping.

Observe that $\gamma(x_1) = 0$ and $\gamma(x_1^{p+1}) = x_1^p - x_1^{qp} \neq 0$. This implies that γ is not a derivation because in the opposite case we have a contradiction: $0 = (p+1)x_1^p\gamma(x_1) = \gamma(x_1^{p+1}) \neq 0$.

Now assume that f is an arbitrary polynomial from A and denote by v its partial derivative $\partial f/\partial x_1$. Then $\gamma(f) = v - v^q$. Let $d : A \to A$ be the derivation such that $d(x_1) = 1 - v^{q-1}$ and $d(x_2) = \ldots = d(x_n) = 0$. Then

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d(x_i) = \frac{\partial f}{\partial x_1} d(x_1) = v(1 - v^{q-1}) = v - v^q = \gamma(f).$$

Therefore, γ is a local derivation of A. ■

As a consequence of Propositions 2 and 3 we get

THEOREM 4. *If k is a field, then the polynomial ring $k[x_1, \dots, x_n]$ is a Kadison algebra if and only if k is infinite.*

In the above theorems and propositions all the algebras of polynomials have a finite set of variables. The same proofs work for algebras of polynomials in an arbitrary set of variables.

If P is a prime ideal of $k[x_1, \dots, x_n]$, then we denote by $k[x_1, \dots, x_n]_P$ the algebra $S^{-1}k[x_1, \dots, x_n]$ of quotients with respect to the multiplicative subset $S = k[x_1, \dots, x_n] \setminus P$.

THEOREM 5. *No algebra of the form $k[x_1, \dots, x_n]_P$, where P is a prime ideal, is a Kadison algebra.*

Proof. Put $k[X] = k[x_1, \dots, x_n]$, $A = k[X]_P$, $B = k[X]/P$. Let $\pi : k[X] \to B$ be the natural homomorphism and let L be the field of quotients of B. Denote by $b_1, \dots, b_n$ the elements $\pi(x_1), \dots, \pi(x_n)$, respectively, and let $b = (b_1, \dots, b_n) \in L^n$. Since $k \subset L$ is a field extension, L is a vector space over k. Consider a basis of L over k containing 1 and let $\mu(v)$ (for every $v \in L$) be the coefficient of 1 in the basis representation of v. Then $\mu : L \to k$, $v \mapsto \mu(v)$, is a k-linear mapping.

It is clear that if $f \in k[X]$, then $f \notin P$ if and only if $f(b) \neq 0$. Thus, for any element $\varphi \in A$, we may define, in a natural way, the value $\varphi(b)$ which belongs to L. Observe that if $\varphi \in A$, then φ is invertible in A if and only if $\varphi(b) \neq 0$.

We define a mapping $\gamma : A \to A$ by

$$\gamma(\varphi) = \mu\left(\frac{\partial\varphi}{\partial x_1}(b)\right) \quad \text{for } \varphi \in A.$$

It is a k-linear mapping. We shall show that γ is a local derivation which is not a derivation.

Let $\varphi \in A$. If $\gamma(\varphi) = 0$, then $\gamma(\varphi) = d(\varphi)$ where d is the zero derivation of A. Now assume that $\gamma(\varphi) \neq 0$. Then $(\partial\varphi/\partial x_1)(b) \neq 0$ and so $\partial\varphi/\partial x_1$ is invertible in A. Put $v = (\partial\varphi/\partial x_1)^{-1}$ and let $d : A \to A$ be the derivation such that $d(x_1) = \gamma(\varphi)v$ and $d(x_i) = 0$ for $i \geq 2$. It is easy to check that $\gamma(\varphi) = d(\varphi)$. This means that γ is a local derivation of A.

Now suppose that γ is a derivation. Observe that $\gamma(x_1) = 1$ and $\gamma(A) \subseteq k$. If $\mathrm{char}(k) \neq 2$, then we have a contradiction: $2x_1 = \gamma(x_1^2) \in k$. If $\mathrm{char}(k) = 2$, then we also have a contradiction: $x_1^2 = \gamma(x_1^3) \in k$. Therefore, γ is not a derivation and consequently, A is not a Kadison algebra. ∎

Corollary 6 ([1], [2]). *No field $k(x_1, \dots, x_n)$ of rational functions is a Kadison algebra.*

Proof. This follows from Theorem 5 for $P = 0$. ∎

The next proposition shows that the class of Kadison algebras is not closed with respect to homomorphic images.

Proposition 7. *Let n be a nonnegative integer and let $R_n = k[t]/(t^n)$.*

(1) *If $n \leq 2$, then R_n is a Kadison algebra.*
(2) *R_3 is a Kadison algebra if and only if* $\mathrm{char}(k) = 2$.
(3) *If $n > 3$, then R_n is not a Kadison algebra.*

Proof. Each element w of R_n has a unique representation of the form

$$w = a_{n-1}\tau^{n-1} + \ldots + a_1\tau^1 + a_0,$$

where $a_0, \dots, a_{n-1} \in k$ and $\tau = t + (t^n)$. In this case we denote by $w(0)$ the constant term a_0, and by w' the element

$$(n-1)a_{n-1}\tau^{n-2} + (n-2)a_{n-2}\tau^{n-3} + \ldots + 2a_2\tau + a_1.$$

Note that if $w \in R_n$, then w is invertible in R_n if and only if $w(0) \neq 0$.

R_1 is a Kadison algebra, because $R_1 = k$. If d is a derivation of R_2, then

$$d(\tau) = \begin{cases} y\tau & \text{if } \mathrm{char}(k) \neq 2, \\ x + y\tau & \text{if } \mathrm{char}(k) = 2, \end{cases}$$

for some $x, y \in k$. This implies that if β is a local derivation of R_2 and $\mathrm{char}(k) \neq 2$, then there exists $y \in k$ such that $\beta(a + b\tau) = by\tau$ for all $a, b \in k$, and if $\mathrm{char}(k) = 2$, then there exist $x, y \in k$ such that $\beta(a + b\tau) = b(x + y\tau)$ for all $a, b \in k$. In both cases such a β is a derivation of R_2. Thus, R_2 is a Kadison algebra.

Now let $n > 2$. Consider the mapping $\gamma : R_n \to R_n$ defined by

$$\gamma(w) = w'(0)\tau \quad \text{for } w \in R_n.$$

We will show that γ is a local derivation of R_n. Let $w \in R_n$. If $w'(0) = 0$, then $\gamma(w) = 0 = d(w)$ where d is the zero derivation of R_n. Assume that $w'(0) \neq 0$. Then w' is invertible in R_n. Put $v = (w')^{-1}w'(0)\tau$ and let $d : R_n \to R_n$ be the mapping $s \mapsto s'v$. It is clear that d is a derivation of R_n and $\gamma(w) = d(w)$. This means that γ is a local derivation.

Note that $\gamma(\tau) = \tau$. Assume that $n > 3$ and suppose that γ is a derivation of R_n. If $\mathrm{char}(k) \neq 2$ then we have a contradiction: $0 = \gamma(\tau^2) = 2\tau\gamma(\tau) = 2\tau^2 \neq 0$. If $\mathrm{char}(k) = 2$ then we also have a contradiction: $0 = \gamma(\tau^3) = 3\tau^2\gamma(\tau) = \tau^3 \neq 0$. Therefore, if $n > 3$, then γ is not a derivation. So we have (3). The same happens in the case when $n = 3$ and $\mathrm{char}(k) \neq 2$.

It remains to prove that if $\mathrm{char}(k) = 2$, then R_3 is a Kadison algebra. Assume that $\mathrm{char}(k) = 2$ and let $\alpha : R_3 \to R_3$ be a local derivation. Then there exists a derivation d of R_3 such that $\alpha(\tau) = d(\tau)$. Put $d(\tau) = x + y\tau + z\tau^2$, where $x, y, z \in k$. Then

$$0 = d(0) = d(\tau^3) = 3\tau^2 d(\tau) = \tau^2(x + y\tau + z\tau^2) = x\tau^2$$

and so $x = 0$, that is, $\alpha(\tau) = y\tau + z\tau^2$ for some $y, z \in k$. Moreover, $\alpha(1) = 0$ and, since $\mathrm{char}(k) = 2$, $\alpha(\tau^2) = 0$. Hence,

$$\alpha(a + b\tau + c\tau^2) = b(y\tau + z\tau^2)$$

for all $a, b, c \in k$, and this implies that α is a derivation of R_3. Therefore, if $\mathrm{char}(k) = 2$, then R_3 is a Kadison algebra. ∎

Let us end this paper with the following theorem.

THEOREM 8. *The algebra $k[[x_1, \ldots, x_n]]$ of formal power series over a field k is not a Kadison algebra.*

Proof. Put $A = k[[x_1, \ldots, x_n]]$. If $f \in A$, then we denote by $\gamma(f)$ the coefficient of the monomial x_1 in f. We shall show that the mapping $\gamma : A \to A$, $f \mapsto \gamma(f)$, is a local derivation of A which is not a derivation. It is clear that γ is k-linear.

Let $f \in A$. If $\gamma(f) = 0$, then $\gamma(f) = d(f)$ where d is the zero derivation of A. Assume now that $\gamma(f) \neq 0$ and denote by v the partial derivative $\partial f/\partial x_1$. The constant term of v is not zero (because it coincides with $\gamma(f)$). Hence v is an invertible element of A. Consider the derivation d of A such that $d(x_1) = v^{-1}\gamma(f)$ and $d(x_i) = 0$ for $i \geq 2$. Then we have

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d(x_i) = \frac{\partial f}{\partial x_1} d(x_1) = vv^{-1}\gamma(f) = \gamma(f).$$

Hence, γ is a local derivation of A. Note that $\gamma(x_1) = 1$ and $\gamma(f) \in k$ for all $f \in A$.

Suppose now that γ is a derivation of A. Then for any $s \geq 2$ we have $sx_1^{s-1} = \gamma(x_1^s) \in k$, which is a contradiction. Therefore, γ is not a derivation of A. ∎

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