SOME ORTHOGONAL DECOMPOSITIONS OF
SOBOLEV SPACES AND APPLICATIONS

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Abstract. Two kinds of orthogonal decompositions of the Sobolev space $W^{1,2}_0$ and hence also of $W^{-1}_2$ for bounded domains are given. They originate from a decomposition of $W^{1,2}_0$ into the orthogonal sum of the subspace of the $\Delta^k$-solenoidal functions, $k \geq 1$, and its explicitly given orthogonal complement. This decomposition is developed in the real as well as in the complex case. For the solenoidal subspace ($k = 0$) the decomposition appears in a little different form.

In the second kind decomposition the $\Delta^k$-solenoidal function spaces are decomposed via subspaces of polyharmonic potentials. These decompositions can be used to solve boundary value problems of Stokes type and the Stokes problem itself in a new manner.

Another kind of decomposition is given for the Sobolev spaces $W^{m,p}$. They are decomposed into the direct sum of a harmonic subspace and its direct complement which turns out to be $\Delta(W^{m+2}_{p} \cap W^{1,2}_{p})$. The functions involved are all vector-valued.

1. Introduction. A series of decompositions of the Sobolev space $W^{1,2}_0$ and its conjugate $W^{-1}_2$ are given which are connected with the solenoidal and potential subspaces. There are two well known such decompositions of the Lebesgue space $L^2$ (see [12]), namely if $G \subset \mathbb{R}^n$, $n \geq 1$, is a smooth enough domain then

$$L^2 = S_2 \oplus \nabla \hat{W}^{1,2}_0,$$

with

$$S_2 = \{u_s \in L^2 : \text{div } u_s = 0 \text{ in } G \},$$

and

$$L^2 = \tilde{S}_2 \oplus \nabla H_1 \oplus \nabla \hat{W}^{1,2}_0$$

where $\tilde{S}_2$ is the closure in $L^2$ of the set of solenoidal functions in $C_0^\infty(G)$ and $H_1 \subset W^{1,2}_0$ is the subspace of harmonic potentials. Here the derivatives are understood in the distributional sense, i.e. in $\mathcal{D}'$.

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The decompositions given in this paper can be viewed as generalizations of both these decompositions. They are connected with the orthogonality not only to “pure” solenoidal subspaces but also to so-called $\Delta^k$-solenoidal subspaces. The latter consist of those elements $u \in \tilde{W}^1_2$ which satisfy $\text{div} \Delta^k u = 0$ in $G$ in the sense of $\mathcal{D}'$. Here $k$ is any natural number. The particular case $k = 1$ was briefly discussed in [4, 5].

Each type of decomposition leads to a corresponding Stokes type boundary value problem or to the Stokes problem itself. In contrast to the traditional methods of finding the solenoidal vector function here at first the potential is determined and then the solenoidal part found as the solution to a Dirichlet problem. Such type of decompositions have also been developed in Clifford analysis (see [2, 3, 6–10, 14]), and applied to boundary value problems of mathematical physics. For decompositions in complex analysis compare [1, 13].

2. Orthogonality to the $\Delta^k$-solenoidal subspaces. Let $G \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with Lipschitz boundary $\partial G$. Let

$$\tilde{W}^1_2 = \tilde{W}^1_2(G; \mathbb{C}^n) = \left\{ u : G \rightarrow \mathbb{C}^n : \|u\|^2_1 = \int_G |\nabla u(x)|^2 \, dx \right\}$$

be a subspace of the well known Sobolev space $W^1_2$ of complex vector-valued functions $u = (u_1, \ldots, u_n)$ in $G$, where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator. For fixed integer $k \geq 1$ let

$$\tilde{S}^1_{\Delta^k, 2} = \left\{ u \in \tilde{W}^1_2 : \text{div} \Delta^k u = 0 \right\}.$$

Here $\text{div}$ and $\Delta$ are operators understood in the sense of distributions in $\mathcal{D}' = \mathcal{D}'(G)$. The set $\tilde{S}^1_{\Delta^k, 2}$ is called the subspace of $\Delta^k$-solenoidal vector functions of $\tilde{W}^1_2$. Since both operators $\text{div}$ and $\Delta$ are closed in $\mathcal{D}'$ it follows that $\tilde{S}^1_{\Delta^k, 2}$ are closed subspaces of $\tilde{W}^1_2$, and the inclusion

$$\tilde{S}^1_{\Delta^k, 2} \subset \tilde{W}^1_2$$

is proper. Hence, $\tilde{W}^1_2$ is representable as the orthogonal sum

$$\tilde{W}^1_2 = \tilde{S}^1_{\Delta^k, 2} \oplus (\tilde{S}^1_{\Delta^k, 2})^\perp$$

in the sense of the inner product in $\tilde{W}^1_2$,

$$\langle u, v \rangle_1 = \langle \nabla u, \nabla v \rangle_0 = \sum_{\nu=1}^n \int_G (\nabla u_\nu(x), \nabla v_\nu(x)) \, dx.$$
First the subspace orthogonal to $\dot{S}_{\Delta^k,2}^1$ will be described. To this end one introduces the following subspace of the scalar Sobolev space $W_2^{2k} = W_2^{2k}(G; \mathbb{C})$:

$$\dot{W}^{2k}_{\Delta^k,2} = \{ p_0 \in W_2^{2k} : \Delta^k p_0|_{\partial G} = 0, \nabla \Delta^k p_0|_{\partial G} = 0 \text{ for } 0 \leq \kappa \leq k - 1 \}.$$  

It is the closure in the $W_2^{2k}$ norm of all scalar functions $p \in C^\infty(G; \mathbb{C})$ satisfying the stated boundary conditions.

**Remark 1.** It is clear that

$$\dot{W}^{2k}_{\Delta^k,2} = \{ p_0 \in W_2^{2k} : \Delta^k p_0 \in \dot{W}^{2k}_2 \text{ for } 0 \leq \kappa \leq k - 1 \}.$$  

**Remark 2.** As the boundary conditions are selfadjoint the bilinear form

\[(1) \quad \langle \Delta^{2k} p_1, p_0 \rangle = \langle \Delta^{k} p_1, \Delta^{k} p_0 \rangle = 0 = \langle p_1, \Delta^{2k} p_0 \rangle, \quad p_1, p_0 \in \dot{W}^{2k}_{\Delta^k,2},\]

is well-posed and selfadjoint. By the classical Lax–Milgram lemma (see e.g. [11]), the map

\[(2) \quad \Delta^{2k} : \dot{W}^{2k}_{\Delta^k,2} \leftrightarrow (\dot{W}^{2k}_{\Delta^k,2})^*\]

is an isometric isomorphism.

**Lemma 1.** Let $q \in (\hat{S}_{\Delta^k,2}^1)^\perp$. Then the boundary value problems

\[(3) \quad \nabla \Delta^{k-1} p_0 = q \quad \text{in } G, \quad p_0 \in \dot{W}^{2k}_{\Delta^k,2},\]

and

\[(4) \quad \Delta^{2k} p_0 = \text{div} \Delta^{k} q \quad \text{in } G, \quad p_0 \in \dot{W}^{2k}_{\Delta^k,2},\]

are equivalent.

**Proof.** (i) If $p_0$ is a solution to (3) then by differentiation, because of $\text{div} \Delta^{k} (\nabla \Delta^{k-1}) = \Delta^{2k}$, the function $p_0$ also solves (4).

(ii) Conversely, let $p_0$ be a generalized solution to (4) in the sense of the duality (2). Then from (4) follows

$$\nabla \Delta^{k-1} p_0 - q \in \ker \text{div} \Delta^{k},$$

i.e.

$$\nabla \Delta^{k-1} p_0 - q \in \dot{S}_{\Delta^k,2}^1.$$  

On the other hand, for any $p_0 \in \dot{W}^{2k}_{\Delta^k,2}$ and arbitrary $\varphi \in \dot{S}_{\Delta^k,2}^1$ the relation

$$\langle \text{div} \Delta^{k} \varphi, p_0 \rangle = -\langle \varphi, \nabla \Delta^{k} p_0 \rangle = 0 = \langle \varphi, \nabla \Delta^{k-1} p_0 \rangle$$

holds in the sense of the duality (2). The condition $\text{div} \Delta^{k} \varphi = 0$ in the sense of (2) means that $\nabla \Delta^{k-1} p_0 \in (\dot{S}_{\Delta^k,2}^1)^\perp$. Thus as $q \in (\dot{S}_{\Delta^k,2}^1)^\perp$, also

\[(6) \quad \nabla \Delta^{k-1} p_0 - q \in (\dot{S}_{\Delta^k,2}^1)^\perp.$$  

Together with (5) this implies $\nabla \Delta^{k-1} p_0 - q = 0$, i.e. $p_0$ solves (3).
Theorem 1. The operator $\nabla \Delta^{k-1}$ establishes an elliptic isomorphism between $\tilde{W}^2_{\Delta^k,2}$ and $(\hat{S}^1_{\Delta^k,2})^\perp$,

$$\nabla \Delta^{k-1} : \tilde{W}^2_{\Delta^k,2} \leftrightarrow (\hat{S}^1_{\Delta^k,2})^\perp.$$ 

This means that for any $p_0 \in \tilde{W}^2_{\Delta^k,2}$ the vector-valued function $\nabla \Delta^{k-1} p_0$ is orthogonal to the subspace $(\hat{S}^1_{\Delta^k,2})^\perp$ and conversely if $q \in (\hat{S}^1_{\Delta^k,2})^\perp$ then there exists a unique potential $p_0 \in \tilde{W}^2_{\Delta^k,2}$ such that $q = \nabla \Delta^{k-1} p_0$. Moreover, there exists some constant $M > 0$ such that

$$\|p_0\|_{\tilde{W}^2_{\Delta^k,2}} \leq M \|\text{div} \Delta^{k} q\|_{(\tilde{W}^2_{\Delta^k,2})^*}. \tag{7}$$ 

Proof. (i) Let $p_0 \in \tilde{W}^2_{\Delta^k,2}$. Then as shown in step (ii) of the preceding proof, $\nabla \Delta^{k-1} p_0 \in (\hat{S}^1_{\Delta^k,2})^\perp$.

(ii) Let $q \in (\hat{S}^1_{\Delta^k,2})^\perp$. Consider the boundary value problem (3). By Lemma 1 it is equivalent to problem (4). The latter problem is uniquely solvable. Its solution satisfies (3).

Finally, the estimate (7) follows from the isometric isomorphism (2).

As a corollary one has

Theorem 2. For any $k \geq 1$ the space $\hat{W}^1_2$ can be decomposed as

$$\hat{W}^1_2 = \hat{S}^1_{\Delta^k,2} \oplus \nabla \Delta^{k-1} (\tilde{W}^2_{\Delta^k,2}). \tag{8}$$ 

Example. For $k = 1$,

$$\hat{W}^2_{\Delta,2} = \{ p_0 \in W^2_2 : p_0|_{\partial G} = 0, \ \nabla p_0|_{\partial G} = 0 \}$$

is the usual Sobolev space $\hat{W}^2_2$ and formula (8) becomes

$$\hat{W}^1_2 = \hat{S}^1_{\Delta,2} \oplus \nabla (\hat{W}^2_2).$$

This means that every function $u \in \hat{W}^1_2$ can be represented as the sum

$$u = u_{\Delta, s} + \nabla p_0$$

where $(\text{div} \Delta) u_{\Delta, s} = 0$ and $p_0 \in \hat{W}^2_2$.

Remark 3. From the representation

$$u = u_{\Delta, s} + \nabla \Delta^{k-1} p_0$$

corresponding to the decomposition (8) it is seen that $p_0$ is the solution to the boundary value problem

$$\Delta^k p_0 = \text{div} \Delta^k u \quad \text{in} \ G, \quad \Delta^\kappa p_0 \in \hat{W}^2_2 \quad \text{for} \ 0 \leq \kappa \leq k - 1.$$ 

Therefore the map $u \mapsto \nabla \Delta^{k-1} p$ is the projector $\nabla \Delta^{k-1} : \hat{W}^1_2 \to (\hat{S}^1_{\Delta^k,2})^\perp$. 

3. Orthogonality to the $\Delta^k$-solenoidal subspaces in the complex case. In this section the above results are extended to the case of several complex variables. Let $G$ be a domain in $\mathbb{C}^n$, $n \geq 2$. It will be identified with $G \subset \mathbb{R}^{2n}$ of the real variables \( \{x_\nu, y_\nu : 1 \leq \nu \leq n\} \) so that $z = (z_1, \ldots, z_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z_\nu = x_\nu + iy_\nu$, $1 \leq \nu \leq n$. The boundary $\partial G$ of the domain $G \subset \mathbb{R}^{2n}$ is supposed to be a smooth hypersurface. Further, as usual, the basic complex differential operators are denoted by

$$
\partial_{z_\nu} = \frac{1}{2} (\partial_{x_\nu} - i \partial_{y_\nu}), \quad \partial_{\bar{z}_\nu} = \frac{1}{2} (\partial_{x_\nu} + i \partial_{y_\nu}), \quad 1 \leq \nu \leq n.
$$

With the same notations as in the real case, one defines the Sobolev function spaces of complex vector-valued functions $u = (u_1, \ldots, u_n)$ vanishing at the boundary $\partial G$:

$$
\hat{W}^1_2 = \left\{ u \in W^1_2(G; \mathbb{C}^n) : \|u\|_1^2 = \int_G |\nabla_z u(z)|^2 \, dx \, dy = \sum_{\mu, \nu=1}^n \int_G |\partial_{z_\nu} u_\mu(z)|^2 \, dx \, dy < \infty, \ u|_{\partial G} = 0 \right\}.
$$

$\nabla_z = (\partial_{z_1}, \ldots, \partial_{z_n})$ is the complex gradient with respect to the variable $z$. Analogously $\nabla_{\bar{z}} = (\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$. The space $\hat{W}^1_2$ endowed with the inner product

$$
\langle u, v \rangle_1 = \int_G (\nabla_z u(z), \nabla_z v(z)) \, dx \, dy = \sum_{\mu, \nu=1}^n \int_G \partial_{z_\nu} u_\mu(z) \partial_{\bar{z}_\nu} v_\mu(z) \, dx \, dy
$$

becomes a Hilbert space. Let $\Delta = \frac{1}{4} \sum_{\nu=1}^n \partial_{z_\nu} \partial_{\bar{z}_\nu}$ denote the Laplace operator and $\text{div}_z u = \sum_{\nu=1}^n \partial_{z_\nu} u_\nu$ the divergence operator with respect to the variable $z$, both understood in the sense of $\mathcal{D}'(G)$ or in the sense of any duality $\langle \cdot, \cdot \rangle$ which is an extension of the usual scalar product in $L_2$. The set

$$
\hat{S}^1_{\Delta^k,2} = \{ u \in \hat{W}^1_2 : \text{div}_z \Delta^k u = 0 \}
$$

is a closed subspace of $\hat{W}^1_2$ for any integer $k \geq 1$.

**Remark 4.** Obviously $\text{div}_z \Delta^k w = 0$ if and only if $\text{div}_x \Delta^k u - \text{div}_y \Delta^k v = 0$, $\text{div}_y \Delta^k u + \text{div}_x \Delta^k v = 0$ for $w = u + iv$. Hence, the subspaces $\hat{S}^1_{\Delta^k,2}$ are not empty.

As usual for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ the notation $\partial^\alpha_{\bar{z}} = \partial^\alpha_{\bar{z}_1} \ldots \partial^\alpha_{\bar{z}_n}$
is used. Then
\[ W_{\Delta_k}^{2k} = \left\{ p_0 \in W_2^{2k} \left(G; \mathbb{C}\right) : \|p_0\|_{2k}^2 = \sum_{|\alpha| \leq 2k} \left| \partial^\alpha_z p_0(z) \right|^2 \, dx \, dy < \infty, \right. \]
\[ \left. \Delta^k p_0|_{\partial G} = 0, \quad \nabla_z \Delta^k p_0|_{\partial G} = 0, \quad \text{for } 0 \leq \kappa \leq k - 1 \right\} \]
is the space of scalar “potentials”.

**Theorem 2’.** For any integer \( k \geq 1 \) the space \( \tilde{W}_2^1 \) has the orthogonal decomposition
\[ \tilde{W}_2^1 = \tilde{S}_{\Delta_k}^1 \oplus \nabla_z \Delta^{k-1} \tilde{W}_2^1. \]

**Proof.** The proof is like the one of Theorem 2.

(i) If a function \( q \in \tilde{W}_2^1 \) has the form \( q = \nabla_z \Delta^{k-1} p_0 \) for some scalar potential \( p_0 \in \tilde{W}_2^{2k} \), then for any \( \varphi \in \tilde{S}_{\Delta_k}^1 \) one has
\[ \langle \text{div}_z \Delta^k \varphi, p_0 \rangle = -\langle \varphi, \nabla_z \Delta^k p_0 \rangle_0 = \langle \varphi, \nabla_z \Delta^{k-1} p_0 \rangle_1. \]
The condition \( \text{div}_z \Delta^k \varphi = 0 \) then implies that \( \nabla_z \Delta^{k-1} p_0 \) is orthogonal to \( \tilde{S}_{\Delta_k}^1 \).

(ii) Let \( q \in (\tilde{S}_{\Delta_k}^1)^\perp \). Then there exists a unique potential \( p_0 \in \tilde{W}_2^{2k} \) such that \( q = \nabla_z \Delta^{k-1} p_0 \), i.e. the boundary value problem
\[ \nabla_z \Delta^{k-1} p_0 = q, \quad p_0 \in \tilde{W}_2^{2k}, \]
has a unique solution. In fact, repeating the proof of Lemma 1 one can establish that for \( q \in (\tilde{S}_{\Delta_k}^1)^\perp \) this problem is equivalent to the well-posed boundary value problem
\[ \Delta^{2k} p_0 = \text{div}_z \Delta^k q, \quad p_0 \in \tilde{W}_2^{2k}. \]
The unique solution \( p_0 \) of this problem is the desired potential \( p_0 \).

**4. Orthogonality to the solenoidal subspace.** In this section the orthogonal complement of the subspace of all solenoidal functions
\[ \tilde{S}_2^1 = \left\{ u_s \in \tilde{W}_2^1 : \text{div} u_s = 0 \text{ in } D' \right\} \]
in \( \tilde{W}_2^1 \) is found. The subspace \( \tilde{S}_2^1 \) is closed in \( \tilde{W}_2^1 \). In order to describe \( (\tilde{S}_2^1)^\perp \) one needs the operator
\[ \Delta_0^{-1} : W_2^{-1} \rightarrow \tilde{W}_2^1 \]
where \( W_2^{-1} = (\tilde{W}_2^1)^* \) and \( \Delta_0^{-1} \) is the inverse operator to \( \Delta : \tilde{W}_2^1 \rightarrow W_2^{-1} \).

**Lemma 2.** The equations
\[ \Delta_0^{-1} \nabla p_1 = q_1 \]
and
\begin{equation}
\text{div } \Delta^{-1}_0 \nabla p_1 = \text{div } q_1
\end{equation}
are equivalent in $\dot{W}^1_2$ if and only if $q_1 \in (S^1_2)\perp$.

Proof. (i) Applying the div operator to (9) shows that any solution to (9) is a solution of (10).

(ii) Let $p_1 \in L_2(G)$ be a solution to (10). Then
\begin{equation}
\Delta^{-1}_0 \nabla p_1 - q_1 \in \ker(\text{div}) = S^1_2.
\end{equation}
On the other hand for any $p_1 \in L_2$ and any $\varphi \in S^1_2$,
\begin{equation}
\langle \Delta^{-1}_0 \nabla p_1, \varphi \rangle_1 = -\langle \nabla p_1, \varphi \rangle_0 = \langle p_1, \text{div } \varphi \rangle_0 = 0.
\end{equation}
Hence, if $q_1 \in (S^1_2)\perp$ then $\Delta^{-1}_0 \nabla p_1 - q_1 \in (S^1_2)\perp$. Thus $\Delta^{-1}_0 \nabla p_1 = q_1$.

Lemma 3. Equation (10) is solvable for any $q_1 \in \dot{W}^1_2$. More precisely, for any $q_1 \in \dot{W}^1_2$ there exists a unique potential $p_1 \in L_2/\mathbb{C}$ such that (10) holds.

Proof. Identifying as usual the factor space $L_2/\mathbb{C}$ with the subspace of $L_2$ of all functions orthogonal to the unity, a solution to (10) in this subspace is found by the Galerkin method. Let $\{v_\nu : \nu \geq 1\}$ be a basis of $L_2/\mathbb{C}$ with smooth functions $v_\nu$. The approximate solutions $p_1^N, N \geq 1$, are defined as
\begin{equation}
p_1^N = \sum_{\nu=1}^N c_\nu^N v_\nu
\end{equation}
where the unknown coefficients $c_\nu^N, 1 \leq \nu \leq N$, are defined from the Galerkin moment equations
\begin{equation}
\langle \text{div } \Delta^{-1}_0 \nabla p_1^N, v_\nu \rangle_0 = \langle \text{div } q_1, v_\nu \rangle_0, \quad \nu = 1, \ldots, N,
\end{equation}
or, what is the same, from
\begin{equation}
(10_N) \quad \langle \Delta^{-1}_0 \nabla p_1^N, \nabla v_\nu \rangle_0 = -\langle \text{div } q_1, v_\nu \rangle_0, \quad 1 \leq \nu \leq N.
\end{equation}
The solvability of this algebraic linear system follows from the a priori estimate which will be deduced next. Multiplying $(10_N)$ by $c_\nu^N$ and summing up gives
\begin{equation}
\langle \Delta^{-1}_0 \nabla p_1^N, \nabla p_1^N \rangle_0 = -\langle \text{div } q_1, p_1^N \rangle_0
\end{equation}
or
\begin{equation}
\|
abla p_1^N\|^2_1 = -\langle \text{div } q_1, p_1^N \rangle_0.
\end{equation}
Therefore, using the Cauchy–Schwarz–Bunyakovsky inequality and the known inequality
\begin{equation}
\|p_1^N\|_0 \leq M\|
abla p_1^N\|_{-1}
\end{equation}
with some $M > 0$ for $p_1^N \in L_2/C$ (Ladyzhenskaya–Babuška–Brezzi–Nečas, see [12], §1, p. 17) from (11) for $N \geq 1$ the estimate
\begin{equation}
\|\nabla p_1^N\|_{-1} \leq M \|\text{div} q_1\|_0
\end{equation}
follows. By (12), for any $N \geq 1$,
\begin{equation}
\|p_1^N\|_0 \leq M^2 \|\text{div} q_1\|_0.
\end{equation}
In particular the constant $M > 0$ does not depend on $N$. These estimates imply the weak compactness of the sequence $(p_1^N)$ in $L_2/C$. Assuming without loss of generality that the sequence itself converges weakly to some $p_1 \in L_2/C$, then also $(\nabla p_1^N)$ weakly converges in $W_2^{-1}$ to $\nabla p_1$. Obviously, $p_1$ is a solution of (10). The uniqueness is obvious too.

**Theorem 3.** The map
\[
\Delta_0^{-1} \nabla : L_2/C \leftrightarrow (\hat{S}_2^1)^\perp
\]
is an elliptic isomorphism, and
\[
\hat{W}_2^1 = \hat{S}_2^1 \oplus \Delta_0^{-1} \nabla (L_2/C).
\]

**Proof.** It has to be shown that for any $p_1 \in L_2/C$ the image $q_1 \equiv \Delta_0^{-1} \nabla p_1$ is orthogonal to $\hat{S}_2^1$ and that on the other hand for any $q_1 \in (\hat{S}_2^1)^\perp$ there exists a (unique) potential $p_1 \in L_2/C$ such that $\Delta_0^{-1} \nabla p_1 = q_1$. Moreover,
\begin{equation}
\|p_1\|_0 \leq M \|\text{div} q_1\|_0
\end{equation}
for some constant $M \geq 0$.

(i) For any $p_1 \in L_2(G)$ and any $\varphi \in \hat{S}_2^1$,
\[
\langle \Delta_0^{-1} \nabla p_1, \varphi \rangle_1 = -\langle \nabla p_1, \varphi \rangle_0 = \langle p_1, \text{div} \varphi \rangle_0 = 0.
\]
Thus $\Delta_0^{-1} \nabla p_1 \in (\hat{S}_2^1)^\perp$. This argument was already used in the proof of Lemma 2.

(ii) Let $q \in (\hat{S}_2^1)^\perp$. It has to be shown that there exists a function $p_1 \in L_2/C$ satisfying (9) in $\hat{W}_2^1$. Indeed, from Lemma 3 combined with Lemma 2 a solution $p_1 \in L_2/C$ can be obtained. Finally, if $\text{div} q_1 = 0$ then
\[
\text{div} \Delta_0^{-1} \nabla p_1 = 0
\]
immediately yields $\langle \nabla p_1, \nabla p_1 \rangle_{-1} = 0$, so that $\nabla p_1 = 0$ and hence $p_1(x) = 0$. This could also be seen from the estimate (15).

The a priori estimate (15) follows from inequality (13) together with (14).

**Remark 5.** If $u \in L_2(G)$ satisfies $\text{div} u = \text{div} q_1$ then the equation
\[
\text{div} \Delta_0^{-1} \nabla p_1 = \text{div} u
\]
has the same solution $p_1$ for all such $u$. This means that the mapping $u \mapsto p_1$ defines a projection $u \mapsto \Delta_0^{-1} \nabla p_1$ of $\hat{W}_2^1$ onto $(\hat{S}_2^1)^\perp$. 
5. Orthogonal decomposition of $\Delta^k$-solenoidal subspaces. Next a detailed representation of the subspace of the $\Delta$-solenoidal functions and of the $\Delta^k$-solenoidal functions for any positive integer $k$ are given. According to Theorem 2 every function $u \in \hat{S}^1_{\Delta^2} \ominus \hat{S}^1_2$ can be written as $u = \Delta^{-1}_0 \nabla p_1$ for some $p_1 \in L_2/\mathbb{C}$. However, in this case $p_1$ is a harmonic potential, i.e. $\Delta p_1 = 0$ in $\mathcal{D}'$.

**Theorem 4.** Denote the subspace of harmonic potentials in $L_2(G)$ by $H_1$. Then

$$\hat{S}^1_{\Delta^2} = \hat{S}^1_2 \oplus \Delta^{-1}_0 \nabla (H_1/\mathbb{C}).$$

**Proof.** (i) If $p_1 \in H_1 \subset L_2$ then of course $q_1 = \Delta^{-1}_0 \nabla p_1$ is orthogonal to $\hat{S}^1_2$ (see the proof of Theorem 2).

(ii) If $q_1 \in (\hat{S}^1_2)\perp$ and $u$ is a solution of $\text{div} \Delta u = 0$ satisfying $\text{div} u = \text{div} q_1$, then the corresponding potential $p_1$, satisfying $\Delta^{-1}_0 \nabla p_1 = q_1$, is an $L_2$-solution to the equation

$$\text{div} \Delta^{-1}_0 \nabla p_1 = \text{div} u$$

(see Remark 5). Applying the Laplace operator $\Delta$ to this equation leads to $\Delta p_1 = 0$. Hence, $p_1 \in H_1$.

**Definition 1.** A potential $p_0$ is called polyharmonic of order $k \geq 1$ if it satisfies the equation $\Delta^k p_0 = 0$ in $\mathcal{D}'$. The set of all polyharmonic potentials of order $k$ is denoted by $H_k$.

**Theorem 5.** For any positive integer $k$,

$$\hat{S}^1_{\Delta^{k+1,2}} = \hat{S}^1_{\Delta^{k,2}} \oplus \nabla \Delta^{k-1}(\hat{W}^{2k}_{\Delta^k,2} \cap H_{2k+1}).$$

In other words, a function $u \in \hat{W}^1_{\Delta^2}$ belongs to the subspace $\hat{S}^1_{\Delta^{k+1,2}} \ominus \hat{S}^1_{\Delta^{k,2}}$ if and only if it has the form $u = \nabla \Delta^{k-1} p_0$ with a potential $p_0 \in \hat{W}^{2k}_{\Delta^k,2}$ which is polyharmonic of order $2k + 1$, i.e. $\Delta^{2k+1} p_0 = 0$.

**Proof.** (i) For any $p_0 \in \hat{W}^{2k}_{\Delta^k,2} \cap H_{2k+1}$ the function $\nabla \Delta^{k-1} p_0$ is obviously orthogonal to $\hat{S}^1_{\Delta^{k,2}}$.

(ii) Let $u \in \hat{W}^1_{\Delta^2}$ be such that $\text{div} \Delta^k u \neq 0$ but $\text{div} \Delta^{k+1} u = 0$. Then in accordance with Theorem 2,

$$u = u_{\Delta^k,s} + q$$

with $u_{\Delta^k,s} \in \hat{S}^1_{\Delta^{k,2}}$ and $q = \nabla \Delta^{k-1} p_0$ where the potential $p_0 \in \hat{W}^{2k}_{\Delta^k,2}$ is the solution to the problem

$$(16) \quad \Delta^{2k} p_0 = \text{div} \Delta^k u, \quad \Delta^{\nu} p_0 \in \hat{W}^{2}_{\Delta^2}, \quad \nu = 0, 1, \ldots, k - 1.$$  

Applying the Laplace operator $\Delta$ yields $\Delta^{2k+1} p_0 = 0$ in $\mathcal{D}'$. 
Corollary 1. Any function \( u \in \dot{W}^1_2 \) can be written as
\[
  u = u_s + \Delta_0^{-1} p_{-1} + \sum_{\nu=0}^{k-1} \nabla \Delta^\nu p_{\nu} + \nabla \Delta^k q_k
\]
where \( p_{-1} \) is a harmonic potential and \( p_{\nu} \), \( 0 \leq \nu \leq k - 1 \), are polyharmonic potentials of order \( 2\nu + 1 \) and \( q_k \) is the unique solution to (15) where \( k \) is replaced by \( k + 1 \).

6. Harmonic decompositions. For \( p > 1 \) and nonnegative integer \( m \) let
\[
  W^m_p = \left\{ u : G \to \mathbb{C}^n : \|u\|_{m,p}^p = \sum_{|\alpha| \leq m} \int_G |D^\alpha u(x)|^p \, dx < \infty \right\}
\]
be the Sobolev space for a bounded domain \( G \subset \mathbb{R}^n \) with smooth boundary. Further let \( H^m_p \subset W^m_p \) be the subspace of harmonic functions,
\[
  H^m_p = \{ u \in W^m_p : \Delta u = 0 \text{ in } D' \}.
\]

Theorem 6. The space \( W^m_p \) can be decomposed into the direct sum
\[
  W^m_p = H^m_p + \Delta(W^{m+2}_p \cap \dot{W}^2_p),
\]
i.e. \( u = u_h + \Delta p_0 \) with \( u_h \in H^m_p \), \( p_0 \in W^{m+2}_p \cap \dot{W}^2_p \). For \( p = 2 \) this is an orthogonal decomposition.

Proof. For given \( u \in W^m_p \) let \( p_0 \in W^{m+2}_p \cap \dot{W}^2_p \) be the solution of the boundary value problem
\[
  \Delta^2 p_0 = \Delta u \quad \text{in } G, \quad p_0 = 0, \quad \nabla p_0 = 0 \quad \text{on } \partial G.
\]
This is a well-posed boundary value problem in the whole scale \( W^m_p \): for any \( u \in W^m_p \) with \( m \geq 0 \), \( p > 1 \) the problem has a unique solution \( p_0 \in W^{m+2}_p \cap \dot{W}^2_p \). Then
\[
  u = u_h + \Delta p_0
\]
where \( u_h \) is defined by this relation. It is clear that \( u_h \in H^m_p \) because
\[
  \Delta u_h = \Delta u - \Delta^2 p_0 = 0
\]
due to the choice of \( p_0 \).

It remains to show that the decomposition is direct, that is, \( u_h \) and \( p_0 \) are uniquely determined. Let \( 0 = u_h + \Delta p_0 \) with \( u_h \in H^m_p \) and \( p_0 \in W^{m+2}_p \cap \dot{W}^2_p \). Then the function \( v = \Delta p_0 = -u_h \) is harmonic as \( u_h \) is. Thus \( \Delta v = \Delta^2 p_0 = 0 \). As \( p_0 \in \dot{W}^2_p \), i.e. \( p_0 = 0 \) and \( \nabla p_0 = 0 \) on \( \partial G \), it follows that \( p_0 = 0 \). Hence, also \( u_h = 0 \). This shows the sum is direct.
Remark 6. From the boundary value problem one has the a priori estimates
\[ \| p_0 \|_{W^{m+2}_p} \leq M \| \Delta u \|_{W^{m-2}_p}. \]

Example. For \( p = 2 \) and \( m = 0 \) the last decomposition reads
\[ L_2 = H_2 \oplus \Delta W^2_2. \]

This orthogonal decomposition will be applied to solving the following variational problem.

Variational Problem. Let \( 0 < a_0 \leq A_0 \) be constants and \( a \) be measurable in \( G \) such that
\[ a_0 \leq a(x) \leq A, \quad x \in G. \]
Moreover, let \( h \in L_2 \). Determine
\[ \inf_{u \in H_2} \Re \int_G \left[ \frac{1}{2} a(x) |u(x)|^2 - h(x) u(x) \right] dx. \]

If \( u \in H_2 \) is minimal then
\[ \int_G \left[ a(x) u(x) - h(x) \overline{\varphi(x)} \right] dx = 0 \]
for all \( \varphi \in H_2 \). In accordance with the decomposition (17) this leads to the next problem.

Problem. Find \( u \in H_2 \) and \( v \in W^2_2 \) such that \( au + \Delta v = h \) in the sense of \( L_2 \) and \( \Delta u = 0 \).

Proposition 1. For any \( h \in L_2 \) there exists a unique pair \((u, v)\) solving the above Problem.

Proof. (i) In order to find \( u \in H_2 \) such that
\[ \langle au, \varphi \rangle_0 = \langle h, \varphi \rangle \quad \text{for all } \varphi \in H_2 \]
the standard Galerkin method is used. Let \( \{ v_k : k \in \mathbb{N} \} \) be a basis of \( H_2 \).
An approximate solution
\[ u^N = \sum_{j=1}^{N} c^N_j v_j, \quad N = 1, 2, \ldots, \]
is determined from the Galerkin moment equations
\[ \langle au^N, v_j \rangle_0 = \langle h, v_j \rangle_0, \quad 1 \leq j \leq N, \]
fixing the coefficients \( c^N_j \), \( 1 \leq j \leq N \). Multiplying (19) by \( c^N_j \) and summing up gives
\[ \langle au^N, u^N \rangle_0 = \langle h, u^N \rangle_0, \]
leading to
\[ a_0 \| u^N \|^2_0 \leq \langle h, u^N \rangle_0 \leq \| h \|_0 \| u^N \|_0 \]
and hence to the a priori estimate
\[ \| u^N \|_0 \leq \frac{1}{a_0} \| h \|_0, \quad N = 1, 2, \ldots \]
Therefore the sequence \((u^N)\) of approximate solutions has at least one weak limit point \(u \in H_2\). This function \(u\) is a solution of the Problem.

(ii) The function \(v \in W_2^{+}\) is found from the decomposition (17). Identity (18) means \(au - h \in H_2^{-}\) in \(L_2\). Then from (17) it is seen that there exists a unique \(v \in W_2^{+}\) such that
\[ au + \Delta v = h \quad \text{in} \quad L_2. \]
To prove the uniqueness of this solution to the Problem, let \(h = 0\). Multiplying \(au + \Delta v = 0\) by \(u \in H_2\) and integrating over \(G\) gives \(\langle au, u \rangle_0 = 0\) because \(\Delta v\) is orthogonal to \(u\) in \(L_2\). Thus \(u = 0\) as also follows directly from the a priori estimate. Hence, \(v = 0\) follows.

7. Applications. The spaces \(W_2^{+}\) and \(W_2^{-}\) are connected by the one-to-one map
\[ \Delta : W_2^{+} \leftrightarrow W_2^{-}. \]
Hence, in the sense of the inner product
\[ \langle u, v \rangle_{-1} = \langle \Delta^{-1} u, v \rangle_0 = \langle u, \Delta^{-1} v \rangle_0 \]
the orthogonal decompositions of the space \(W_2^{-}\) are automatically obtained from those of \(W_2^{+}\). In particular, according to Section 2, one has the chain of decompositions
\[ W_2^{-} = \Delta S_\Delta^{-1,2} \oplus \nabla \Delta k W_\Delta^{2k,2}, \quad k = 1, 2, \ldots \]
Thus, any function \(h \in W_2^{-}\) can be represented as
\[ h = \Delta u_{\Delta k,s} + \nabla \Delta k p_k, \]
where \(\text{div} \Delta k u_{\Delta k,s} = 0\) in \(\mathcal{D}'\) and \(p_k \in W_\Delta^{2k,2}\). This suggests the following problem.

**Boundary Value Problem 1.** Let \(h \in W_2^{-}\). Find functions \(u_{\Delta k,s}, p_k\) satisfying
\[ \Delta u_{\Delta k,s} + \nabla \Delta k p_k = h, \quad \text{div} \Delta k u_{\Delta k,s} = 0 \quad \text{in} \quad G, \]
\[ u_{\Delta k,s}|_{\partial G} = 0, \quad \Delta^\nu p_k|_{\partial G} = 0, \quad \nabla \Delta^\nu p_k|_{\partial G} = 0, \quad \nu = 0, 1, \ldots, k - 1. \]
For any \(k\) this problem is well-posed and the solution \(u_{\Delta k,s}, p_k\) can be found from the decompositions given in Section 2.
In connection with Sections 4 and 5, one also has the chain of decompositions
\[
W^{-1} \rightarrow 2 = \Delta \tilde{S} + \nabla [H_1 / C \oplus \Delta \tilde{W}^2_2],
\]
\[
W^{-1} \rightarrow 2 = \Delta \tilde{S} + \nabla [H_1 / C + \Delta \tilde{W}^2_2 \cap H_3 \oplus \Delta \tilde{W}^4_{\Delta 2, 2}]
\]
etc. Hence, any \( h \in W^{-1} \) can be represented as
\[
h = \Delta u_s + \nabla (p_{-1} + \Delta p_0),
\]
\[
h = \Delta u_s + \nabla [p_{-1} + \Delta (p_0 + \Delta p_1)]
\]
etc. Any of these representations again corresponds to a boundary value problem.

**Boundary Value Problem 2.** For given \( h \in W^{-1} \) find three functions \( u_s, p_{-1}, p_0 \) such that
\[
\Delta u_s + \nabla (p_{-1} + \Delta p_0) = h, \quad \text{div} \, u_s = 0, \quad \Delta p_{-1} = 0 \quad \text{in } G,
\]
\[
u_s|_{\partial G} = 0, \quad p_0|_{\partial G} = 0, \quad \nabla p_0|_{\partial G} = 0.
\]

**Boundary Value Problem 3.** For given \( h \in W^{-1} \) find four functions \( u_s, p_{-1}, p_0, p_1 \) satisfying
\[
\Delta u_s + \nabla (p_{-1} + \Delta p_0 + \Delta^2 p_1) = h,
\]
\[
\text{div} \, u_s = 0, \quad \Delta p_{-1} = 0, \quad \Delta^3 p_0 = 0 \quad \text{in } G,
\]
\[
u_s|_{\partial G} = 0, \quad p_0|_{\partial G} = 0, \quad \nabla p_0|_{\partial G} = 0,
\]
\[
p_1|_{\partial G} = 0, \quad \nabla p_1|_{\partial G} = 0, \quad \Delta p_1|_{\partial G} = 0, \quad \nabla \Delta p_1|_{\partial G} = 0.
\]

All these problems are well-posed. The solutions can be found by using the decompositions of \( W^{-1} \) given in (20), (21). Obviously, these problems are some versions of the classical Stokes problem
\[
\Delta u_s + \nabla p = h, \quad \text{div} \, u_s = 0 \quad \text{in } G,
\]
\[
u_s|_{\partial G} = 0,
\]
revealing the structure of the potential \( p \in L_2 / C \) in more detail. Contrary to the classical procedure of finding the divergence part \( u_s \), here the potential is found first e.g. in the form \( p = p_{-1} + \Delta p_0 \) by solving the problems
\[
\Delta^2 p_0 = \text{div} \, h \quad \text{in } G, \quad p_0 \in \tilde{W}^2_2,
\]
and
\[
\text{div} (\Delta^{-1}_0 p_{-1}) = \text{div} (\Delta^{-1}_0 h) \quad \text{in } G, \quad p_1 \in L_2 / C.
\]
In the second step the divergence part \( u_s \) can be found as the solution to the Dirichlet problem
\[
\Delta u_s = h - \nabla p \quad \text{in } G, \quad u_s|_{\partial G} = 0
\]
(compare [4, 5]).
REFERENCES


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