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ON UNIFORM DIMENSIONS OF FINITE GROUPS

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Abstract. Let G be any finite group and $L(G)$ the lattice of all subgroups of G. If $L(G)$ is strongly balanced (globally permutable) then we observe that the uniform dimension and the strong uniform dimension of $L(G)$ are well defined, and we show how to calculate these dimensions.

1. Strongly balanced lattices. All lattices considered in this paper are finite. We denote by 0 and 1 the least and the greatest element of the lattice respectively. We will also use some other notation and terminology about lattices, as for example in [2, 9]. However we will change the terminology proposed in [7] and used in [1] to the form as in [6]. The latter is more convenient for lattices of subgroups.

Let L be a lattice. As in [6] we will say that L is *balanced* (permutable in [1, 7]) if for all $x, y, z \in L$ we have

$$x \wedge y = 0 \ \& \ (x \vee y) \wedge z = 0 \ \Rightarrow \ (y \vee z) \wedge x = 0 \ \ \& \ \ (z \vee x) \wedge y = 0$$

and, consequently, L is *strongly balanced* (globally permutable in [1, 7]) if all nonempty intervals of L are balanced.

Clearly any sublattice with 0 of a balanced lattice is balanced and any sublattice of a strongly balanced lattice is strongly balanced. Furthermore, modular lattices are always strongly balanced, but not conversely (see Sections 2 and 3 below).

If $a, u \in L$ then, as in [4, 7], we will say that a is *essential in L* if $a \wedge x \neq 0$ for every $0 \neq x \in L$, and u is *uniform in L* if $u \neq 0$ and every element from $(0, u]$ is essential in $[0, u]$. It is obvious that $1 \in L$ is always essential and any atom in L is a uniform element.

Let $X = \{x_1, \ldots, x_n\} \subset L \setminus \{0\}$. Then, as in [7], we will say that X is *independent* if for every $1 \leq i \leq n$ we have $x_i \wedge \bigvee_{k \neq i} x_k = 0$, and a subset $B \subset L$ will be called a *base* of L if any element of B is uniform and B is a maximal independent subset of L. If B is a base of L then the cardinality of B will be called the *uniform dimension* of L and denoted by $\mathrm{u}(L)$.

Because our lattices are finite, below any nontrivial element there exists a uniform element, for example an atom. Hence from [7] we have:

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THEOREM 1.1. *Let L be a finite balanced lattice.*

(a) *There exists a base in L.*

(b) *Every independent set of uniform elements in L can be extended to a base of L.*

(c) *Any two bases of L have the same cardinality. Hence* $\mathrm{u}(L)$ *is well defined.*

One can observe that for nonbalanced lattices the uniform dimension cannot be well defined. Examples of such lattices are presented in [1, 7] (see also Example 2.2).

It is easy to verify the following lemma (see [7]) which for the modular case was proved in [3].

LEMMA 1.2. *Let L be a balanced lattice and let $a, b \in L$. If $a \wedge b = 0$ and $a \vee b$ is essential in L, then* $\mathrm{u}(L) = \mathrm{u}([0,a]) + \mathrm{u}([0,b])$. *If $K \subseteq L$ is a sublattice with the same* 0 *then* $\mathrm{u}(K) \leq \mathrm{u}(L)$.

Now let L be a strongly balanced lattice. Then the smallest number α such that $\mathrm{u}([a,b]) \leq \alpha$ for every nontrivial interval $[a,b] \subseteq L$ will be called the *strong uniform dimension* of L (global uniform dimension of L in [7]), and will be denoted by $\mathrm{u_s}(L)$.

Let L be a strongly balanced lattice. Because L is finite, each of its nonempty intervals is balanced and finite. Hence, from Theorem 1.1, we know that its uniform dimension exists. This means that for L the strong uniform dimension is well defined and we have

$$\mathrm{u_s}(L) = \sup_{a \in L} \mathrm{u}([a,1]).$$

From the facts mentioned above one can see that the uniform dimension and the strong uniform dimension are well connected with some algebraic operations on lattices. For example we have

PROPOSITION 1.3. *Let L be a strongly balanced lattice. If $K \subseteq L$ is any sublattice then* $\mathrm{u_s}(K) \leq \mathrm{u_s}(L)$.

PROPOSITION 1.4. *Let L_1 and L_2 be lattices. Then*:

(a) *L_1 and L_2 are* (*strongly*) *balanced if and only if $L_1 \times L_2$ is* (*strongly*) *balanced.*

(b) *If L_1 and L_2 are balanced*, *then* $\mathrm{u}(L_1 \times L_2) = \mathrm{u}(L_1) + \mathrm{u}(L_2)$.

(c) *If L_1 and L_2 are strongly balanced*, *then* $\mathrm{u_s}(L_1 \times L_2) = \mathrm{u_s}(L_1) + \mathrm{u_s}(L_2)$.

We will need the following result about the connection of uniform dimensions with some mappings of lattices.

LEMMA 1.5. *Let L, M be lattices and let $\varphi : L \to M$ be a meet preserving mapping.*

(a) *If* L, M *are balanced and* $\varphi^{-1}(0) = \{0\}$ *then* $\mathrm{u}(L) \le \mathrm{u}(M)$.

(b) *If* L, M *are strongly balanced and* φ *is an injection then* $\mathrm{u_s}(L) \le \mathrm{u_s}(M)$.

Proof. Let $\varphi : L \to M$ be a meet preserving mapping. Then φ preserves the order and in particular, $\varphi(x) \vee \varphi(y) \le \varphi(x \vee y)$ for any $x, y \in L$. Hence for all $x_1, \dots, x_n \in L$,

$$\varphi(x_i) \wedge \bigvee_{k \neq i} \varphi(x_k) \le \varphi\Big(x_i \wedge \bigvee_{k \neq i} x_k\Big).$$

Now if the set $\{x_1, \dots, x_n\}$ is independent and $\varphi^{-1}(0) = \{0\}$, then the above inequality implies that $\{\varphi(x_1), \dots, \varphi(x_n)\}$ is independent in M. This means, with the help of Lemma 1.2, that $\mathrm{u}(L) \le \mathrm{u}(M)$.

Now let L, M be strongly balanced, and let φ be an injection. For a fixed $a \in L$ put $L' = [a, 1] \subseteq L$ and $M' = [\varphi(a), 1] \subseteq M$. If we put $\varphi' = \varphi|_{L'}$ then, by assumptions, the triple (L', M', φ') satisfies the conditions of the first part of our lemma. Hence $\mathrm{u}([a, 1]) \le \mathrm{u}([\varphi(a), 1]) \le \mathrm{u_s}(M)$. This means that $\mathrm{u_s}(L) \le \mathrm{u_s}(M)$, because $a \in L$ was arbitrarily chosen. ∎

2. Lattices of subgroups. All groups considered here will be finite with trivial element e. If G is a group then, as in [2, 9], by $L(G)$ we denote the lattice of all subgroups of G. This lattice is certainly finite.

Let G be a group. Then G will be called (*strongly*) *balanced* if the lattice $L(G)$ is (strongly) balanced and the uniform dimension (strong uniform dimension) of $L(G)$ will be called the *uniform dimension* (*strong uniform dimension*) *of the group* G and will be denoted by $\mathrm{u}(G)$ ($\mathrm{u_s}(G)$ respectively).

Now we formulate some simple properties of these dimensions of groups.

PROPOSITION 2.1. *Let* G *be a strongly balanced group.*

(a) *If* $H \subseteq G$ *is a subgroup then* $L(H)$ *is a sublattice in* $L(G)$ *with the same* 0. *Hence* $\mathrm{u}(H) \le \mathrm{u}(G)$ *and* $\mathrm{u_s}(H) \le \mathrm{u_s}(G)$.

(b) *If* $H \subseteq G$ *is a normal subgroup then* $L(G/H) \simeq [H, G] \subseteq L(G)$. *Hence* $\mathrm{u_s}(G/H) \le \mathrm{u_s}(G)$.

EXAMPLE 2.2. Let G be a nonabelian group of order 8. If G is a dihedral group then it is easy to check that G is not strongly balanced and even not balanced.

On the other hand, if $G \simeq Q_8$ is a quaternion group of order 8 then G is strongly balanced, and even modular, with $\mathrm{u}(G) = 1$ and $\mathrm{u_s}(G) = \mathrm{u}(G/G') = 2$, where G' is the commutator subgroup of G.

From a property of subgroups of direct products of groups with coprime orders (see [9]) and from Proposition 1.4 we immediately obtain

LEMMA 2.3. *Let G_1 and G_2 be groups with coprime orders and let $G = G_1 \times G_2$. Then $L(G) \simeq L(G_1) \times L(G_2)$. Moreover*:

(a) *If G_1 and G_2 are balanced then* $\mathrm{u}(G) = \mathrm{u}(G_1) + \mathrm{u}(G_2)$.

(b) *If G_1 and G_2 are strongly balanced then* $\mathrm{u}_\mathrm{s}(G) = \mathrm{u}_\mathrm{s}(G_1) + \mathrm{u}_\mathrm{s}(G_2)$.

For further considerations we need the following result, which is in some sense stronger than Proposition 2.1.

LEMMA 2.4. *Let G be a strongly balanced group. If P is a normal subgroup of G, then*

$$\mathrm{u}(G) \le \mathrm{u}(P) + \mathrm{u}(G/P) \quad \textit{and} \quad \mathrm{u}_\mathrm{s}(G) \le \mathrm{u}_\mathrm{s}(P) + \mathrm{u}_\mathrm{s}(G/P).$$

Proof. Let $P \subseteq G$ be a normal subgroup and $K \subseteq G$ be a maximal subgroup with the property $K \cap P = \{e\}$. Then from the assumption we know that $K \vee P = KP$ is essential in $L(G)$, because G is balanced. Hence, by Lemma 1.2 we get $\mathrm{u}(G) = \mathrm{u}(P) + \mathrm{u}(K)$.

Put $L = L(K)$, $M = L(G/P) = [P, G]$ and $\varphi(X) = X \vee P = XP$ for any subgroup $X \in L$. By the isomorphism theorem the triple (L, M, φ) satisfies the assumptions of Lemma 1.5. Hence $\mathrm{u}(K) \le \mathrm{u}(G/P)$ and consequently

$$\mathrm{u}(G) \le \mathrm{u}(P) + \mathrm{u}(G/P).$$

Now let us consider the strong uniform dimension. For this let $H \subseteq G$ be any subgroup. We need to estimate the uniform dimension of the interval $[H, G]$.

Let, as assumed, $P \subseteq G$ be a normal subgroup. Certainly $H \subseteq HP$. Let $K \subseteq G$ be a maximal subgroup with the property $K \cap HP = H$. Because, by assumption, $[H, G]$ is a balanced lattice, by definition the element $K \vee HP = KHP = KP$ is essential in $[H, G]$. Hence, by Lemma 1.2 we obtain

$$\mathrm{u}([H, G]) = \mathrm{u}([H, K]) + \mathrm{u}([H, HP]). \tag{1}$$

Put $L = [H, K]$, $M = L(G/P) = [P, G]$ and $\varphi(X) = XP$ for any subgroup $X \in L$. If $X, Y \in L$ then by an elementary coset calculation one can see that $\varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y)$ and $\varphi(X) = \varphi(Y)$ if and only if $X = Y$. Hence, by Lemma 1.5,

$$\mathrm{u}([H, K]) = \mathrm{u}(L) \le \mathrm{u}(M) = \mathrm{u}(G/P). \tag{2}$$

To estimate $\mathrm{u}([H, HP])$ put $L = [H, HP]$, $M = L(P)$ and $\varphi(X) = X \wedge P$ for any $X \in L$. Clearly φ is a meet preserving mapping. Moreover, it can be calculated that φ is an injection of L into M. Applying Lemma 1.5 we obtain

$$\mathrm{u}([H, HP]) = \mathrm{u}(L) \le \mathrm{u}(M) = \mathrm{u}(P). \tag{3}$$

Now our claim is a consequence of formulas (1)–(3). ∎

The above lemma cannot be extended to a general result for strongly balanced lattices. As in [7] let us consider the following example:

EXAMPLE 2.5. Let $n \geq 2$, and let B_n be the Boolean algebra with atoms $b_1, \ldots, b_n$. Denote by L_n the lattice obtained form B_n by adding an element x such that $0 < x < b_1$. From the construction of L_n it follows that this lattice is strongly balanced and it can be checked that $\mathrm{u}(L_n) = \mathrm{u_s}(L_n) = n$.

On the other hand, $\mathrm{u}([0,x]) + \mathrm{u}([x,1]) = 1 + 1 = 2$ and $\mathrm{u_s}([0,x]) + \mathrm{u_s}([x,1]) = 1 + n - 1 = n$.

The above example and Example 2.2 show that formulas connected with dimensions of extensions of lattices and groups should be applied carefully. However, as a consequence of Lemma 2.4 we obtain the following stronger version of Lemma 2.3.

THEOREM 2.6. *Let G_1 and G_2 be subgroups of a group G such that $G = G_1 \rtimes G_2$ and any subgroup of G_1 is invariant under conjugation by elements from G_2.*

(a) *If G is balanced then* $\mathrm{u}(G) = \mathrm{u}(G_1) + \mathrm{u}(G_2)$.

(b) *If G is strongly balanced then* $\mathrm{u_s}(G) = \mathrm{u_s}(G_1) + \mathrm{u_s}(G_2)$.

Proof. Clearly $G_1 \wedge G_2 = \{e\}$ and $G_1 \vee G_2 = G$ is essential in $L(G)$. Hence, by Lemma 1.2 we have $\mathrm{u}(G) = \mathrm{u}(G_1) + \mathrm{u}(G_2)$.

Now let G be strongly balanced. For $G_1 = P$ and $G_2 \simeq G/P$ we obtain, by Lemma 2.4, $\mathrm{u_s}(G) \leq \mathrm{u_s}(G_1) + \mathrm{u_s}(G_2)$.

Let $H \subseteq G_1$ be a subgroup such that $\mathrm{u_s}(G_1) = \mathrm{u}([H, G_1])$ and $K \subseteq G_2$ be a subgroup such that $\mathrm{u_s}(G_2) = \mathrm{u}([K, G_2])$. Then, by assumption about the action of G_2 on subgroups of G_1, we have $G_1K \wedge HG_2 = HK$ and, by Lemma 1.2,

$$\mathrm{u}([HK, G]) = \mathrm{u}([HK, G_1K]) + \mathrm{u}([HK, HG_2]).$$

On the other hand, one can check that $[H, G_1] \simeq [HK, G_1K]$ and $[K, G_2] \simeq [HK, HG_2]$. Hence the result follows. ∎

3. Uniform dimensions of groups. In this section we are going to calculate $\mathrm{u}(G)$ and $\mathrm{u_s}(G)$ for all strongly balanced groups G. We start with the characterization of uniform elements in $L(G)$. From ([8], 5.3.6), we immediately have

LEMMA 3.1. *Let G be a group and $H \in L(G)$. Then H is uniform in $L(G)$ ($\mathrm{u}(H) = 1$) if and only if H is isomorphic either to a cyclic p-group or to a generalized quaternion group.*

As a consequence of this lemma and Example 2.2 we deduce that for any $n > 3$ the generalized quaternion group Q_{2^n} is balanced, but not strongly balanced, because it has the dihedral group of order 8 as its homomorphic image.

Now we will characterize all groups G with $\mathrm{u_s}(G) = 1$. Since $L(G)$ is a chain if and only if G is a cyclic p-group (see [9]) and the strong uniform dimension has value 1 exactly for chains, we have

LEMMA 3.2. *Let G be a group. Then $\mathrm{u_s}(G) = 1$ if and only if G is a cyclic p-group.*

To consider the general case we recall the description of strongly balanced groups given in [1]. For brevity, call a group G an *exceptional strongly balanced group* (ESB-group) if $G = P \rtimes Q$, where P is an elementary abelian p-group, $Q = \langle y \mid y^{q^m} = 1\rangle$ is a cyclic q-group and $y^{-1}xy = x^k$ for all $x \in P$, where k is an integer such that $k^{q^m} \equiv 1 \pmod p$ but $k \not\equiv 1 \pmod p$.

THEOREM 3.3 ([1]). *Let G be a group. Then G is strongly balanced if and only if it is one of the following groups*:

(a) *a modular p-group*;

(b) *an ESB-group*;

(c) *a direct product of groups given in* (a) *and* (b) *with pairwise coprime orders.*

REMARKS. It is known that an ESB-group is modular exactly in the case when $m = 1$ can be taken in the above definition. For the smallest example of a nonmodular ESB-group, see page 68 in [1].

It can be checked that the direct decomposition from (c) in the above theorem is unique up to the order of components.

Further, if G is a p-group, then by $\Phi(G)$, $\Omega(G)$, G^p we will denote, respectively, the Frattini subgroup, the subgroup generated by all elements of order p and the subgroup generated by all pth powers of elements of G.

Let G be a strongly balanced group. Then, due to Lemma 2.3 and Theorem 3.3(c) we have to calculate the uniform dimensions of G only if G is as in (a) or (b) of the above theorem. According to a classical result of Iwasawa (see [9]), case (a) can be divided into the following subcases:

(a_1) G is an abelian p-group.

(a_2) G is a hamiltonian 2-group.

(a_3) G is a nonabelian p-group containing an abelian normal subgroup A and an element b such that $G = A\langle b\rangle$. Moreover, there exists a positive integer s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in the case $p = 2$, and $A \cap \langle b\rangle \subseteq A^p$.

(a_4) G is as in (a_3), but $A \cap \langle b\rangle \not\subseteq A^p$.

It is well known that the presentation of a modular p-group G in cases (a_3) and (a_4) need not be unique (see [9], p. 65). Now we show that these cases are not disjoint.

EXAMPLE 3.4. Let p be a prime number and $1 \leq s < r$ ($2 \leq s < r$ if $p = 2$) be natural numbers and let

$$G = \langle a, b \mid a^{p^r} = b^{p^r} = e,\ b^{-1}ab = a^{1+p^s} \rangle.$$

Put $A_1 = \langle a \rangle$ and $B = \langle b \rangle$. Then for any $x \in A_1$ we have $b^{-1}xb = x^{1+p^s}$ and $G = A_1 \rtimes B$. This means that $G = A_1B$ satisfies the conditions of case (a_3), because $A_1 \cap B = \{e\} \subseteq A_1^p$.

Now let $C = \langle c \rangle$ where $c = b^{p^{r-s}}$, and let $A_2 = \langle a, c \rangle$. Then $|C| = p^s$, $b^{-1}xb = x^{1+p^s}$ for any $x \in A_2$, and $A_2 = A_1C \simeq A_1 \oplus C$ is an abelian normal subgroup of G with $G = A_2B$. However in this case $A_2 \cap B = C \not\subseteq A_2^p$. This means that this presentation of G fits case (a_4).

THEOREM 3.5. *Let G be a modular p-group. Then $\mathrm{u_s}(G)$ is equal to the cardinality of a minimal system of generators of G.*

Proof. Let G be any modular p-group. From the Iwasawa characterization of such groups mentioned above it is easy to deduce that $\Phi(G) = G^p$, $G/\Phi(G)$ is an elementary abelian p-group and it is known that the cardinality of a minimal system of generators of G is just the dimension of $G/\Phi(G)$ viewed as a vector space over the field F_p with p elements (see [5]). From the same characterization we also infer that $\Omega(G)$ is an elementary abelian p-group. Clearly in the proof it is enough to consider the four cases listed above.

Case (a_1). Let G be an abelian p-group and let G_i, $i = 1, \ldots, n$, be cyclic subgroups of G such that $G = G_1 \times \cdots \times G_n$. Then, by Theorem 2.6 and Lemma 3.2, we have $\mathrm{u}(G) = \mathrm{u_s}(G) = n$.

On the other hand, it is evident that n is exactly the minimal number of generators of G.

Case (a_2). Let G be any hamiltonian 2-group. Then $G = Q_8 \times A$, where A is an elementary abelian 2-group (see [8]). Hence, by Theorem 2.6 and Example 2.2 we obtain $\mathrm{u}(G) = \mathrm{u}(A) + 1$ and $\mathrm{u_s}(G) = \mathrm{u_s}(A) + 2$.

Let $|A| = 2^n$. Then, by case (a_1) of this proof, we have

$$\mathrm{u}(G) = n + 1 \quad \text{while} \quad \mathrm{u_s}(G) = n + 2.$$

On the other hand we know that $\Phi(G) = G^2$. Hence the group $G/\Phi(G)$ is elementary abelian of order 2^{n+2}. This means that G is generated by exactly $n + 2$ elements.

Case (a_3). Now we can assume that our group G is nonabelian of the form $G = AB$ where A is an abelian p-group and $B = \langle b \rangle$ is a cyclic p-group such that $b^{-1}ab = a^{1+p^s}$ for some $s \geq 1$. Moreover, $C = A \cap B \subseteq A^p$.

Let $\mathrm{u_s}(A) = n$. Then, by case (a_1), a minimal system of generators of A has n elements. Moreover, $A^p \subseteq \Phi(G)$ is normal in G and G/A^p is an abelian group with exactly $n+1$ generators because, by assumption, $C \subseteq A^p$. Hence, it can be calculated that $G/\Phi(G)$ is an elementary abelian group of order p^{n+1} and a minimal system of generators of G has $n+1$ elements.

On the other hand, by assumption, there exists $a \in A$ such that $a \notin C$ but $a^p \in C$. Let $D = \langle a, b\rangle$. Then $D \cap A = \langle a\rangle$ is cyclic. Hence $|D \cap \Omega(A)| = p$. From Lemma 3.1 we have $\mathrm{u}(D) > 1$ and consequently, $\Omega(D) \not\subseteq \Omega(A)$. This means that $\Omega(G)$ is elementary abelian of order at least p^{n+1}. Hence, $n+1 \le \mathrm{u}(G)$. But from Theorem 2.6, $\mathrm{u_s}(G) = n+1$. These facts together give

$$\mathrm{u}(G) = \mathrm{u_s}(G) = n+1.$$

Case (a_4). Let us keep the notation of the previous case. However, by assumption, we have $C \not\subseteq A^p$. Let $C = \langle c\rangle$. Then $c \in A \setminus A^p$. Hence, by case (a_1) and since $\mathrm{u}(A) = n$, for some $a_1, \ldots, a_{n-1} \in A$ we have $A = \langle a_1, \ldots, a_{n-1}, c\rangle$. This means that $G = \langle a_1, \ldots, a_{n-1}, b\rangle$ and the images of these generators modulo A^p are independent over F_p. Hence G has exactly n generators.

On the other hand, let $A_1 = \langle a_1, \ldots, a_{n-1}\rangle$. Then $\mathrm{u}(A_1) = n-1$ and $G = A_1 \rtimes B$. From this equality and Theorem 2.6 we have

$$\mathrm{u}(G) = \mathrm{u_s}(G) = n$$

and the proof of the theorem is complete. ∎

THEOREM 3.6. *Let G be an ESB-group. Then $\mathrm{u_s}(G)$ is equal to the cardinality of a minimal system of generators of G.*

Proof. Let G be an ESB-group. Then, under the notation of the definition, $G = P \rtimes Q$, where P is an elementary abelian p-group of order p^n and Q is a cyclic q-group. Then, by Theorem 3.5, $\mathrm{u}(P) = n$ and $\mathrm{u}(Q) = 1$. Hence, by Theorem 2.6,

$$\mathrm{u}(G) = \mathrm{u_s}(G) = \mathrm{u_s}(P) + \mathrm{u_s}(Q) = n+1.$$

On the other hand, let $g_1, \ldots, g_r$ be any system of generators of G. Because $Q \simeq G/P$ is a cyclic q-group, we can assume that for example the coset g_1P generates G/P. Hence, after small modifications if necessary, we can assume that $g_2, \ldots, g_r \in P$. Now, by induction on the value of $n = \mathrm{u}(P)$ one can prove that the cardinality of any minimal system of generators of G is equal to $n+1$. ∎

As a consequence of the results of this section and their proofs we immediately have

THEOREM 3.7. *Let G be a strongly balanced group. Then* $\mathrm{u_s}(G) = \mathrm{u}(G) + 1$ *if and only if G has a direct factor which is the quaternion group of order* 8. *In any other case* $\mathrm{u_s}(G) = \mathrm{u}(G)$.

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REFERENCES

[1] C. Bagiński and A. Sakowicz, *Finite groups with permutable lattices of subgroups*, Colloq. Math. 82 (1999), 65–77.
[2] G. Birkhoff, *Lattice Theory*, 3rd ed., Amer. Math. Soc., Providence, RI, 1967.
[3] P. Grzeszczuk, J. Okniński and E. R. Puczyłowski, *Relations between some dimensions of modular lattices*, Comm. Algebra 17 (1989), 1723–1737.
[4] P. Grzeszczuk and E. R. Puczyłowski, *On Goldie and dual Goldie dimensions*, J. Pure Appl. Algebra 31 (1984), 47–54.
[5] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1983.
[6] J. Ježek, *Review of the paper*: *A.P. Zolotarev*, *Balanced lattices and Goldie numbers in balanced lattices*, MR 95i:06008.
[7] J. Krempa and B. Terlikowska-Osłowska, *On uniform dimension of lattices*, in: Contributions to General Algebra 9, Holder-Pichler-Tempsky, Wien, 1995, 219–230.
[8] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, Berlin, 1982.
[9] R. Schmidt, *Subgroup Lattices of Groups*, de Gruyter, Berlin, 1994.
[10] A. P. Zolotarev, *Balanced lattices and Goldie numbers in balanced lattices*, Sibirsk. Mat. Zh. 35 (1994), 602–611 (in Russian).

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