ON THE SET REPRESENTATION OF AN ORTHOMODULAR POSET

BY

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Abstract. Let $P$ be an orthomodular poset and let $B$ be a Boolean subalgebra of $P$. A mapping $s : P \to \langle 0, 1 \rangle$ is said to be a centrally additive $B$-state if it is order preserving, satisfies $s(a') = 1 - s(a)$, is additive on couples that contain a central element, and restricts to a state on $B$. It is shown that, for any Boolean subalgebra $B$ of $P$, $P$ has an abundance of two-valued centrally additive $B$-states. This answers positively a question raised in [13, Open question, p. 13]. As a consequence one obtains a somewhat better set representation of orthomodular posets and a better extension theorem than in [2, 12, 13]. Further improvement in the Boolean vein is hardly possible as the concluding example shows.

Our notation is standard. We use OMP to abbreviate orthomodular poset, OML to abbreviate orthomodular lattice, $Z$ to denote the centre of an orthomodular poset, $\subset$ for set inclusion, and $\langle 0, 1 \rangle$ for the real unit interval. We remind the reader that a subset $B$ of an orthomodular poset $P$ is called a Boolean subalgebra of $P$ if $B$ is closed under orthocomplementation and finite orthogonal joins and $B$ forms a Boolean algebra under these inherited operations. It is well known that any two elements of $B$ also have a join (resp., a meet) in $P$ and that the join (resp., the meet) taken in $B$ coincides with the join (resp., the meet) taken in $P$. For general background on orthomodular posets the reader should consult [11], on orthomodular lattices [1, 6], and for various results related to set representations of orthomodular posets [2, 5, 7, 8, 9, 13, 14].

Definition 1. Let $P$ be an OMP and $s : P \to \langle 0, 1 \rangle$ be a map that satisfies

1. $s(0) = 0$,
2. $s(a') = 1 - s(a)$ for all $a \in P$,
3. if $a \leq b$ then $s(a) \leq s(b)$.

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We say \( s \) is a state if it satisfies

(4) if \( a \leq b' \), then \( s(a \lor b) = s(a) + s(b) \).

We say \( s \) is a centrally additive state if it satisfies

(4') if \( a \leq b' \) and \( b \in Z \), then \( s(a \lor b) = s(a) + s(b) \).

If \( B \) is a Boolean subalgebra of \( P \) we say \( s \) is a \( B \)-state if it satisfies

(4'') if \( a \leq b' \) and \( a, b \in B \), then \( s(a \lor b) = s(a) + s(b) \).

Centrally additive states are obtained by weakening the additivity requirement for states to those orthogonal pairs where at least one element belongs to the centre, and \( B \)-additive states are obtained by weakening the additivity requirement for states to those orthogonal pairs where both elements belong to the subalgebra \( B \). Note that a centrally additive state is more than just a \( B \)-additive state for \( B \) being the Boolean algebra \( Z \). We shall call a state two-valued if its range is \( \{0, 1\} \). The following notion is key to the study of two-valued centrally additive states.

**Definition 2.** Let \( P \) be an OMP. We say \( I \subset P \) is a central ideal if

1. \( b \in I \) and \( a \leq b \) imply \( a \in I \),
2. if \( a \in I \) then \( a' \notin I \) for every \( a \in P \),
3. if \( a \leq b' \), \( a, b \in I \), and \( b \in Z \) then \( a \lor b \in I \),
4. \( I \) contains a prime ideal of \( Z \).

**Lemma 3.** Let \( I \) be a central ideal of \( P \) and \( a' \notin I \). Then

\[
J = \{ x \in L \mid x \leq m \lor a \text{ for some } m \in I \cap Z \} \cup I
\]

is a central ideal of \( P \) containing \( I \) and the element \( a \).

**Proof.** Let \( Q = I \cap Z \). By assumption (4), \( Q \) contains a prime ideal of the centre, hence by assumption (2), \( Q \) is a prime ideal of the centre.

As \( J \) is the union of two order ideals, it is an order ideal. Hence \( J \) satisfies the first condition.

For the second condition suppose \( x, x' \in J \). Obviously not both \( x, x' \in I \). If \( x \leq m_1 \lor a \) and \( x' \leq m_2 \lor a \), then \( 1 = (m_1 \lor m_2) \lor a \), giving \( a' \leq m_1 \lor m_2 \). As \( m_1 \lor m_2 \) belongs to \( Q \), we have the contradiction \( a' \in I \). We are left with the possibility that \( x \in I \) and \( x' \leq m \lor a \) for some \( m \in Q \). This implies \( m' \land a' \leq x \), hence \( m' \land a' \in I \). As \( m \in Q \) and \( I \) is a central ideal we see that \( m \lor (m' \land a') = m \lor a' \in I \), yielding the contradiction \( a' \in I \). Note that the second condition implies \( J \cap Z = I \cap Z \) since \( I \cap Z \) is a prime ideal of \( Z \).

For the third condition suppose \( x, y \in J \), \( x \leq y' \) and \( y \in Z \). Then \( y \in Q \). If \( x \in I \), then as \( I \) is a central ideal we have \( x \lor y \in I \). Otherwise \( x \leq m \lor a \) for some \( m \in Q \). Then \( x \lor y \leq m \lor a \lor y = (m \lor y) \lor a \) and since both \( m, y \in Q \) it follows that \( x \lor y \in J \).
Finally, the fourth condition follows trivially as $I$ contains a prime ideal of $Z$. ■

**Corollary 1.** For $I$ a central ideal of $P$ these are equivalent:

1. $I$ is a maximal central ideal.
2. For each $a \in P$ exactly one of $a, a'$ belongs to $I$.

The connection between maximal central ideals and two-valued centrally additive states can now be made clear.

**Proposition 4.** Let $P$ be an OMP and $B$ be a Boolean subalgebra of $P$. For $s : P \to \{0, 1\}$ these are equivalent:

1. $s$ is a centrally additive $B$-state.
2. $s^{-1}(0)$ is a maximal central ideal which contains a prime ideal of $B$.

For $I \subset P$ these are equivalent:

1. $I$ is a maximal central ideal which contains a prime ideal of $B$.
2. $I = s^{-1}(0)$ for some two-valued centrally additive $B$-state $s$.

**Proof.** (1)$\Rightarrow$(2). Set $I = s^{-1}(0)$. As $s$ restricts to a state on $Z$, $I \cap Z$ is a prime ideal of $Z$. Similarly, as $s$ restricts to a state on $B$, $I \cap B$ is a prime ideal of $B$. Obviously, $I$ is a downset and for each $a \in P$ exactly one of $a, a'$ belongs to $I$. Finally, if $x, y \in I$, $x \leq y'$ and $y \in Z$, then as $s$ is centrally additive, $s(x \lor y) = s(x) + s(y) = 0$, yielding $x \lor y \in I$.

(2)$\Rightarrow$(1). Set $I = s^{-1}(0)$. As $0 \in I$ we have $s(0) = 0$, and as $I$ is a downset, $s$ is order preserving. As $I$ is maximal, exactly one of $a, a'$ belongs to $I$ for each $a \in P$, so $s(a') = 1 - s(a)$. Assume $x \leq y'$ with either $x, y \in B$ or $y \in Z$. To show $s(x \lor y) = s(x) + s(y)$ it suffices to show this under the assumption that $x, y \in I$. The result follows from the assumptions that $I \cap B$ is a prime ideal of $B$ and that $I$ is a central ideal.

(3)$\Rightarrow$(4). Define $s : P \to \{0, 1\}$ by setting $s(x) = 0$ if $x \in I$ and $s(x) = 1$ if $x \notin I$. Then $I = s^{-1}(0)$. That $s$ is a centrally additive $B$-state then follows from the equivalence of (1) and (2).

(4)$\Rightarrow$(3). This follows directly from the equivalence of (1) and (2). ■

The following result is crucial for the representation theorem.

**Lemma 5.** Let $L$ be an OMP. Let $B$ be a Boolean subalgebra of $L$ containing $Z$ and let $a, b \in L$ with $a \nleq b$. Then there is a central ideal $I$ with $a', b \in I$ such that $I \cap B$ is a prime ideal of $B$.

**Proof.** Set $X = \{x \in B \mid a \leq x\} \cup \{y \in B \mid b' \leq y\} \cup \{z \in Z \mid a \leq z \lor b\}$. We first claim that $X$ generates a proper filter of $B$. As each of the three sets involved in the definition of $X$ is closed under finite meets, it suffices to show that for $x, y \in B$ and $z \in Z$ with $a \leq x, b' \leq y, a \leq z \lor b$ we have $x \land y \land z \neq 0$. Assume to the contrary that $x \land y \land z = 0$. We want to derive
the contradiction \( a \leq b \). Certainly, \( a \leq z \lor b \) implies by the centrality of \( z \) that \( a \land z' \leq b \land z' \). Also, \( x \land y \land z = 0 \) implies \( z \leq x' \lor y' \). As \( a \leq x \) and \( z \leq x' \lor y' \) we have \( a \land z \leq x \land (x' \lor y') = x \land y' \leq y' \leq b \), so \( a \land z \leq b \land z \). As \( a \land z \leq b \land z \) and \( a \land z' \leq b \land z' \), the centrality of \( z \) yields \( a \leq b \).

Since \( X \) generates a proper filter, there is a prime ideal \( Q \) of \( B \) which is disjoint from \( X \). Let \( I_0 = \{ x \in L \mid x \leq p \) for some \( p \in Q \} \). We claim that \( I_0 \) is a central ideal. The first condition is trivial from the definition. The second follows as \( I_0 \) is the downset generated by a proper ideal of \( B \). The third condition also follows: \( I_0 \) is closed under all finite joins. The fourth follows as \( I_0 \) contains a prime ideal of \( B \) and the centre is contained in \( B \). We next want to show that \( a, b' \not\in I_0 \). Indeed, if \( a \in I_0 \) then \( a \leq x \) for some \( x \in Q \). But then \( x \in X \cap Q \), a contradiction. Similarly, if \( b' \in I_0 \) then \( b' \leq y \) for some \( y \in Q \) and \( y \in X \cap Q \), a contradiction. Let us set

\[
I_1 = \{ x \in L \mid x \leq m \lor b \) for some \( m \in I_0 \cap Z \} \cup I_0.
\]

By Lemma 3, \( I_1 \) is a central ideal of \( L \). We claim that \( a \not\in I_1 \). Indeed, \( a \in I_1 \) would imply that \( a \leq z' \lor b \) for some \( z \in I_0 \cap Z \). But this \( z \) would then belong to \( X \cap Q \), which is absurd. As \( a \not\in I_1 \), we apply Lemma 3 again to extend \( I_1 \) to a central ideal containing both \( a', b \).

**Theorem 6.** Let \( P \) be an OMP, \( B \) be a Boolean subalgebra of \( P \), and \( a \not\leq b \) be elements of \( P \). Then there is a centrally additive \( B \)-state \( s : P \to \{0,1\} \) such that \( s(a) = 1 \) and \( s(b) = 0 \).

**Proof.** Taking the subalgebra generated by \( B \cup Z \) if necessary, we may assume without loss of generality that \( B \) contains the centre of \( P \). Use Lemma 5 to produce a central ideal \( I \) with \( a', b \in I \) such that \( I \cap Z \) is a prime ideal of \( B \). By a standard Zorn’s lemma argument extend \( I \) to a maximal central ideal \( M \). By Proposition 4 there is a centrally additive \( B \)-state \( s : P \to \{0,1\} \) with \( M = s^{-1}(0) \). Then \( a', b \in M \) yield \( s(a) = 1 \) and \( s(b) = 0 \).

**Theorem 7.** Let \( P \) be an OMP and let \( B \) be a Boolean subalgebra of \( P \). Then there is a set \( S \) and a mapping \( \sigma : P \to \exp S \) into the power set of \( S \) such that, for any \( a, b \in L \),

1. \( a \leq b \) if and only if \( \sigma(a) \subset \sigma(b) \),
2. \( \sigma(a') = S - \sigma(a) \),
3. if \( a, b \in B \) then \( \sigma(a \lor b) = \sigma(a) \cup \sigma(b) \) and \( \sigma(a \land b) = \sigma(a) \cap \sigma(b) \),
4. if \( a \in Z \), then \( \sigma(a \lor b) = \sigma(a) \cup \sigma(b) \) and \( \sigma(a \land b) = \sigma(a) \cap \sigma(b) \).

**Proof.** The proof closely follows the Boolean patterns and we therefore omit the details. Let \( S \) be the set of all two-valued centrally additive \( B \)-states on \( P \). Define \( \sigma : P \to \exp S \) by setting \( \sigma(a) = \{ s \in S \mid s(a) = 1 \} \).
The “topological” version of the above representation theorem is also in force. Again, the technique is similar to the Boolean case. The resulting Stone space will however be a closure space only (see [13] for details; recall that a closure space (see [3]) differs from a topological space in that the union of two closed sets need not be closed).

**Theorem 8.** Let $P$ be an OMP and let $B$ be a Boolean subalgebra of $P$. Then there exists a compact Hausdorff closure space $C$ and a mapping $\sigma : L \to \text{Clop}(C)$ to the collection $\text{Clop}(C)$ of all clopen subspaces of $C$ such that

1. $a \leq b$ if and only if $\sigma(a) \subset \sigma(b)$,
2. $\sigma(a') = S - \sigma(a)$,
3. if $a, b \in B$ then $\sigma(a \lor b) = \sigma(a) \cup \sigma(b)$ and $\sigma(a \land b) = \sigma(a) \cap \sigma(b)$,
4. if $a \in Z$, then $\sigma(a \lor b) = \sigma(a) \cup \sigma(b)$ and $\sigma(a \land b) = \sigma(a) \cap \sigma(b)$.

Further, if $P$ is an OML then the map $\sigma$ is onto $\text{Clop}(C)$.

**Proof.** Let $S$ and $\sigma$ be as in the previous theorem. Let $C$ be the closure space whose underlying set is $S$ and whose basic closed sets are $\{\sigma(a) \mid a \in P\}$. As each $\sigma(a)$ and its complement are closed, each $\sigma(a)$ is clopen. For distinct states $s, t \in S$ there is $a \in P$ with $s(a) \neq t(a)$ hence $\sigma(a)$ is a clopen set separating these points. Therefore $C$ is Hausdorff. As the state space $S$ is compact under the subspace topology inherited from $\langle 0,1 \rangle^P$, and each $\sigma(a)$ is closed in this subspace topology, the collection $\{\sigma(a) \mid a \in P\}$ has the finite intersection property, and it follows that $C$ is also compact. Conditions (1) through (4) of the theorem are established in the previous result.

For the further remark assume $P$ is an OML. Let $A \subset S$ be a clopen set of $C$. Using the compactness of $C$ and the fact that $A$ is open, we have $A = \sigma(a_1) \cup \ldots \cup \sigma(a_n)$ for some $a_1, \ldots, a_n \in P$. But $A$ is closed so for some $T \subset P$ we have $A = \bigcap\{\sigma(a) \mid a \in T\}$. It follows from (1) that $A \subset \sigma(a_1 \lor \ldots \lor a_n) \subset \bigcap\{\sigma(a) \mid a \in T\}$ hence equality. This shows $\sigma$ is onto.

Our next theorem generalizes the extension property for Boolean states.

**Theorem 9.** Let $P$ be an OMP and $B_1, B_2$ be Boolean subalgebras of $P$. Let $s : B_1 \to \langle 0,1 \rangle$ be a (Boolean) state on $B_1$. Then there is a centrally additive $B_2$-state $t : P \to \langle 0,1 \rangle$ that restricts to $s$ on $B_1$.

**Proof.** Assume first $s$ is two-valued. From well known properties of states on Boolean algebras, $s$ can be extended to a two-valued state on the Boolean subalgebra of $P$ generated by $B_1 \cup Z$, so we may assume without loss of generality that $B_1$ contains $Z$. Also, from the form of the problem, we may assume that $B_2$ contains $Z$. Let $J = s^{-1}(0)$, a prime ideal of $B_1$. Note that
\(J\) contains a prime ideal of \(Z\). By the prime ideal theorem, there is a prime ideal \(K\) of \(B_2\) containing \(\{x \in B_2 \mid x \leq j \text{ for some } j \in J\}\). Then \(K\) contains \(J \cap B_2\). Hence \(K \cap Z = J \cap Z\) and as both are prime ideals of \(Z\) we have \(K \cap Z = J \cap Z\). Let

\[
I = \{x \in P \mid x \leq y \text{ for some } y \in J \cup K\}.
\]

We claim \(I\) is a central ideal. Obviously, \(I\) is a downset. Suppose \(x, x' \in I\). Then as both \(J, K\) are closed under finite joins and neither contains 1 we deduce that \(x \leq j\) for some \(j \in J\) and \(x' \leq k\) for some \(k \in K\). Then \(k' \leq j\). But this would imply \(k' \in K\), contrary to \(K\) being a prime ideal. Since \(I \supset J, K\) it follows that \(I\) contains \(J \cap Z = K \cap Z\), a prime ideal of \(Z\), and as we have shown that \(I\) never contains an element and its orthocomplement, \(I \cap Z = J \cap Z = K \cap Z\). Suppose \(x, y \in I\) with \(x \leq y'\) and \(y \in Z\). If \(x \leq j\) for some \(j \in J\), then as \(y \in I \cap Z = J \cap Z\) we have \(j, y \in J\) hence \(j \vee y \in J\), and as \(x \vee y \leq j \vee y\) we have \(x \vee y \in I\). If \(x \leq k\) for some \(k \in K\) the argument is similar. Therefore \(I\) is a central ideal of \(P\).

Taking the two-valued centrally additive state \(t : P \to \{0, 1\}\) associated with \(I\) we see that \(t\) extends \(s\) since \(I \supset J\) and \(t\) is a \(B_2\) state since \(I\) contains a prime ideal of \(B_2\). We have proved every two-valued state \(s\) on \(B_1\) can be extended to a two-valued centrally additive \(B_2\)-state on \(P\). The general result then follows from the compactness and convexity of the space of all centrally additive \(B_2\)-states on \(P\) by using a standard argument found e.g. in [13].

To conclude this note, let us show by an example that our results are in a sense best possible. Let \(P\) be an OMP and \(B\) be a Boolean subalgebra of \(P\). Let us call a mapping \(s : P \to \langle 0, 1 \rangle\) a strong \(B\)-state if

(1) \(s(0) = 0\),
(2) \(s(a') = 1 - s(a)\) for any \(a \in P\),
(3) if \(a \leq b\) then \(s(a) \leq s(b)\), and
(4) if \(a \leq b'\) and \(b \in B\), then \(s(a \vee b) = s(a) + s(b)\).

It turns out that there is no hope for a representation theorem via these states—there are finite OMP’s which do not have an order determining set of two-valued strong \(B\)-states. We will show this using the Greechie paste technique (see [4]).

**Example 10.** Let us consider the OMP, \(P\), given by the Greechie diagram indicated below. Let us consider elements \(a, b\) therein. Then \(a \not\leq b'\). Let \(B\) be the maximal Boolean subalgebra of \(P\) containing the atom \(a\). Then there is no two-valued strong \(B\)-state with \(s(a) = 1\) and \(s(b') = 0\).
Proof. If \( s(a) = 1 \), then \( s(c) = s(d) = 0 \) (the elements \( c, a, d \) constitute all atoms of \( B \)). Suppose \( s(b') = 0 \). Then \( s(b) = 1 \). Since \( e \leq b' \), we see that \( s(e) = 0 \). This implies that \( s(f) = 1 \), and therefore \( s(g) = 0 \). As \( s(c) = s(g) = 0 \), we infer that \( s(h) = 1 \). This yields \( s(i) = 0 \), and therefore \( s(j) = 1 \). As a consequence, \( s(k) = 0 \). Since \( s(c) = s(k) = 0 \), we have \( s(l) = 1 \). But \( s(f) = s(l) = 1 \), a contradiction. Thus, there is no two-valued strong \( B \)-state on \( P \) with \( s(a) = 1 \) and \( s(b') = 0 \).

REFERENCES


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