COLLOQUIUM MATHEMATICUM

VOL. 89 2001 NO. 2

ON COMPACT ASTHENO-KÄHLER MANIFOLDS

BY

KOJI MATSUO (Ichinoseki) and TAKAO TAKAHASHI (Hirosaki)

Abstract. We prove that every compact balanced astheno-Kähler manifold is Kähler, and that there exists an astheno-Kähler structure on the product of certain compact normal almost contact metric manifolds.

1. Introduction. A complex m-dimensional Hermitian manifold M endowed with the Kähler form Ω is called an *astheno-Kähler manifold* if $\Omega^{m-2} = \Omega \wedge \ldots \wedge \Omega$ ($m-2$ times) is pluriharmonic, that is, $\partial\overline{\partial}\Omega^{m-2} = 0$, where ∂ and $\overline{\partial}$ are the complex exterior differentials (cf. [7], [9]). S.-T. Yau says in Open Problem 93 of [13] that such a manifold seems to be particularly interesting for many analytic arguments to be useful. For example, it is known that every holomorphic 1-form on a compact astheno-Kähler manifold is closed. So a compact complex parallelizable manifold cannot be astheno-Kähler unless it is a complex torus, because over such a manifold, there exist, by definition, m linearly independent global holomorphic 1-forms (cf. [15]). On the other hand, it is well known (Boothby [3]) that a complex parallelizable manifold has a natural Hermitian-flat metric, and the existence of a Hermitian-flat metric is a somewhat weaker property than complex parallelizability. Li, Yau, and Zheng [9] conjecture that compact non-Kähler Hermitian-flat manifolds or similarity Hopf manifolds of complex dimension ≥ 3 do not admit any astheno-Kähler metrics.

In this paper, we prove the following theorem.

THEOREM 1.1. *Every compact balanced astheno-Kähler manifold is Kähler.*

A Hermitian manifold is said to be *balanced* if the torsion 1-form of its Hermitian connection vanishes everywhere. As a corollary, since a compact Hermitian-flat manifold is balanced, we have

COROLLARY 1.1. *Every compact Hermitian-flat astheno-Kähler manifold is Kähler.*

2000 *Mathematics Subject Classification*: 53C55, 53C15, 53C25.

Key words and phrases: astheno-Kähler manifolds, Hermitian connection, Sasakian manifolds, cosymplectic manifolds.

The only known examples of compact astheno-Kähler manifolds are trivial ones (cf. [9]). It is also known (cf. [12]) that any product of normal almost contact manifolds is a complex manifold. Another purpose of this paper is to prove that there exists a non-trivial astheno-Kähler structure on the product of certain compact normal almost contact metric manifolds. Consequently, by means of the result of the second author [14], we also know that $S^3 \times S^3$ admits many astheno-Kähler structures.

Throughout this paper, we always assume the differentiability of class C^∞, and manifolds to be connected and without boundary.

2. Hermitian connections and curvatures. Let M be a complex manifold of complex dimension $m \geq 3$ with the complex structure J. A *Hermitian metric* g on M is a Riemannian metric such that $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M. The triple (M, J, g) is called a *Hermitian manifold*. The *Kähler form* Ω of (M, J, g) is defined by $\Omega(X, Y) = g(X, JY)$ for all vector fields X, Y on M. It is well known (cf. [5]) that every Hermitian manifold (M, J, g) admits a unique linear connection D such that $DJ = 0$, $Dg = 0$, and the torsion tensor field T satisfies $T(JX, Y) = JT(X, Y)$ for all vector fields X, Y on M. This connection D is called the *Hermitian connection* of (M, J, g). The curvature tensor field H of the Hermitian connection D, called the *Hermitian curvature tensor field*, is defined by

$$H(X, Y) = [D_X, D_Y] - D_{[X,Y]}$$

for all vector fields X, Y on M.

LEMMA 2.1 (cf. [11]). *The Hermitian curvature tensor field H satisfies the following equations: For all vector fields X, Y, Z, W on M,*

$$g(H(X,Y)Z,W) = -g(H(Y,X)Z,W) = -g(H(X,Y)W,Z),$$
$$H(JX,JY)Z = H(X,Y)Z, \quad H(X,Y)JZ = JH(X,Y)Z,$$

(first Bianchi's identity)

$$\mathbf{Cyc}_{X,Y,Z}[H(X,Y)Z] = \mathbf{Cyc}_{X,Y,Z}[T(T(X,Y),Z) + (D_X T)(Y,Z)],$$

where $\mathbf{Cyc}_{X,Y,Z}$ denotes the cyclic sum over X, Y, and Z.

A Hermitian manifold (M, J, g) is said to be *Hermitian-flat* if the Hermitian curvature tensor field H vanishes everywhere on M.

We define three tensor fields S_i $(i = 1, 2, 3)$ which are analogous to the Ricci tensor field in Kähler geometry:

$$S_1(X,Y) = \tfrac{1}{2}\operatorname{trace}[Z \mapsto H(X,JY)JZ],$$
$$S_2(X,Y) = \tfrac{1}{2}\, g(\operatorname{trace} H^*(X), Y),$$
$$S_3(X,Y) = \tfrac{1}{2}\operatorname{trace}[Z \mapsto H(Z,X)Y + H(Z,Y)X]$$

for all vector fields X, Y on M, where $H^*(X) : (Z, W) \to H(Z, JW)JX$. Then, by Lemma 2.1, we have

LEMMA 2.2 (cf. [11]). *The Ricci-type tensor fields* S_i $(i = 1, 2, 3)$ *defined above are symmetric and compatible with* J, *i.e.*,

$$S_i(X, Y) = S_i(Y, X), \quad S_i(JX, JY) = S_i(X, Y)$$

for all vector fields X, Y *on* M.

We moreover define two *scalar curvatures* s and $\widehat{s}$ which are analogous to the scalar curvature in Kähler geometry:

$$s = \operatorname{trace} S_1 = \operatorname{trace} S_2, \qquad \widehat{s} = \operatorname{trace} S_3.$$

On a compact Hermitian manifold (M, J, g), it is well known (Gauduchon [5]) that

$$s - \widehat{s} = \delta\tau + \|\tau\|^2, \tag{2.1}$$

where τ denotes the *torsion* 1-*form* defined by $\tau(X) = \operatorname{trace}[Y \mapsto T(X, Y)]$, δ the codifferential, i.e., $\delta = - * d *$, and $\|\tau\|$ the g-norm of τ.

3. Proof of Theorem 1.1. We shall use the following real differential operator d^c (cf. [1]) to judge whether the manifolds are astheno-Kähler. Extend the complex structure J to p-forms φ on M as follows:

$$J\varphi = \varphi \quad \text{for } p = 0,$$
$$J\varphi(X_1, \ldots, X_p) = (-1)^r \varphi(JX_1, \ldots, JX_p) \quad \text{for } p > 0,$$

where $X_1, \ldots, X_p$ are vector fields on M. Then the operator d^c is given by

$$d^c\varphi = -J^{-1}dJ\varphi = (-1)^p JdJ\varphi \quad \text{for any } p\text{-form } \varphi \text{ on } M.$$

It is well known that $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$. Therefore an astheno-Kähler manifold may be defined as follows.

DEFINITION 3.1. A complex m-dimensional Hermitian manifold (M, J, g) endowed with the Kähler form Ω is called an *astheno-Kähler manifold* if $dd^c\Omega^{m-2} = 0$.

With the help of the Kähler form Ω, we have two linear operators L and Λ acting on forms. L is defined by $L\varphi = \Omega \wedge \varphi$ for any form φ, and Λ is the adjoint operator of L with respect to the global scalar product defined on M by $\langle \varphi, \psi \rangle = p! \int_M (\varphi, \psi)\, v_g$ for any p-forms φ, ψ on M, where (φ, ψ) is the pointwise g-scalar product, and v_g is the volume element of g. Then Λ can be locally written as follows: For any p-form φ on M,

$$\Lambda\varphi = \begin{cases} 0 & \text{for } p = 0, 1, \\ \dfrac{p!}{(p-2)!} \displaystyle\sum_{\alpha=1}^{2m} i(e_\alpha) i(Je_\alpha)\varphi & \text{for } p \geq 2, \end{cases}$$

where $i(X)$ denotes the interior product by X, and $\{e_\alpha\}_{\alpha=1}^{2m}$ is a local adapted g-orthonormal frame field of M such that $e_{m+j} = Je_j$ for $j = 1, \ldots, m$. For any p-form φ, we have

$$\Lambda L\varphi = L\Lambda\varphi + 4(m-p)\varphi. \tag{3.1}$$

Moreover, we inductively obtain

$$\Lambda L^k\varphi = L^k\Lambda\varphi + 4k(m-p-k+1)L^{k-1}\varphi. \tag{3.2}$$

LEMMA 3.1. *Let r be a positive integer such that $r \leq k$. Then, for any p-form φ on M,*

$$\Lambda^r L^k\varphi = L^k\Lambda^r\varphi + \sum_{i=1}^{r} 4^i (i!)^2 \binom{k}{i}\binom{r}{i}\binom{m-p-k+r}{i} L^{k-i}\Lambda^{r-i}\varphi,$$

where $\binom{r}{i}$ is a binomial coefficient.

Proof. This is easily proved by induction on r. ∎

In particular, we have, from Lemma 3.1,

$$\Lambda^k L^k\varphi = L^k\Lambda^k\varphi + \sum_{i=1}^{k} 4^i\,(i!)^2 \binom{k}{i}\binom{k}{i}\binom{m-p}{i} L^{k-i}\Lambda^{k-i}\varphi.$$

Moreover, we obtain

$$\begin{aligned} \Lambda^{k+3}L^k\varphi &= L^k\Lambda^{k+3}\varphi + \sum_{i=1}^{k-1} 4^i (i!)^2 \binom{k}{i}\binom{k+3}{i}\binom{m-p+3}{i} L^{k-i}\Lambda^{k+3-i}\varphi \\ &\quad + 4^{k+3}k!\,\frac{(k+3)!}{3!}\binom{m-p+3}{k}\Lambda^3\varphi. \end{aligned} \tag{3.3}$$

Now, if $m > 3$, then

$$\begin{aligned} dd^c\Omega^{m-2} &= (m-2)d[d^c\Omega \wedge \Omega^{m-3}] \\ &= (m-2)[dd^c\Omega \wedge \Omega^{m-3} - d^c\Omega \wedge d\Omega^{m-3}] \\ &= (m-2)[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega] \wedge \Omega^{m-4} \\ &= (m-2)L^{m-4}[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega]. \end{aligned} \tag{3.4}$$

By (3.3) and the fact that $\Lambda^r[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega] = 0$ for $r > 3$, we then get

$$\begin{aligned} \Lambda^{m-1}dd^c\Omega^{m-2} \\ = 4^{m-1}\frac{(m-1)!}{3!}(m-2)\,\Lambda^3[dd^c\Omega \wedge \Omega + (m-3)d\Omega \wedge d^c\Omega]. \end{aligned} \tag{3.5}$$

On the other hand, by the direct calculation using the Hermitian connection D, we have the following lemma.

Lemma 3.2. *On a Hermitian manifold* (M, J, g),

$$d\Omega(X,Y,Z) = \tfrac{1}{3}\,\mathbf{Cyc}_{X,Y,Z}[\Omega(T(X,Y),Z)], \tag{3.6}$$

$$d^{c}\Omega(X,Y,Z) = -\tfrac{1}{3}\,\mathbf{Cyc}_{X,Y,Z}[g(T(X,Y),Z)], \tag{3.7}$$

$$\begin{aligned} dd^{c}\Omega(X,Y,Z,W) = -\tfrac{1}{6}\,\mathbf{Cyc}_{X,Y,Z}[&g(T(X,Y),T(Z,W)) \\ &+ g(H(X,Y)Z,W) + g(H(Z,W)X,Y)] \end{aligned} \tag{3.8}$$

for all vector fields X, Y, Z, W *on* M.

Then, by means of Lemmas 2.1, 2.2, 3.2, and (2.1), we obtain

$$\Lambda^3(dd^{c}\Omega\wedge\Omega) = 96(m-2)\,[2(\delta\tau+\|\tau\|^2)-\|T\|^2],$$
$$\Lambda^3(d\Omega\wedge d^{c}\Omega) = 96\,[\|T\|^2-2\|\tau\|^2].$$

Therefore, by (3.5), we conclude

Lemma 3.3. *On a Hermitian manifold* (M, J, g) *of* $\dim_{\mathbb{C}} M = m > 3$,

$$\Lambda^{m-1}dd^{c}\Omega^{m-2} = 4^{m-3}(m-1)!(m-2)[2(m-2)\delta\tau+2\|\tau\|^2-\|T\|^2].$$

Similarly, we have

Lemma 3.4. *On a Hermitian manifold* (M, J, g) *of* $\dim_{\mathbb{C}} M = m = 3$,

$$\Lambda^2 dd^{c}\Omega^{m-2} = \Lambda^2 dd^{c}\Omega = 8[2(\delta\tau+\|\tau\|^2)-\|T\|].$$

Let (M, J, g) be a compact Hermitian manifold of complex dimension $m \geq 3$. Integrate the equality in Lemma 3.3 or Lemma 3.4 under the conditions $\tau = 0$ and $dd^{c}\Omega^{m-2} = 0$. Then we conclude that $T = 0$. This completes the proof of Theorem 1.1.

4. Examples of compact astheno-Kähler manifolds. Let M be a $(2n+1)$-dimensional almost contact manifold with the structure tensor fields (ϕ, ξ, η), that is, ϕ is a $(1,1)$-tensor field, η a 1-form, and ξ a vector field on M such that

$$\eta(\xi) = 1, \qquad \phi^2 = -I + \eta\otimes\xi,$$

where I denotes the identity transformation of the tangent spaces and $n \geq 1$. An almost contact structure (ϕ, ξ, η) is said to be *normal* (cf. [2]) if $[\phi,\phi] + 2d\eta\otimes\xi = 0$, where $[\phi,\phi]$ denotes the Nijenhuis tensor field of ϕ, i.e., $[\phi,\phi](X,Y) = [\phi X,\phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X, Y]$ for all vector fields X, Y on M. A Riemannian metric g on M is said to be *compatible* if

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M. An almost contact manifold M with a compatible Riemannian metric g is said to have an *almost contact metric structure* (ϕ, ξ, η, g). It is known that there always exists an almost contact metric structure on an almost contact manifold. The *fundamental 2-form* Φ

on an almost contact metric manifold M is defined by

$$\Phi(X,Y) = g(X,\phi Y)$$

for all vector fields X, Y on M. Then we have $\eta \wedge \Phi^n \neq 0$. If $\Phi = d\eta$, then M is, by definition, a *contact manifold*. Such an almost contact metric structure is called a *contact metric structure*. Moreover, if a contact metric structure is normal, then it is called a *Sasakian structure*. It is well known (cf. [2]) that there is the standard Sasakian structure on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. On the other hand, if $d\Phi = 0$ and $d\eta = 0$, then M is said to have an *almost cosymplectic structure*. Moreover, if an almost cosymplectic structure is normal, then it is called a *cosymplectic structure*. If N is a compact Kähler manifold, then $N \times S^1$ is the trivial example of compact cosymplectic manifolds. The non-trivial examples of compact cosymplectic manifolds are found in [4] and [10].

Let M_i $(i = 1, 2)$ be a $(2m_i + 1)$-dimensional compact normal almost contact metric manifold with the structure tensor fields (ϕ_i, ξ_i, η_i). On the product manifold $M = M_1 \times M_2$, we consider an almost complex structure J defined by

$$J = \phi_1 - \eta_2 \otimes \xi_1 + \phi_2 + \eta_1 \otimes \xi_2 \quad \text{(see [12])}.$$

This almost complex structure J is integrable since each almost contact structure is normal. Thus M endowed with J is a compact complex manifold of complex dimension $m = m_1 + m_2 + 1$. Moreover, if g_i is the compatible Riemannian metric on M_i for each $i = 1, 2$, then the Riemannian product metric $g = g_1 + g_2$ on M is compatible with J, that is, g is a Hermitian metric on M. Then its Kähler form Ω is given by

$$\Omega = \Phi_1 + \Phi_2 - 2\eta_1 \wedge \eta_2, \tag{4.1}$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$.

Now, in the case where M_1 and M_2 are more special, we investigate the Hermitian structure (4.1) of $M = M_1 \times M_2$.

THEOREM 4.1. *Let (M_i, g_i) be a 3-dimensional compact Sasakian manifold for each $i = 1, 2$. Then the product manifold $M = M_1 \times M_2$ with the Hermitian structure* (4.1) *is astheno-Kähler.*

Proof. Since M_1 and M_2 are both Sasakian, we have

$$d\Omega = -2(\Phi_1 \wedge \eta_2 - \Phi_2 \wedge \eta_1),$$

and

$$d^c\Omega = Jd\Omega = -2(J\Phi_1 \wedge J\eta_2 - J\Phi_2 \wedge J\eta_1) = 2(\Phi_1 \wedge \eta_1 + \Phi_2 \wedge \eta_2).$$

Here we used the fact that Ω and Φ_i are J-invariant, and $J\eta_1 = \eta_2$, $J\eta_2 = -\eta_1$. Thus we get

$$dd^c\Omega = 2(\Phi_1^2 + \Phi_2^2).$$

Since $\dim_{\mathbb{C}} M = m = 3$, i.e., $\dim M_i = m_i = 1$ for each $i = 1, 2$, $\Phi_i^2 = 0$ on M_i, and hence we obtain

$$dd^{\mathrm{c}}\Omega^{m-2} = dd^{\mathrm{c}}\Omega = 2(\Phi_1^2 + \Phi_2^2) = 0 \quad \text{on } M.$$

Therefore we conclude that the Hermitian structure (4.1) on M is astheno-Kähler. ∎

REMARK 4.1. Let (M_i, g_i) be a Sasakian manifold for each $i = 1, 2$. If $\dim_{\mathbb{C}} M = m > 3$, then

$$\begin{aligned} dd^{\mathrm{c}}\Omega^{m-2} &= (m-2)[dd^{\mathrm{c}}\Omega \wedge \Omega + (m-3)d\Omega \wedge d^{\mathrm{c}}\Omega] \wedge \Omega^{m-4} \\ &= 2(m-2)(\Phi_1^2 + \Phi_2^2) \wedge [\Phi_1 + \Phi_2 + 2(m-4)\eta_1 \wedge \eta_2] \wedge \Omega^{m-4} \\ &= 2(m-2)(\Phi_1^2 + \Phi_2^2) \wedge (\Phi_1 + \Phi_2)^{m-3} \\ &= 2(m-2)\sum_{k=0}^{m-1} C(m,k)\, \Phi_1^{(m-1)-k} \wedge \Phi_2^k, \end{aligned}$$

where $C(m,k)$ are given as follows:

$$C(m,0) = C(m,m-1) = 0, \qquad C(m,1) = C(m,m-2) = m-3,$$
$$C(m,k) = \binom{m-3}{k} + \binom{m-3}{k-2} \quad \text{for } 2 \le k \le m-3.$$

Since $\Phi_i^p = 0$ on M_i for $p > m_i$, $\Phi_1^{(m-1)-k} = 0$ on M_1 if $0 \le k < m_2$, and $\Phi_2^k = 0$ on M_2 if $m_2 < k \le m-1$. Thus

$$\Phi_1^{(m-1)-k} \wedge \Phi_2^k = 0 \quad \text{on } M \text{ if } k \neq m_2,$$

and hence we obtain

$$dd^{\mathrm{c}}\Omega^{m-2} = 2(m-2)C(m,m_2)\Phi_1^{m_1} \wedge \Phi_2^{m_2} \neq 0 \quad \text{on } M.$$

Therefore if $m > 3$, then the Hermitian structure (4.1) on M is not astheno-Kähler.

THEOREM 4.2. *Let (M_1, g_1) be a 3-dimensional compact Sasakian manifold, and (M_2, g_2) a compact cosymplectic manifold of dimension ≥ 3. Then the product manifold $M = M_1 \times M_2$ with the Hermitian structure* (4.1) *is astheno-Kähler.*

Proof. Since M_1 is Sasakian and M_2 cosymplectic, we have

$$d\Omega = -2\Phi_1 \wedge \eta_2 \quad \text{and} \quad d^{\mathrm{c}}\Omega = 2\Phi_1 \wedge \eta_1.$$

Thus we get

$$dd^{\mathrm{c}}\Omega = 2\Phi_1^2.$$

Since $m_1 = 1$, $\Phi_1^2 = 0$ on M_1, that is, $dd^{\mathrm{c}}\Omega = 0$ and $d\Omega \wedge d^{\mathrm{c}}\Omega = 0$ on M, and hence we obtain

$$dd^{\mathrm{c}}\Omega \wedge \Omega + (m-3)d\Omega \wedge d^{\mathrm{c}}\Omega = 0.$$

Therefore, by (3.4), we conclude that the Hermitian structure (4.1) on M is astheno-Kähler. ■

REMARK 4.2. Let (M_1, g_1) be a Sasakian manifold of dimension > 3, and (M_2, g_2) a cosymplectic manifold. Then

$$\begin{aligned} dd^c \Omega^{m-2} &= 2(m-2)\Phi_1^2 \wedge (\Phi_1 + \Phi_2)^{m-3} \\ &= 2(m-2) \sum_{k=0}^{m-3} \binom{m-3}{k} \Phi_1^{(m-1)-k} \wedge \Phi_2^k. \end{aligned}$$

Since $m-3 \geq m_2$, $\Phi_1^{(m-1)-k} = 0$ on M_1 if $0 \leq k < m_2$, and $\Phi_2^k = 0$ on M_2 if $m_2 < k \leq m-3$. Thus

$$\Phi_1^{(m-1)-k} \wedge \Phi_2^k = 0 \quad \text{on } M \text{ if } k \neq m_2,$$

and hence we obtain

$$dd^c \Omega^{m-2} = 2(m-2) \binom{m-3}{m_2} \Phi_1^{m_1} \wedge \Phi_2^{m_2} \neq 0 \quad \text{on } M.$$

Therefore if $m_1 > 1$, then the Hermitian structure (4.1) on M is not astheno-Kähler.

REFERENCES

[1] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.

[2] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer, Berlin, 1976.

[3] W. M. Boothby, *Hermitian manifolds with zero curvature*, Michigan Math. J. 5 (1958), 229–233.

[4] A. Fujimoto and H. Mutō, *On cosymplectic manifolds*, Tensor (N.S.) 28 (1974), 43–52.

[5] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267 (1984), 495–518.

[6] S. I. Goldberg, *Curvature and Homology*, Academic Press, New York, 1962.

[7] J. Jost and S.-T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. 170 (1993), 221–254.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience, 1969.

[9] J. Li, S.-T. Yau and F. Zheng, *On projectively flat hermitian manifolds*, Comm. Anal. Geom. 2 (1994), 103–109.

[10] J. C. Marrero and E. Padron, *New examples of compact cosymplectic solvmanifolds*, Arch. Math. (Brno) 34 (1998), 337–345.

[11] K. Matsuo, *On local conformal Hermitian-flatness of Hermitian manifolds*, Tokyo J. Math. 19 (1996), 499–515.

[12] A. Morimoto, *On normal almost contact structures*, J. Math. Soc. Japan 15 (1963), 420–436.

[13] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, Conf. Proc. Lecture Notes Geom. Topology 1, International Press, 1994.
[14] T. Takahashi, *Deformations of Sasakian structures and its application to the Brieskorn manifolds*, Tôhoku Math. J. 30 (1978), 37–43.
[15] H. C. Wang, *Complex parallelisable manifolds*, Proc. Amer. Math. Soc. 5 (1954), 771–776.

Department of Mathematics
Ichinoseki National College of Technology
Ichinoseki 021-8511, Japan
E-mail: matsuo@ichinoseki.ac.jp

Department of Mathematics
Hirosaki University
Hirosaki 036-8560, Japan
E-mail: takao@cc.hirosaki-u.ac.jp

Received 2 January 2001 (4013)