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A CLASSIFICATION OF TWO-PEAK SINCERE POSETS OF FINITE PRINJECTIVE TYPE AND THEIR SINCERE PRINJECTIVE REPRESENTATIONS

BY

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Abstract. Assume that K is an arbitrary field. Let $(I, \preceq)$ be a two-peak poset of finite prinjective type and let KI be the incidence algebra of I. We study sincere posets I and sincere prinjective modules over KI. The complete set of all sincere two-peak posets of finite prinjective type is given in Theorem 3.1. Moreover, for each such poset I, a complete set of representatives of isomorphism classes of sincere indecomposable prinjective modules over KI is presented in Tables 8.1.

1. Introduction. Throughout this paper K is a field and $I = (I, \preceq)$ is a finite poset (i.e. partially ordered set). We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$. All posets are assumed to be connected. Following [21] we denote by $\max I$ the set of all maximal elements of I (called *peaks* of I). The poset I is called an *r-peak poset* if $|\max I| = r$, where $|X|$ denotes the cardinality of the set X. A subposet J of I is said to be a *peak subposet* if $J \cap \max I = \max J$.

Usually we view I as a quiver with the commutativity relations induced by the ordering $\prec$ (see [20, Example 10, p. 281]). Given a field K we denote by KI the *incidence K-algebra* of I, that is, KI is the K-subalgebra of the full $I \times I$ matrix algebra $\mathbb{M}_I(K)$ consisting of all matrices $[\lambda_{ij}]$ in $\mathbb{M}_I(K)$ such that $\lambda_{ij} = 0$ if $i \not\preceq j$ (see [20], [21]). Given $j \in I$ we denote by $e_j \in KI$ the standard primitive idempotent corresponding to j. The reader is referred to [20]–[22], [24] for a discussion of incidence algebras and their applications in integral representation theory.

A right KI-module is identified with a system

$$X = (X_i;\ {}_jh_i)_{i,j \in I}$$

where $X_i = Xe_i$ is a finite-dimensional vector space over K and ${}_jh_i : X_i \to X_j$, $i \prec j$, are K-linear maps with ${}_ih_i = \mathrm{id}$ and ${}_th_j \cdot {}_jh_i = {}_th_i$ for $i \prec j \prec t$. We denote by $\mathrm{mod}(KI)$ the category of finitely generated right KI-modules.

Following [17] and [30] we call a right KI-module X *prinjective* if X is finitely generated and the right module Xe_- over the algebra $KI^- \cong$

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$e_-(KI)e_-$ is projective, where $I^- = I \setminus \max I$ and $e_- = \sum_{j\in I^-} e_j$. It is easy to prove that $X \in \mathrm{mod}(KI)$ is prinjective if and only if there exists a short exact sequence

$$0 \to P_1 \to P_0 \to X \to 0 \tag{1.1}$$

in $\mathrm{mod}(KI)$, where P_0, P_1 are projective and P_1 is semisimple of the form $P_1 = \bigoplus_{p\in\max I}(e_pKI)^{t_p}$, $t_p \geq 0$.

We denote by $\mathrm{prin}(KI)$ the full subcategory of $\mathrm{mod}(KI)$ consisting of the prinjective modules. It follows from [17] that the category $\mathrm{prin}(KI)$ is additive, has the finite unique decomposition property, is closed under extensions in $\mathrm{mod}(KI)$, has Auslander–Reiten sequences, source maps and sink maps, and has enough relative projective and relative injective objects. An interpretation of $\mathrm{prin}(KI)$ as a bimodule matrix problem in the sense of Chapter 17 of [20] is given in [21].

Following [21] we define a poset I to be of *finite prinjective type* if the category $\mathrm{prin}(KI)$ is of finite representation type, that is, the number of isomorphism classes of indecomposable modules in $\mathrm{prin}(KI)$ is finite. It follows from [21, Theorem 3.1] that this definition does not depend on the choice of K.

We recall from [21] that the category I-spr of *peak I-spaces* (or *socle projective representations* of I) over the field K is defined as follows. The objects of I-spr are systems $\mathbf{M} = (M_j)_{j\in I}$ of finite-dimensional K-vector spaces M_j such that $M_j \subseteq M^\bullet = \bigoplus_{p\in\max I} M_p$ for all $j \in I$, $\pi_p(M_j) = 0$ for $j \not\preceq p \in \max I$ and $\pi_j(M_i) \subseteq M_j$ for $i \prec j$, where π_j is the composite map

$$M^\bullet \xrightarrow{\pi'_j} \bigoplus_{j\preceq p\in\max I} M_p \hookrightarrow M^\bullet$$

and π'_j is the direct summand projection. By a morphism $f : \mathbf{M} \to \mathbf{M}'$ in I-spr we mean a K-linear map $f : M^\bullet \to M'^\bullet$ such that $f(M_j) \subseteq M'_j$ for all $j \in I$. It was shown in [21, (2.4)] that there is an equivalence of categories with the finite unique decomposition property

$$\boldsymbol{\varrho} : I\text{-spr} \to \mathrm{mod}_{\mathrm{sp}}(KI) \tag{1.2}$$

where $\mathrm{mod}_{\mathrm{sp}}(KI)$ is the full subcategory of $\mathrm{mod}(KI)$ consisting of all socle projective modules (see [19]). Throughout this paper we shall identify I-spr with $\mathrm{mod}_{\mathrm{sp}}(KI)$ along $\boldsymbol{\varrho}$. It is clear that I-spr is an additive category, has the finite unique decomposition property and is closed under taking kernels and extensions. The category I-spr has Auslander–Reiten sequences, source maps and sink maps (see [21]). It was shown in [21, Section 2] that the study of peak I-spaces is equivalent to the study of prinjective modules.

One of the motivations for the study of peak I-spaces is the fact that many problems of representation theory can be reduced to the corresponding problems for peak I-spaces (see [2], [18], [20]–[22], [28]). Peak I-spaces play an important role in the representation theory of finite groups (see [29]), finite-dimensional algebras (see [18], [20, Chapter 17]) and lattices over orders (see [20, Chapter 13], [21], [22], [24]–[28]). Moreover sincere posets of finite prinjective type and their sincere representations provide us an important tool in constructing embeddings of Kronecker modules into the category of prinjective peak I-spaces (see [6], [7], [15], [16]).

The main aim of this paper is to give the complete set of all sincere two-peak posets I of finite prinjective type (Tables 3.2) and a complete set of representatives of isomorphism classes of indecomposable modules in $\mathrm{prin}(KI)$ and their coordinate vectors (Tables 8.1 and Tables 3.3). We prove in Theorem 3.1 that there are precisely 60 sincere two-peak posets $\mathcal{F}_1^{(2)}, \ldots, \mathcal{F}_{60}^{(2)}$ of finite prinjective type, listed in Tables 3.2. Moreover every sincere indecomposable prinjective module over the K-algebra $K\mathcal{F}_j^{(2)}$, $1 \leq j \leq 60$, is isomorphic to one of the sincere prinjective modules $M_j^{(i)}$ listed in Tables 8.1. The total number of sincere prinjective representations of two-peak posets of finite prinjective type is 328.

We recall that the complete set of 14 sincere one-peak posets of finite prinjective type and their 42 sincere prinjective representations was given by M. Kleiner [12], [13] (see also [20, Theorem 10.2 and Tables 10.7] for a correction of Kleiner's list [13]).

Our construction of posets is presented in Section 6 and applies two computer implementable algorithms (Algorithms 6.5 and 6.8). We inductively construct sincere posets I of finite prinjective type such that $|I| = n + 1$ from sincere posets J of finite prinjective type such that $|J| = n$.

The main results of this paper were presented to the International Conference "Representation Theory of Algebras" in Bielefeld (September 1998). A similar result in a more general setting of weakly positive quadratic forms was announced by M. V. Zeldich [31].

The paper is a part of the author's doctoral dissertation [14] written under the supervision of Professor Daniel Simson at Nicholas Copernicus University in Toruń. In [14] all r-peak posets of finite prinjective type and their sincere prinjective representations are given, for all $r \geq 2$. The result for $r \geq 3$ will be published in a subsequent paper.

The author would like to express her sincere thanks and appreciation to Professor Daniel Simson for formulating the problem, for his careful reading of the manuscript and for helpful suggestions and comments.

2. Preliminaries and notation. We recall from [17] and [21] that the *coordinate vector* $\mathbf{cdn}\, X \in \mathbb{N}^I$ of a module X in $\mathrm{mod}(KI)$ is defined by

$$(2.1)\qquad (\mathbf{cdn}\,X)(j) = \begin{cases} \dim_K(X_j) & \text{for } j \in \max I,\\ \dim_K(\operatorname{top} X)e_j & \text{for } j \in I \setminus \max I, \end{cases}$$

where $\operatorname{top} X = X/XJ(KI)$ and $J(KI)$ is the Jacobson radical of the algebra KI. We view $\mathbf{cdn}\,X$ as a map $\mathbf{cdn}\,X : I \to \mathbb{N}$.

The poset I is said to be *sincere* if there exists an indecomposable module X in $\operatorname{prin}(KI)$ such that $(\mathbf{cdn}\,X)(j) \neq 0$ for all $j \in I$.

Following [21] we associate with any poset I the *Tits quadratic form* $q_I : \mathbb{Z}^I \to \mathbb{Z}$,

$$(2.2)\qquad q_I(x) = \sum_{i\in I} x_i^2 + \sum_{\substack{i\prec j\\ j\in I\setminus\max I}} x_i x_j - \sum_{p\in\max I}\Big(\sum_{i\prec p} x_i\Big)x_p.$$

The following important characterization was proved in [21, Theorem 3.1] and completed in [15].

THEOREM 2.3. *For every finite poset J and every field K, the following conditions are equivalent.*

(a) *The poset J is of finite prinjective type.*

(b) *The quadratic form q_J is weakly positive, i.e. $q_J(v) > 0$ for any non-zero vector $v \in \mathbb{N}^J$.*

(c) *There exist preprojective components $\widetilde{\mathcal{P}}(J)$ and $\mathcal{P}(J)$ of the Auslander–Reiten translation quivers $\Gamma(\operatorname{prin}(KJ))$ and $\Gamma(J\text{-spr})$ of the categories* $\operatorname{prin}(KJ)$ *and* J-spr, *respectively, and $\Gamma(\operatorname{prin}(KJ)) = \widetilde{\mathcal{P}}(J)$, $\Gamma(J\text{-spr}) = \mathcal{P}(J)$.*

(d) *The set $\mathcal{R}^+_{q_J} = \{v \in \mathbb{N}^J : q_J(v) = 1\}$ of positive roots of q_J is finite.*

(e) *$q_J(\mathbf{cdn}\,X) = 1$ for every indecomposable module X in* $\operatorname{prin}(KJ)$.

(e′) *$q_J(\mathbf{cdn}\,X) = 1$ for every indecomposable object X in J-spr.*

(f) *The map $X \mapsto \mathbf{cdn}X$ defines a bijection $\Gamma(\operatorname{prin}(KJ)) \to \mathcal{R}^+_{q_J}$.* ∎

For the definition of the Auslander–Reiten translation quiver of $\operatorname{mod}(KI)$, $\operatorname{prin}(KI)$ and a preprojective component the reader is referred to [3], [5], [18] and [20].

We recall that a vector $x \in \mathbb{N}^n$ is said to be *sincere* if $x_j \neq 0$ for $j \in \{1, \dots, n\}$.

3. The main classification theorem. One of the main aims of this paper is the following theorem proved in Section 7.

THEOREM 3.1. *Let K be an arbitrary field.*

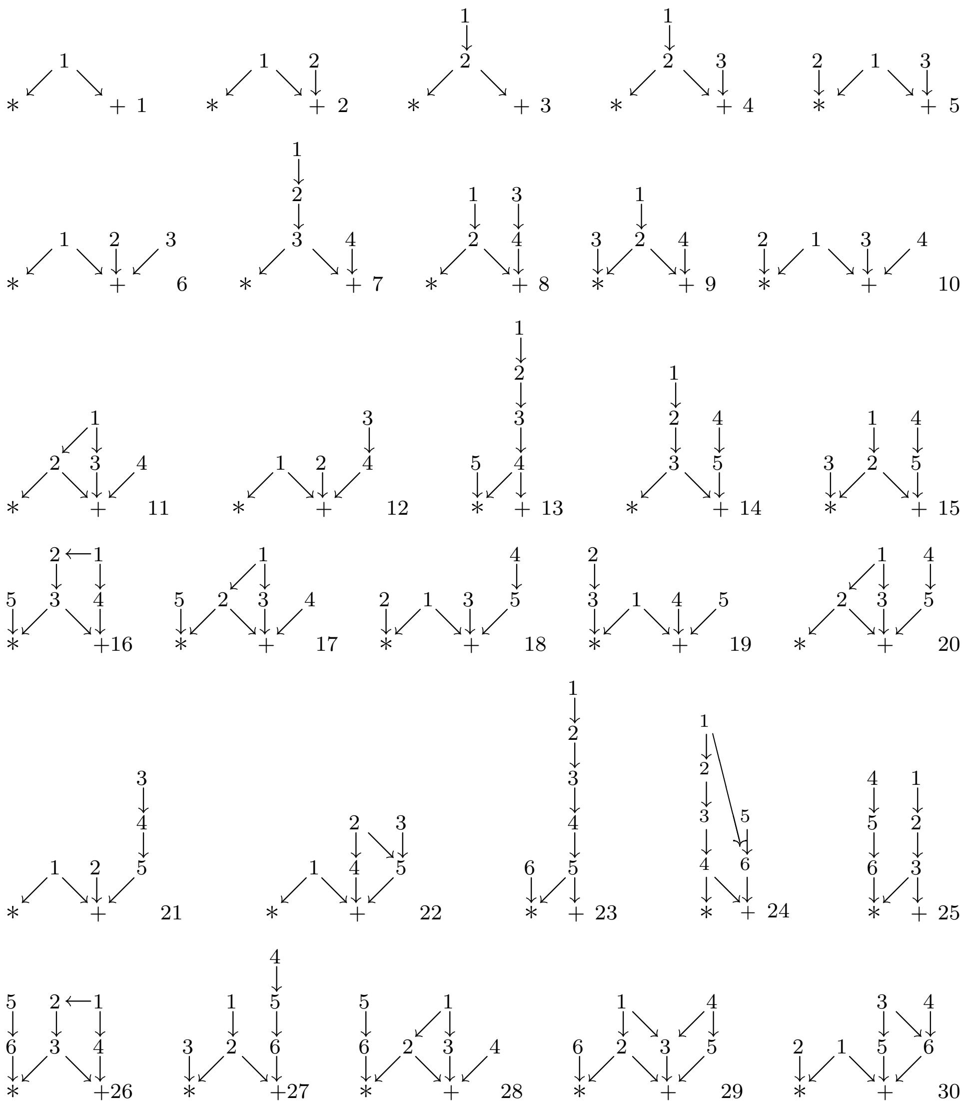
(a) *A two-peak poset I of finite prinjective type is sincere if and only if I has one of the* 60 *forms presented in Tables* 3.2 *below.*

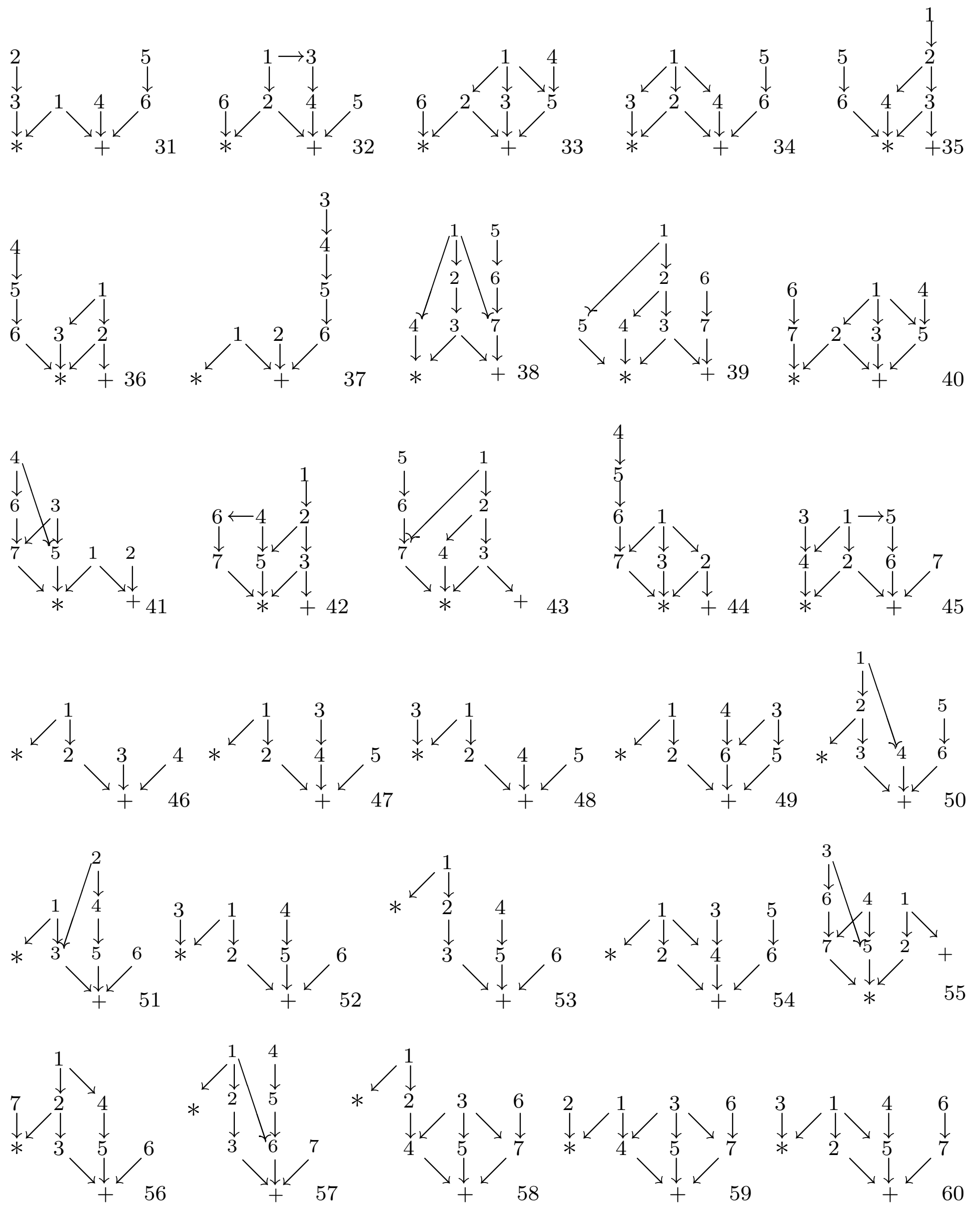
(b) *Let* $\mathcal{F}_i^{(2)}$ *be a sincere poset of Tables* 3.2 *and let* $q_{\mathcal{F}_i^{(2)}}$ *be the associated quadratic form. A sincere positive vector* $z \in \mathbb{N}^{\mathcal{F}_i^{(2)}}$ *satisfies* $q_{\mathcal{F}_i^{(2)}}(z) = 1$ *if and only if* z *is one of the vectors* z_i^j *presented in Tables* 3.3 *below.*

(c) *An indecomposable prinjective* $K\mathcal{F}_i^{(2)}$*-module* X *is sincere if and only if* X *is isomorphic to a module* $M_i^{(j)}$ *presented in Tables* 8.1 *of Section* 8.

Tables 3.2

SINCERE TWO-PEAK POSETS $\mathcal{F}_1^{(2)}, \ldots, \mathcal{F}_{60}^{(2)}$ OF FINITE PRINJECTIVE TYPE





Tables 3.3

Sincere positive roots of the quadratic forms $q_{\mathcal{F}_1^{(2)}}, \ldots, q_{\mathcal{F}_{60}^{(2)}}$

Given a sincere poset $\mathcal{F}_i^{(2)}$ of Tables 3.2 we denote by $z_i^1, \ldots, z_i^{s_j}$ the sincere positive roots of the quadratic form $q_{\mathcal{F}_i^{(2)}}$ of $\mathcal{F}_i^{(2)}$ (see (2.2) and Section 4).

$z_1^1 = (1,1,1)$	$z_2^1 = (1,1,1,1)$	$z_3^1 = (1,1,1,1)$	$z_4^1 = (1,1,1,1,1)$
$z_4^2 = (1,1,1,1,2)$	$z_5^1 = (1,1,1,1,1)$	$z_6^1 = (1,1,1,1,1)$	$z_6^2 = (1,1,1,1,2)$
$z_6^3 = (2,1,1,1,2)$	$z_7^1 = (1,1,1,1,1,2)$	$z_7^2 = (1,1,1,1,2,2)$	$z_8^1 = (1,1,1,1,1,2)$
$z_9^1 = (1,1,1,1,2,2)$	$z_9^2 = (1,2,1,1,2,2)$	$z_9^3 = (2,1,1,1,2,2)$	$z_9^4 = (1,1,1,1,1,1)$

$z_9^5 = (1,1,1,1,2,1)$	$z_9^6 = (1,1,1,1,1,2)$	$z_{10}^1 = (1,1,1,1,1,2)$	$z_{10}^2 = (1,1,1,1,1,1)$
$z_{10}^3 = (2,1,1,1,2,2)$	$z_{10}^4 = (2,1,1,1,1,2)$	$z_{11}^1 = (1,1,1,1,1,2)$	$z_{12}^1 = (1,1,1,1,1,2)$
$z_{12}^2 = (2,1,1,1,1,3)$	$z_{12}^3 = (2,2,1,1,1,3)$	$z_{12}^4 = (2,1,1,1,1,2)$	$z_{13}^1 = (1,1,1,1,1,3,2)$

$z_{13}^2 = (1,1,1,1,2,3,2)$	$z_{13}^3 = (1,1,1,1,1,2,2)$	$z_{14}^1 = (1,1,1,1,1,2,3)$
$z_{14}^2 = (1,1,2,1,1,2,3)$	$z_{14}^3 = (1,2,1,1,1,2,3)$	$z_{14}^4 = (2,1,1,1,1,2,3)$
$z_{14}^5 = (1,1,1,1,1,1,2)$	$z_{14}^6 = (1,1,1,1,1,1,3)$	$z_{14}^7 = (1,1,1,1,1,2,2)$
$z_{15}^1 = (1,2,1,1,1,2,3)$	$z_{15}^2 = (2,1,1,1,1,2,3)$	$z_{15}^3 = (2,2,1,1,1,3,3)$
$z_{15}^4 = (2,2,2,1,1,3,3)$	$z_{15}^5 = (2,2,1,1,1,2,3)$	$z_{15}^6 = (1,2,1,1,1,2,2)$
$z_{15}^7 = (2,1,1,1,1,2,2)$	$z_{15}^8 = (1,1,1,1,1,2,2)$	$z_{15}^9 = (1,1,1,1,1,1,2)$
$z_{16}^1 = (1,1,1,1,1,2,2)$	$z_{17}^1 = (1,1,1,1,1,2,2)$	$z_{17}^2 = (1,1,1,1,1,1,2)$
$z_{17}^3 = (1,2,1,1,1,2,3)$	$z_{17}^4 = (1,2,1,2,1,2,3)$	$z_{17}^5 = (1,2,1,1,1,2,2)$
$z_{18}^1 = (2,1,1,1,1,1,3)$	$z_{18}^2 = (2,1,2,1,1,1,3)$	$z_{18}^3 = (2,1,1,1,1,2,2)$
$z_{18}^4 = (1,1,1,1,1,1,2)$	$z_{18}^5 = (3,1,1,1,1,2,3)$	$z_{18}^6 = (3,1,2,1,1,2,3)$
$z_{18}^7 = (2,1,1,1,1,2,3)$	$z_{18}^8 = (2,1,2,1,1,2,3)$	$z_{18}^9 = (3,1,2,1,1,2,4)$
$z_{18}^{10} = (3,1,2,1,2,2,4)$	$z_{18}^{11} = (3,1,2,2,1,2,4)$	$z_{18}^{12} = (2,1,1,1,1,1,2)$
$z_{19}^1 = (2,1,1,1,1,2,2)$	$z_{20}^1 = (1,1,1,1,1,1,3)$	$z_{20}^2 = (1,2,1,1,1,1,3)$
$z_{20}^3 = (1,2,1,1,1,2,3)$	$z_{20}^4 = (1,1,1,1,1,1,2)$	$z_{21}^1 = (2,1,1,1,1,1,3)$
$z_{21}^2 = (2,2,1,1,1,1,3)$	$z_{21}^3 = (2,2,1,1,1,1,4)$	$z_{21}^4 = (3,2,1,1,1,1,4)$
$z_{21}^5 = (3,2,1,1,1,2,4)$	$z_{22}^1 = (2,1,1,1,1,1,3)$	$z_{23}^1 = (1,1,1,1,1,1,4,3)$
$z_{23}^2 = (1,1,1,1,2,2,4,3)$	$z_{23}^3 = (1,1,1,2,1,2,4,3)$	$z_{23}^4 = (1,1,2,1,1,2,4,3)$
$z_{23}^5 = (1,2,1,1,1,2,4,3)$	$z_{23}^6 = (2,1,1,1,1,2,4,3)$	$z_{23}^7 = (1,1,1,1,1,2,3,2)$
$z_{23}^8 = (1,1,1,1,1,1,3,3)$	$z_{23}^9 = (1,1,1,1,1,2,3,3)$	$z_{23}^{10} = (1,1,1,1,1,2,4,2)$
$z_{23}^{11} = (1,1,1,1,1,1,3,2)$	$z_{24}^1 = (1,1,1,1,1,1,2,3)$	$z_{25}^1 = (1,1,1,1,1,1,3,1)$
$z_{25}^2 = (1,1,1,1,1,1,3,2)$	$z_{25}^3 = (1,1,2,1,1,1,4,2)$	$z_{25}^4 = (1,2,1,1,1,1,4,2)$
$z_{25}^5 = (2,1,1,1,1,1,4,2)$	$z_{25}^6 = (2,2,1,1,1,1,4,2)$	$z_{25}^7 = (2,1,2,1,1,1,4,2)$
$z_{25}^8 = (1,2,2,1,1,1,4,2)$	$z_{25}^9 = (2,2,1,1,1,1,4,3)$	$z_{25}^{10} = (2,1,2,1,1,1,4,3)$
$z_{25}^{11} = (1,2,2,1,1,1,4,3)$	$z_{25}^{12} = (1,1,2,1,1,1,3,2)$	$z_{25}^{13} = (1,2,1,1,1,1,3,2)$
$z_{25}^{14} = (2,1,1,1,1,1,3,2)$	$z_{25}^{15} = (2,2,2,1,1,1,5,3)$	$z_{25}^{16} = (2,2,2,1,1,2,5,3)$
$z_{25}^{17} = (2,2,2,1,2,1,5,3)$	$z_{25}^{18} = (2,2,2,2,1,1,5,3)$	$z_{25}^{19} = (2,2,2,1,1,1,4,3)$
$z_{26}^1 = (1,2,2,1,1,1,4,2)$	$z_{26}^2 = (1,2,2,1,1,2,4,3)$	$z_{26}^3 = (1,2,2,1,2,1,4,3)$
$z_{26}^4 = (1,2,2,1,1,1,3,3)$	$z_{26}^5 = (1,1,2,1,1,1,3,3)$	$z_{26}^6 = (1,2,1,1,1,1,3,3)$
$z_{26}^7 = (1,1,2,1,1,1,3,2)$	$z_{26}^8 = (1,2,1,1,1,1,3,2)$	$z_{26}^9 = (1,1,1,1,1,1,3,2)$
$z_{26}^{10} = (1,1,1,1,1,1,2,2)$	$z_{27}^1 = (1,2,1,1,1,1,2,3)$	$z_{27}^2 = (2,1,1,1,1,1,2,3)$

$z_{27}^{3} = (2,2,1,1,1,1,2,3)$	$z_{27}^{4} = (2,2,1,1,1,1,3,3)$	$z_{27}^{5} = (2,2,2,1,1,1,3,3)$
$z_{27}^{6} = (2,2,1,1,1,1,2,4)$	$z_{27}^{7} = (2,2,1,1,1,1,3,4)$	$z_{27}^{8} = (2,2,2,1,1,1,3,4)$
$z_{27}^{9} = (2,3,2,1,1,1,3,4)$	$z_{27}^{10} = (3,2,2,1,1,1,3,4)$	$z_{27}^{11} = (2,3,1,1,1,1,3,4)$
$z_{27}^{12} = (3,2,1,1,1,1,3,4)$	$z_{27}^{13} = (2,3,2,1,1,1,4,4)$	$z_{27}^{14} = (3,2,2,1,1,1,4,4)$
$z_{27}^{15} = (3,3,2,1,1,1,4,4)$	$z_{27}^{16} = (3,3,2,1,1,1,4,5)$	$z_{27}^{17} = (3,3,2,1,1,2,4,5)$
$z_{27}^{18} = (3,3,2,1,2,1,4,5)$	$z_{27}^{19} = (3,3,2,2,1,1,4,5)$	$z_{28}^{1} = (1,2,1,1,1,1,3,2)$
$z_{28}^{2} = (1,1,1,1,1,1,2,2)$	$z_{28}^{3} = (1,2,1,1,1,1,2,2)$	$z_{28}^{4} = (1,2,1,1,1,1,2,3)$
$z_{28}^{5} = (1,2,1,2,1,1,2,3)$	$z_{28}^{6} = (1,2,1,1,1,1,3,3)$	$z_{28}^{7} = (1,2,1,2,1,1,3,3)$
$z_{28}^{8} = (1,3,1,1,1,1,3,3)$	$z_{28}^{9} = (1,3,1,2,1,1,3,3)$	$z_{28}^{10} = (2,2,1,1,1,1,3,3)$
$z_{28}^{11} = (2,2,1,2,1,1,3,3)$	$z_{28}^{12} = (2,2,1,2,1,1,3,4)$	$z_{28}^{13} = (1,3,1,2,1,1,3,4)$
$z_{28}^{14} = (1,3,2,2,1,1,3,4)$	$z_{28}^{15} = (2,3,1,2,1,1,3,4)$	$z_{28}^{16} = (2,3,1,2,1,1,4,4)$
$z_{28}^{17} = (2,3,1,2,1,2,4,4)$	$z_{28}^{18} = (2,3,1,2,2,1,4,4)$	$z_{29}^{1} = (1,2,1,1,1,1,2,3)$
$z_{30}^{1} = (2,1,1,1,1,1,1,3)$	$z_{30}^{2} = (2,1,1,1,1,1,2,3)$	$z_{30}^{3} = (3,1,1,1,1,1,2,3)$
$z_{30}^{4} = (3,1,1,1,1,1,2,4)$	$z_{30}^{5} = (3,1,1,1,2,1,2,4)$	$z_{30}^{6} = (3,1,1,2,1,1,2,4)$
$z_{30}^{7} = (3,1,1,2,2,1,2,4)$	$z_{30}^{8} = (3,1,1,2,2,1,2,5)$	$z_{30}^{9} = (4,1,1,2,2,1,2,5)$
$z_{30}^{10} = (4,1,1,2,2,1,3,5)$	$z_{30}^{11} = (4,2,1,2,2,1,3,5)$	$z_{31}^{1} = (2,1,1,1,1,1,2,3)$
$z_{31}^{2} = (2,1,1,2,1,1,2,3)$	$z_{31}^{3} = (2,1,1,1,1,1,2,2)$	$z_{31}^{4} = (3,1,1,2,1,1,2,3)$
$z_{31}^{5} = (3,1,1,1,1,1,2,3)$	$z_{31}^{6} = (3,1,1,1,1,1,3,3)$	$z_{31}^{7} = (3,1,1,2,1,1,3,3)$
$z_{31}^{8} = (3,1,1,2,1,1,2,4)$	$z_{31}^{9} = (3,1,1,2,1,2,2,4)$	$z_{31}^{10} = (3,1,1,2,2,1,2,4)$
$z_{31}^{11} = (3,1,1,2,1,2,3,4)$	$z_{31}^{12} = (3,1,1,2,2,1,3,4)$	$z_{31}^{13} = (3,1,1,2,1,1,3,4)$
$z_{31}^{14} = (4,1,1,2,1,1,3,4)$	$z_{31}^{15} = (4,1,1,2,1,2,3,4)$	$z_{31}^{16} = (4,1,1,2,2,1,3,4)$
$z_{31}^{17} = (4,1,1,2,1,2,3,5)$	$z_{31}^{18} = (4,1,1,2,2,1,3,5)$	$z_{31}^{19} = (4,1,1,3,1,2,3,5)$
$z_{31}^{20} = (4,1,1,3,2,1,3,5)$	$z_{31}^{21} = (4,1,1,2,2,2,3,5)$	$z_{31}^{22} = (4,1,1,3,2,2,3,5)$
$z_{31}^{23} = (4,1,1,3,2,2,3,6)$	$z_{31}^{24} = (5,1,1,3,2,2,3,6)$	$z_{31}^{25} = (5,1,1,3,2,2,4,6)$
$z_{31}^{26} = (5,1,2,3,2,2,4,6)$	$z_{31}^{27} = (5,2,1,3,2,2,4,6)$	$z_{32}^{1} = (1,2,1,1,1,1,2,3)$
$z_{32}^{2} = (1,2,1,1,2,1,2,3)$	$z_{32}^{3} = (1,2,1,1,2,1,2,4)$	$z_{32}^{4} = (1,3,1,1,2,1,2,4)$
$z_{32}^{5} = (1,3,1,1,2,1,3,4)$	$z_{32}^{6} = (1,3,1,1,2,2,3,4)$	$z_{33}^{1} = (1,2,1,1,1,1,2,3)$
$z_{34}^{1} = (1,2,1,1,1,1,2,4)$	$z_{35}^{1} = (1,1,1,1,1,1,3,2)$	$z_{35}^{2} = (1,1,2,1,1,1,3,2)$
$z_{35}^{3} = (1,1,1,1,1,1,3,1)$	$z_{35}^{4} = (1,1,2,1,1,1,4,2)$	$z_{35}^{5} = (1,1,2,1,1,2,4,2)$
$z_{35}^{6} = (1,1,2,1,2,1,4,2)$	$z_{36}^{1} = (1,1,1,1,1,1,3,1)$	$z_{36}^{2} = (1,2,1,1,1,1,3,1)$
$z_{36}^{3} = (1,2,1,1,1,1,4,1)$	$z_{36}^{4} = (1,2,2,1,1,1,4,1)$	$z_{36}^{5} = (1,2,2,1,1,1,4,2)$
$z_{36}^{6} = (1,2,1,1,1,1,4,2)$	$z_{36}^{7} = (1,3,1,1,1,1,4,2)$	$z_{36}^{8} = (2,2,1,1,1,1,4,2)$
$z_{36}^{9} = (1,3,2,1,1,1,4,2)$	$z_{36}^{10} = (1,3,2,1,1,1,5,2)$	$z_{36}^{11} = (1,3,2,1,1,2,5,2)$
$z_{36}^{12} = (1,3,2,1,2,1,5,2)$	$z_{36}^{13} = (1,3,2,2,1,1,5,2)$	$z_{36}^{14} = (1,2,1,1,1,1,3,2)$
$z_{37}^{1} = (2,2,1,1,1,1,1,4)$	$z_{37}^{2} = (3,2,1,1,1,1,1,4)$	$z_{37}^{3} = (3,2,1,1,1,1,2,4)$
$z_{37}^{4} = (3,2,1,1,1,1,1,5)$	$z_{37}^{5} = (3,2,1,1,1,1,2,5)$	$z_{37}^{6} = (3,3,1,1,1,1,1,5)$

$z_{37}^{7} = (3, 3, 1, 1, 1, 1, 2, 5)$	$z_{37}^{8} = (4, 2, 1, 1, 1, 1, 2, 5)$	$z_{37}^{9} = (4, 3, 1, 1, 1, 1, 2, 5)$
$z_{37}^{10} = (4, 3, 1, 1, 1, 1, 2, 6)$	$z_{37}^{11} = (4, 3, 1, 1, 1, 2, 2, 6)$	$z_{37}^{12} = (4, 3, 1, 1, 2, 1, 2, 6)$
$z_{37}^{13} = (4, 3, 1, 2, 1, 1, 2, 6)$	$z_{37}^{14} = (4, 3, 2, 1, 1, 1, 2, 6)$	$z_{38}^{1} = (1, 2, 2, 1, 1, 1, 1, 3, 4)$
$z_{39}^{1} = (1, 1, 2, 1, 1, 1, 1, 3, 3)$	$z_{40}^{1} = (1, 2, 1, 1, 1, 1, 1, 3, 3)$	$z_{40}^{2} = (1, 3, 1, 1, 1, 1, 1, 3, 3)$
$z_{40}^{3} = (1, 3, 1, 1, 1, 1, 1, 3, 4)$	$z_{40}^{4} = (1, 3, 2, 1, 1, 1, 1, 3, 4)$	$z_{40}^{5} = (1, 3, 1, 2, 1, 1, 1, 3, 4)$
$z_{40}^{6} = (1, 3, 2, 2, 1, 1, 1, 3, 4)$	$z_{40}^{7} = (1, 3, 2, 2, 1, 1, 1, 3, 5)$	$z_{40}^{8} = (1, 4, 2, 2, 1, 1, 1, 3, 5)$
$z_{40}^{9} = (1, 4, 2, 2, 1, 1, 1, 4, 5)$	$z_{40}^{10} = (1, 4, 2, 2, 1, 1, 2, 4, 5)$	$z_{40}^{11} = (1, 4, 2, 2, 1, 2, 1, 4, 5)$
$z_{40}^{12} = (1, 2, 1, 1, 1, 1, 1, 2, 3)$	$z_{41}^{1} = (3, 1, 1, 1, 1, 1, 1, 4, 2)$	$z_{42}^{1} = (1, 1, 2, 1, 1, 1, 1, 4, 2)$
$z_{43}^{1} = (1, 1, 2, 1, 1, 1, 1, 4, 2)$	$z_{44}^{1} = (1, 3, 2, 1, 1, 1, 1, 5, 2)$	$z_{45}^{1} = (1, 3, 1, 1, 1, 1, 2, 3, 4)$
$z_{46}^{1} = (1, 1, 1, 1, 1, 2)$	$z_{47}^{1} = (1, 1, 1, 1, 1, 1, 2)$	$z_{47}^{2} = (1, 1, 1, 1, 1, 1, 3)$
$z_{47}^{3} = (1, 1, 1, 1, 2, 1, 3)$	$z_{47}^{4} = (2, 1, 1, 1, 1, 1, 3)$	$z_{47}^{5} = (2, 1, 1, 1, 2, 1, 3)$
$z_{47}^{6} = (2, 1, 1, 1, 2, 1, 4)$	$z_{47}^{7} = (2, 1, 1, 2, 2, 1, 4)$	$z_{47}^{8} = (2, 1, 2, 1, 2, 1, 4)$
$z_{48}^{1} = (1, 1, 1, 1, 1, 1, 2)$	$z_{49}^{1} = (1, 1, 1, 1, 1, 1, 1, 3)$	$z_{49}^{2} = (2, 1, 1, 1, 1, 1, 1, 3)$
$z_{49}^{3} = (2, 1, 1, 1, 1, 1, 1, 4)$	$z_{49}^{4} = (2, 1, 1, 1, 2, 1, 1, 4)$	$z_{49}^{5} = (2, 1, 1, 2, 1, 1, 1, 4)$
$z_{49}^{6} = (2, 1, 1, 2, 2, 1, 1, 4)$	$z_{49}^{7} = (2, 1, 1, 2, 2, 1, 1, 5)$	$z_{49}^{8} = (3, 1, 1, 2, 2, 1, 1, 5)$
$z_{49}^{9} = (2, 2, 1, 2, 2, 1, 1, 5)$	$z_{49}^{10} = (3, 1, 1, 2, 2, 1, 2, 5)$	$z_{50}^{1} = (1, 1, 1, 1, 1, 1, 1, 3)$
$z_{51}^{1} = (2, 1, 1, 1, 1, 2, 1, 4)$	$z_{52}^{1} = (1, 1, 1, 1, 1, 1, 1, 2)$	$z_{52}^{2} = (1, 1, 1, 1, 1, 1, 1, 3)$
$z_{52}^{3} = (1, 1, 1, 1, 1, 2, 1, 3)$	$z_{52}^{4} = (2, 1, 1, 1, 1, 2, 1, 3)$	$z_{52}^{5} = (2, 1, 1, 1, 1, 2, 2, 3)$
$z_{52}^{6} = (2, 1, 1, 1, 1, 1, 1, 3)$	$z_{52}^{7} = (2, 1, 1, 1, 1, 1, 2, 3)$	$z_{52}^{8} = (2, 1, 1, 1, 1, 2, 1, 4)$
$z_{52}^{9} = (2, 1, 1, 1, 1, 2, 2, 4)$	$z_{52}^{10} = (2, 1, 1, 1, 2, 2, 1, 4)$	$z_{52}^{11} = (2, 1, 1, 2, 1, 2, 1, 4)$
$z_{52}^{12} = (2, 1, 1, 1, 2, 2, 2, 4)$	$z_{52}^{13} = (2, 1, 1, 2, 1, 2, 2, 4)$	$z_{52}^{14} = (3, 1, 1, 1, 2, 2, 2, 4)$
$z_{52}^{15} = (3, 1, 1, 2, 1, 2, 2, 4)$	$z_{52}^{16} = (3, 1, 1, 1, 2, 2, 2, 5)$	$z_{52}^{17} = (3, 1, 1, 2, 1, 2, 2, 5)$
$z_{52}^{18} = (3, 1, 1, 1, 2, 3, 2, 5)$	$z_{52}^{19} = (3, 1, 1, 2, 1, 3, 2, 5)$	$z_{52}^{20} = (3, 1, 1, 2, 2, 3, 2, 5)$
$z_{52}^{21} = (3, 1, 1, 2, 2, 2, 2, 5)$	$z_{52}^{22} = (3, 1, 1, 2, 2, 3, 2, 6)$	$z_{52}^{23} = (3, 2, 1, 2, 2, 3, 2, 6)$
$z_{52}^{24} = (4, 1, 1, 2, 2, 3, 2, 6)$	$z_{52}^{25} = (4, 1, 1, 2, 2, 3, 3, 6)$	$z_{52}^{26} = (4, 1, 2, 2, 2, 3, 3, 6)$
$z_{52}^{27} = (3, 1, 1, 1, 1, 2, 2, 4)$	$z_{53}^{1} = (1, 1, 1, 1, 1, 1, 1, 3)$	$z_{53}^{2} = (1, 1, 1, 1, 1, 2, 1, 3)$
$z_{53}^{3} = (1, 1, 1, 1, 1, 2, 1, 4)$	$z_{53}^{4} = (1, 1, 1, 1, 2, 2, 1, 4)$	$z_{53}^{5} = (1, 1, 1, 2, 1, 2, 1, 4)$
$z_{53}^{6} = (2, 1, 1, 1, 2, 2, 1, 4)$	$z_{53}^{7} = (2, 1, 1, 2, 1, 2, 1, 4)$	$z_{53}^{8} = (2, 1, 1, 1, 1, 2, 1, 4)$
$z_{53}^{9} = (2, 1, 1, 1, 2, 2, 1, 5)$	$z_{53}^{10} = (2, 1, 1, 2, 1, 2, 1, 5)$	$z_{53}^{11} = (2, 1, 1, 2, 2, 2, 1, 5)$
$z_{53}^{12} = (2, 1, 1, 2, 2, 3, 1, 5)$	$z_{53}^{13} = (2, 1, 1, 1, 2, 3, 1, 5)$	$z_{53}^{14} = (2, 1, 1, 2, 1, 3, 1, 5)$
$z_{53}^{15} = (2, 1, 1, 2, 2, 3, 1, 6)$	$z_{53}^{16} = (2, 1, 2, 2, 2, 3, 1, 6)$	$z_{53}^{17} = (2, 2, 1, 2, 2, 3, 1, 6)$
$z_{53}^{18} = (3, 1, 1, 2, 2, 3, 1, 6)$	$z_{53}^{19} = (3, 1, 1, 2, 2, 3, 2, 6)$	$z_{54}^{1} = (1, 1, 1, 1, 1, 1, 1, 3)$
$z_{55}^{1} = (2, 1, 1, 1, 1, 1, 1, 4, 1)$	$z_{56}^{1} = (1, 2, 1, 1, 1, 2, 1, 2, 4)$	$z_{57}^{1} = (1, 1, 1, 1, 1, 1, 2, 1, 4)$
$z_{58}^{1} = (2, 1, 1, 1, 2, 2, 1, 1, 5)$	$z_{59}^{1} = (3, 1, 1, 1, 2, 2, 1, 2, 5)$	$z_{60}^{1} = (1, 1, 1, 1, 1, 1, 1, 1, 3)$

4. Integral quadratic forms and their positive roots. In this section we collect the fundamental definitions and facts about quadratic forms and their roots, which will be needed in Sections 5 and 6. For more information about this topic the reader is referred to [18, Section 1], [8].

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be an *integral quadratic form*, that is, q has the form

$$q(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} \alpha_{ij} x_i x_j, \tag{4.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ and $\alpha_{ij} \in \mathbb{Z}$.

We call the vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ *positive* if $x \neq 0$ and $x_i \geq 0$ for all $i = 1, \dots, n$. The integral quadratic form (4.1) is said to be *weakly positive* provided $q(x) > 0$ for all positive vectors $x \in \mathbb{Z}^n$. A vector $x \in \mathbb{Z}^n$ satisfying $q(x) = 1$ is called a *root* of q. It is well known that a weakly positive integral quadratic form has only finitely many positive roots (see [18, Section 1]).

A vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ is called *sincere* if $x_i \neq 0$ for all $i = 1, \dots, n$. A weakly positive integral quadratic form is said to be *sincere* provided there exists a sincere positive root x of q.

For each $i = 1, \dots, n$ we denote by $e(i)$ the vector of $\mathbb{Z}^n$ having 1 at the ith position and zeros elsewhere.

Let $(-,-)_q$ be the symmetric $\mathbb{Z}$-bilinear form corresponding to q, that is,

$$2(x, y)_q = q(x + y) - q(x) - q(y).$$

Following [18] we define the $\mathbb{Z}$-linear form $D_i q : \mathbb{Z}^n \to \mathbb{Z}$ by the formula

$$D_i q(x) := 2(e(i), x) = 2x_i + \sum_{i \neq j} \alpha_{ij} x_j.$$

Let q be the quadratic form (4.1). We start with the following three lemmata proved in [18]. Then we prove several technical facts which are fundamental for our algorithms of Section 6.

Lemma 4.2. *If x is a root of an integral quadratic form* (4.1), *then* $\sum_{i=1}^{n} x_i D_i q(x) = 2$.

Proof. For a root $x = (x_1, \dots, x_n)$ of q we have

$$\sum_{i=1}^{n} x_i D_i q(x) = 2 \sum_{i=1}^{n} x_i (e(i), x)_q = 2 \Big(\sum_{i=1}^{n} x_i e(i), x \Big)_q = 2(x, x)_q = 2q(x) = 2,$$

because $q(x) = 1$. ∎

Lemma 4.3. *Let $x = (x_1, \dots, x_n)$ be a positive root of a weakly positive integral quadratic form* (4.1) *and let $i \in \{1, \dots, n\}$ be such that $x_i \neq 0$. If $x \neq e(i)$, then $|D_i q(x)| \leq 1$.*

Proof. Let $i \in \{1, \ldots, n\}$ be such that $x_i > 0$ and $x \neq e(i)$. From our assumptions it follows that $x + e(i)$ and $x - e(i)$ are positive and so

$$0 < q(x \pm e(i)) = q(x) + q(e(i)) \pm 2(e(i), x)_q = 1 + 1 \pm D_i q(x).$$

This yields $-2 < D_i q(x) < 2$. ∎

COROLLARY 4.4. *Let $x = (x_1, \ldots, x_n)$ be a positive root of a weakly positive integral quadratic form* (4.1). *If $x \neq e(j)$ for $j = 1, \ldots, n$, then there exists $i \in \{1, \ldots, n\}$ such that $x_i \neq 0$ and $D_i q(x) = 1$.*

Proof. By Lemma 4.2, there is $i \in \{1, \ldots, n\}$ such that $x_i D_i q(x) \geq 1$. Since x is positive, we have $x_i \geq 1$ and $D_i q(x) \geq 1$. Then Lemma 4.3 yields $D_i q(x) = 1$. ∎

LEMMA 4.5. *If $x = (x_1, \ldots, x_n)$ is a root of an integral quadratic form* (4.1)*, then $x - D_i q(x) e(i)$ is a root of q for each $i = 1, \ldots, n$.*

Proof. Let $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} q(x - D_i q(x) e(i)) &= (x - D_i q(x) e(i), x - D_i q(x) e(i))_q \\ &= q(x) - 2(D_i q(x) e(i), x)_q + q(D_i q(x) e(i)) \\ &= 1 - 2 D_i q(x)(e(i), x)_q + (D_i q(x))^2 q(e(i)) \\ &= 1 - (D_i q(x))^2 + (D_i q(x))^2 = 1. \ \blacksquare \end{aligned}$$

As an immediate consequence of Corollary 4.4 and Lemma 4.5 we get

COROLLARY 4.6. *Let $x = (x_1, \ldots, x_n)$ be a positive root of a weakly positive integral quadratic form* (4.1). *Suppose $x \neq e(i)$ for all i. Then there exists $i \in \{1, \ldots, n\}$ such that $x_i \neq 0$ and $x - e(i)$ is a positive root of q.* ∎

Throughout this paper the following fact will be used.

COROLLARY 4.7. *If there exists a sincere positive root $x = (x_1, \ldots, x_n)$ of a weakly positive integral quadratic form* (4.1)*, then there exists a sincere positive root $y = (y_1, \ldots, y_n)$ of q such that*

$$D_i q(y) = 1 \quad \textit{implies} \quad y_i = 1. \tag{4.8}$$

Proof. Let $y^{(0)} = x$ be a sincere positive root of q. If x does not satisfy (4.8) for each $i = 1, \ldots, n$, then there exists $i \in \{1, \ldots, n\}$ such that $D_i q(x) = 1$ and $x_i > 1$, and according to Lemma 4.5 the vector $y^{(1)} = x - e(i)$ is a sincere positive root of q. If $y^{(1)}$ does not satisfy (4.8) for each $i = 1, \ldots, n$, then there exists $j \in \{1, \ldots, n\}$ such that $y^{(2)} = y^{(1)} - e(j)$ is a sincere positive root of q. Continuing, we find a sincere positive root y satisfying (4.8), because otherwise we get an infinite chain $y^{(0)}, y^{(1)}, \ldots$ of pairwise different sincere positive roots of q. This is a contradiction, because q is weakly positive and therefore it has only finitely many positive roots. ∎

LEMMA 4.9. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$, $n > 0$, be an integral quadratic form, and let z and x be roots of q. If $x = z \pm e(i)$ for some $i \in \{1, \dots, n\}$, then $D_i q(z) = \mp 1$.*

Proof. By our assumptions,

$$\begin{aligned} 1 = q(x) &= q(z \pm e(i)) = (z \pm e(i), z \pm e(i))_q \\ &= q(z) \pm 2(z, e(i))_q + q(e(i)) = 1 + 1 \pm D_i q(z) = 2 \pm D_i q(z). \end{aligned}$$

It follows that $D_i q(z) = \mp 1$. ∎

The following fact is an immediate consequence of Lemmata 4.5 and 4.9.

COROLLARY 4.10. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$, $n > 0$, be a quadratic form and let z be a root of q. Then $z \pm e(i)$ is a root of q if and only if $D_i q(z) = \mp 1$.* ∎

PROPOSITION 4.11. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive sincere quadratic form, and let z be a sincere positive root of q. There exists a chain*

$$x^{(1)}, \dots, x^{(m)} \tag{4.12}$$

of positive roots of q such that

(a) *$x^{(1)} = z$, $x^{(i)} = x^{(i-1)} - e(j_i)$ for any $i = 2, \dots, m$ and for some $j_i \in \{1, \dots, n\}$,*

(b) *the roots $x^{(1)}, \dots x^{(m-1)}$ are sincere,*

(c) *$x^{(m)}_{j_m} = 0$ and $x^{(m)}_j > 0$ for $j \neq j_m$.*

Proof. Let $z = x^{(1)}$ be a sincere positive root of q. By Corollary 4.4, there exists $j_2 \in \{1, \dots, n\}$ such that $D_{j_2} q(z) = 1$. Then according to Corollary 4.10 the vector $x^{(2)} = x^{(1)} - e(j_2)$ is a positive root of q. If $x^{(2)}_{j_2} \neq 0$, then $x^{(2)}$ is sincere and there exists $j_3 \in \{1, \dots, n\}$ such that $x^{(3)} = x^{(2)} - e(j_3)$ is a positive root of q. Continuing, we construct a chain $x^{(1)}, x^{(2)}, \dots$ of sincere positive roots of q satisfying the required conditions. Since q is weakly positive, it has only finitely many positive roots. Therefore there exists m such that $x^{(m)}_{j_m} = 0$. ∎

We recall from [18, Section 1] the following fact.

COROLLARY 4.13. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive quadratic form, and let $z \neq 0$ be a positive root of q. There exists a chain*

$$x^{(1)}, \dots, x^{(m)} \tag{4.14}$$

of positive roots of q such that

(a) *$x^{(1)} = z$, $x^{(i)} = x^{(i-1)} - e(j_i)$ for $i \in \{2, \dots, m\}$ and for some $j_i \in \{1, \dots, n\}$,*

(b) *$x^{(m)} = e(j)$ for some $j \in \{1, \dots, n\}$.*

Proof. Apply Proposition 4.11 and simple induction arguments. ■

Let z be a root of an integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$. If there exists a chain (4.14) satisfying (a) and (b) of Corollary 4.13, then z is called a *Weyl root* of q (see [18, Section 1]). By Corollary 4.13, all positive roots of a weakly positive quadratic form are Weyl roots. Let us indicate how we can construct all Weyl roots of an integral quadratic form, and all positive roots of a weakly positive integral quadratic form.

REMARK 4.15. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly positive quadratic form. To construct all positive roots of q it is enough to construct all possible chains $x^{(1)}, \ldots, x^{(m)}$ of the form (4.14) satisfying conditions (a) and (b) of Corollary 4.13. The elements of these chains form the set of all positive roots of q. We construct these chains inductively as follows. If we have constructed a chain $y^{(1)}, \ldots, y^{(m)}$ of the form (4.14) and of length m, then for any $i \in \{1, \ldots, n\}$ such that $D_i q(y^{(1)}) = -1$ we form the chain $y^{(0,i)}, y^{(1)}, \ldots, y^{(m)}$, where $y^{(0,i)} = y^{(1)} + e(i)$. Obviously this chain is of the form (4.14), of length $m+1$ and satisfies (a) and (b). Since q has only finitely many positive roots this procedure will stop and all chains of the form (4.14) satisfying (a) and (b) will be constructed.

5. Sincere posets of finite prinjective type. The following lemma shows that there are only finitely many sincere two-peak posets of finite prinjective type. A similar result was given by S. Kasjan [9].

LEMMA 5.1. *Let I be a sincere poset of finite prinjective type such that $|\max I| = 2$. Then $|I| \le 25$.*

Proof. Let z be a sincere positive root of the quadratic form (2.2). We can assume that $D_i q_I(z) \ge 0$ for each $i \in I$. Indeed, if there exists $i_1 \in I$ such that $D_{i_1} q_I(z) = -1$, then $z^{(1)} = z + e(i_1) \ne z$ is a positive root of q_I such that $z^{(1)}_{i_1} > z_{i_1}$ (see Lemma 4.5). If there exists $i_2 \in I$ such that $D_{i_2} q_I(z^{(1)}) = -1$, then $z^{(2)} = z^{(1)} + e(i_2) \ne z^{(1)}$ is a positive root of q_I such that $z^{(2)}_{i_2} > z^{(1)}_{i_2}$. In this way we construct a chain $z, z^{(1)}, z^{(2)}, \ldots$ of pairwise different positive roots of q_I. Since I is of finite prinjective type, q_I has only finitely many positive roots and so there exists $j \in \mathbb{N}$ such that $D_i q_I(z^{(j)}) > -1$ for all $i \in I$. It follows that there exists a positive root z satisfying $D_i q_I(z) \ge 0$ for each $i \in I$. Such a root is maximal in the sense of [18, Section 1].

In particular, z satisfies $D_* q_I(z) \ge 0$, $D_+ q_I(z) \ge 0$, where $\max I = \{*, +\}$. On the other hand, Theorem 1 of Ovsienko [18] yields $D_p q_I(z) = 2z_p - \sum_{i \prec p} z_i \le 12 - \sum_{i \prec p} z_i$ for $p = *$ and $p = +$. Hence there exist at most 12 elements $i \in I$ such that $i \prec p$. As I is connected, there exists at least one $i \in I$ satisfying $i \prec *$ and $i \prec +$. It follows that $|I| \le 12 + 12 - 1 + 2 = 25$. ■

The following theorem provides the main tool for our algorithms of Section 6.

THEOREM 5.2. *Let I be a sincere two-peak poset of finite prinjective type such that $|I| > 3$. There exists $j \in I^- = I \setminus \max I$ such that $J = I \setminus \{j\}$ is a sincere connected two-peak poset of finite prinjective type.*

Proof. Let q_I be the Tits quadratic form of I. Since I is sincere, by Theorem 2.3 the quadratic form q_I has a sincere positive root z. In view of Corollary 4.7 we assume that z satisfies the condition (4.8) for all $i \in I$.

We claim that there exists $i \in I^-$ such that $D_i q_I(z) = 1$. Indeed, by Corollary 4.4, there exists $j \in I$ such that $D_j q_I(z) = 1$. Note that $z_j = 1$ by (4.8). Assume that $j \in \max I$. Then

$$1 = D_j q_I(z) = 2z_j - \sum_{k \prec j} z_k = 2 - \sum_{k \prec j} z_k$$

and we get $\sum_{k \prec j} z_k = 1$. Since the root z is sincere and positive, we conclude that there is exactly one $k \in I$ such that $k \prec j$. Moreover $z_j D_j q_I(z) = 1$, hence by Lemma 4.2 there exists $t \in I$, $t \neq j$, satisfying $D_t q_I(z) = 1$. From our assumptions it follows that $z_t = 1$. If we assume that $t \in \max I$, then as above we show that there exists exactly one $k \in I$ satisfying $k \prec t$. Consequently, $|I| = 3$, contrary to our assumption. Therefore there is an $i \in I^-$ which satisfies $D_i q_I(z) = 1$ and our claim is proved.

Let $i \in I^-$ be such that $D_i q_I(z) = 1$ and $z_i = 1$. By Lemma 4.5, $x = z - e(i)$ is a positive root of q_I satisfying $x_i = 0$. View $J = I \setminus \{i\}$ as a peak subposet of I. We show that J is sincere and connected. It is easy to see that $x' \in \mathbb{N}^J$ is a positive sincere root of q_J, where $x'_j = x_j$ for $j \in J$. It follows that J is sincere and of course $|J| = |I| - 1$, $|\max J| = 2$. Moreover, J is connected. Indeed, if $J = J_1 \cup J_2$, where $J_1 \cap J_2 = \emptyset$, then $1 = q_J(x') = q_{J_1}(x^{(1)}) + q_{J_2}(x^{(2)})$, where $x^{(i)}_j = x'_j$ for $i = 1, 2$, $j \in J_i$. The quadratic forms q_{J_1}, q_{J_2} are weakly positive, hence $q_{J_1}(x^{(1)}) = 0$ or $q_{J_2}(x^{(2)}) = 0$. Hence $x^{(1)} = 0$ or $x^{(2)} = 0$. Consequently $J_1 = \emptyset$ or $J_2 = \emptyset$, because the vector x' is sincere. ■

Throughout this section I is a sincere two-peak poset of finite prinjective type. Assume that $I = \{1, \ldots, n, *, +\}$ and $\max I = \{*, +\}$. Moreover let q_I be the Tits form of I and z be a positive sincere root of q_I. In addition, z is assumed to satisfy the condition (4.8) for all $i \in I$. For $i \in I$ we define the following two subsets of I:

$$i^{\vartriangle} = \{s \in I^- : s \succeq i\}, \qquad i^{\triangledown} = \{s \in I^- : s \preceq i\}.$$

Theorem 5.2 implies that if $|I| > 3$, then there exists $i \in I^-$ such that $I \setminus \{i\}$ is a sincere two-peak poset. For simplicity the i will be chosen in a special way. For this the following facts will be needed.

LEMMA 5.3. *Let I and z be as above and let $i \in I^- = I \setminus \max I$ be such that $D_i q_I(z) = 1$. If the subset $i^\triangledown = \{c_1 \prec \ldots \prec c_k = i\}$ of I is linearly ordered and satisfies the condition*

$$c_1 \prec * \ (\textit{resp. } c_1 \prec +) \ \Rightarrow \ i \prec * \ (\textit{resp. } i \prec +), \tag{5.4}$$

then $D_{c_j} q_I(z) = 1$ for $j = 1, \ldots, k$.

Proof. First we note that our assumptions yield

$$D_{c_j} q_I(z) = D_i q_I(z) - z_i + z_{c_j} + d_j,$$

where $j = 1, \ldots, k$ and $d_j \geq 0$ is an integer. This implies that $D_{c_j} q_I(z) = z_{c_j} + d_j$, because $z_i = 1$ and $D_i q_I(z) = 1$. The root z is sincere, hence $D_{c_j} q_I(z) = z_{c_j} + d_j > 0$. On the other hand, Lemma 4.3 gives $D_{c_j} q_I(z) < 2$. Consequently, $D_{c_j} q_I(z) = 1$. ∎

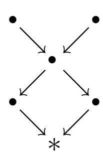
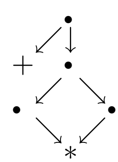
LEMMA 5.5. *Let I and z be as above and let $i \in I^- = I \setminus \max I$ be such that $D_i q_I(z) = 1$. If the subset $i^\vartriangle = \{i = c_1 \prec \ldots \prec c_k\}$ of I is linearly ordered and satisfies the condition*

$$i \prec * \ (\textit{resp. } i \prec +) \ \Rightarrow \ c_k \prec * \ (\textit{ resp. } c_k \prec +), \tag{5.6}$$

then $D_{c_j} q_I(z) = 1$ for $j = 1, \ldots, k$.

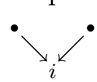
Proof. The proof, similar to that of Lemma 5.3, is left to the reader. ∎

LEMMA 5.7. *Let I and z be as above. Assume that the assumptions of Lemmata* 5.3 *and* 5.5 *are not satisfied for any $i \in I^-$ such that $D_i q_I(z) = 1$. Then I contains a two-peak proper subposet J, which contains one of the following two posets:*

(5.8)  (5.9) 

where $\max I = \max J = \{*, +\}$.

Proof. Let $i \in I^-$ be such that $D_i q_I(z) = 1$. Since the assumptions of Lemmata 5.3 and 5.5 are not satisfied, we have $i \notin *^\triangledown \cap +^\triangledown$. Indeed, by [21, Theorem 3.1] the poset $*^\triangledown \cap +^\triangledown$ is linearly ordered (because I is of finite prinjective type). Therefore, if $i \in *^\triangledown \cap +^\triangledown$, then $i^\triangledown$ is linearly ordered and (5.4) is satisfied, contrary to our assumptions. Hence $i \notin *^\triangledown \cap +^\triangledown$; say $i \prec *$ and $i \not\prec +$. Therefore if the assumptions of Lemma 5.3 are not satisfied, then either $i^\triangledown$ is not linearly ordered or (5.4) is not satisfied. If $i^\triangledown$ is not linearly ordered, then I contains the poset



and if (5.4) is not satisfied, then I contains

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ i & & + \end{tikzcd}$$

Moreover if the assumptions of Lemma 5.5 are not satisfied, then $i^{\vartriangle} \subseteq I^-$ is not linearly ordered, because in view of $i \prec *$ and $i \not\prec +$, the condition (5.6) is automatically satisfied. Therefore I contains (5.8) or (5.9).

As in the proof of Theorem 5.2 we show that there exists $j \in I$, $j \neq i$, such that $D_j q_I(z) = 1$. Then $z_j = 1$ by (4.8). Therefore $J = I \setminus \{j\}$ is a sincere poset (cf. the proof of Theorem 5.2). If $j \in \max I$, then J is a sincere one-peak poset which contains one of the following posets:

$$\begin{tikzcd}[column sep=small] \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & * & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \bullet \arrow[d] & \\ & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & * & \end{tikzcd}$$

and this is impossible (see [20, Theorem 10.2]). Thus we can assume that $j \in I^-$. Hence clearly $J = I \setminus \{j\}$ contains (5.8) or (5.9). ∎

6. Algorithms for the construction of sincere posets of finite prinjective type. In this section we give two algorithms for the construction of sincere two-peak posets of finite prinjective type. We start with some definitions and preliminary constructions.

Definition 6.1. We say that a sincere poset $J = (J, \preceq_J)$ is *dominated* by a poset $I = (I, \preceq_I)$ (or I *dominates* J) if $|J| = |I| + 1$ and

(a) I is sincere.

(b) There exists $j \in J^-$ such that $J \setminus \{j\} = I$ and the relation $\preceq_I$ is the restriction of $\preceq_J$ to I.

(c) There exists a sincere positive root z of the quadratic form q_I such that $\widetilde{z}$ is a sincere positive root of q_J, where $\widetilde{z}_i = z_i$ for $i \neq j$ and $\widetilde{z}_j = 1$.

If in addition $j \in J^-$ is a minimal (resp. maximal) element in J^-, then we say that J is *min-dominated* (resp. *max-dominated*) by I.

Proposition 6.2. *If J is a sincere two-peak poset of finite prinjective type such that $|J| > 3$, then there exists a sincere two-peak poset I which dominates J.*

Proof. Let $\widetilde{z}$ be a sincere positive root of q_J. Assume that $\widetilde{z}$ satisfies (4.8) for all $i \in J$. There exists $i \in J^-$ such that $D_i q_J(\widetilde{z}) = 1$ (cf. the proof of Theorem 5.2). The poset $I = J \setminus \{i\}$ with relation $\preceq_I$ induced from $\preceq_J$ is connected. Moreover, since $\widetilde{z}_i = 1$, the positive root $\widetilde{z} - e(i)$ of q_J has ith coordinate zero. From this we easily conclude that z is a sincere positive root of q_I, where $z_j = \widetilde{z}_j$ for $j \in I$. Therefore I dominates J. This finishes the proof, because obviously $|\max I| = 2$. ∎

A family $\mathcal{X}_I$. Let $I = (I, \preceq)$ be such that $|I| = n$, $\max I = \{*, +\}$. Let $X_1, \ldots, X_k$ be all pairwise different sets consisting of pairwise incomparable elements of I^-. We define new posets $(I_1, \preceq_1), \ldots, (I_k, \preceq_k), (I_*, \preceq_*), (I_+, \preceq_+)$ as follows:

• For $j = 1, \ldots, k$ we set $I_j = I \cup \{c\}$, where $c \notin I$ and $\preceq_j$ is the smallest partial order relation satisfying:

(a) $c \preceq_j i$ for all $i \in X_j$,
(b) $i \preceq_j s$ if and only if $i \preceq s$, for $i, s \in I$.

• For $j = *, +$ we set $I_j = I \cup \{c\}$, where $c \notin I$ and $\preceq_j$ is the smallest partial order relation satisfying:

(a) $c \preceq_j j$ $(j = *, +)$,
(b) $i \preceq_j s$ if and only if $i \preceq s$, for $i, s \in I$.

We set $\mathcal{X}_I = \{I_1, \ldots, I_k, I_*, I_+\}$.

LEMMA 6.3. *If J is a poset of finite prinjective type and J is min-dominated by a poset I, then $J \in \mathcal{X}_I$.*

Proof. By Definition 6.1 there exists a minimal element c of J such that $I = J \setminus \{c\}$ and $\preceq_I$ is the restriction of $\preceq_J$. Let X be the set of all $i \in J^-$, $i \neq c$, such that $c \preceq_J i$ and the relation $c \preceq_J i$ is minimal (i.e. if there exists $k \in J$ such that $c \preceq_J k \preceq_J i$, then $k = c$ or $k = i$). Note that elements of X are pairwise incomparable in J^-, because otherwise there exist $i, j \in X$, $i \neq j$, such that $i \preceq_J j$. Then the relation $c \preceq_J j$ is not minimal, contrary to assumption.

First we consider the case $X \neq \emptyset$. Since X consists of pairwise incomparable elements of I^- we conclude that $X = X_l$, for some $l \geq 1$, is one of the sets associated with I in the definition of $\mathcal{X}_I$ above. In this case $J = I_l$.

In the case $X = \emptyset$ we have $J = I_*$ (resp. $J = I_+$) if $c \preceq_J *$ and $c \npreceq_J +$ (resp. $c \preceq_J +$ and $c \npreceq_J *$). Moreover if $c \preceq_J *$ and $c \preceq_J +$ then the subposet $*^\nabla \cap +^\nabla$ of J contains two incomparable elements, because $I = J \setminus \{c\}$ is connected and c is a minimal element in J. Then J is of infinite prinjective type by [21, Theorem 3.1]. This contradicts our assumptions, and therefore $c \notin *^\nabla \cap +^\nabla$. Hence $J \in \mathcal{X}_I$. ∎

A family $\mathcal{X}^I$. Now we dually define a finite family $\mathcal{X}^I$ as follows:

We form new two-peak posets $(I^1, \preceq^1), \ldots, (I^k, \preceq^k), (I^*, \preceq^*), (I^+, \preceq^+)$, $(I^{1,*}, \preceq^{1,*}), \ldots, (I^{k,*}, \preceq^{k,*}), (I^{1,+}, \preceq^{1,+}), \ldots, (I^{k,+}, \preceq^{k,+})$, in the following way:

• For $j = 1, \dots, k$ we set $I^j = I \cup \{c\}$, where $c \notin I$, $c \notin \max I^j$ and $\preceq^j$ is the smallest partial order relation satisfying:

(a) $i \preceq^j c$ for all $i \in X_j$,
(b) $i \preceq^j s$ if and only if $i \preceq s$, for $i, s \in I$,
(c) $c \preceq^j *$ (resp. $c \preceq^j +$) if for all $i \in X_j$ we have $i \preceq *$ (resp. $i \preceq +$).

• For $j = *, +$ we set $I^j = I \cup \{c\}$, where $c \notin I$, $c \notin \max I^j$ and $\preceq^j$ is the smallest partial order relation satisfying:

(a) $c \preceq^j j$ $(j = *, +)$,
(b) $i \preceq^j s$ if and only if $i \preceq s$, for $i, s \in I$.

• For $j = 1, \dots, k$ and $p = *, +$ we set $I^{j,p} = I \cup \{c\}$, where $c \notin I$, $c \notin \max I^{j,p}$ and $\preceq^{j,p}$ is the smallest partial order relation satisfying:

(a) $i \preceq^{j,p} c$ for all $i \in X_j$,
(b) $i \preceq^{j,p} s$ if and only if $i \preceq s$, for $i, s \in I$.
(c) $c \preceq^{j,p} p$ if for all $i \in X_j$ we have $i \preceq p$.

We set $\mathcal{X}^I = \{I^1, \dots, I^k, I^*, I^+, I^{1,*}, \dots, I^{k,*}, I^{1,+}, \dots, I^{k,+}\}$ and note that $I_p = I^p$ for $p = *, +$.

Lemma 6.4. *If J is a poset of finite prinjective type and J is max-dominated by a poset I, then $J \in \mathcal{X}^I$.*

Proof. By Definition 6.1 there exists a maximal element c of J^- such that $I = J \setminus \{c\}$ and $\preceq_I$ is the restriction of $\preceq_J$. Let X be the set of all $c \neq i \in J^-$ such that the relation $i \preceq_J c$ is minimal (i.e. if there exists $k \in J$ such that $i \preceq_J k \preceq_J c$, then $k = c$ or $k = i$). Note that elements of X are pairwise incomparable in J^-, because otherwise there exist $i, j \in X$, $i \neq j$, such that $i \preceq_J j$. Then the relation $i \preceq_J c$ is not minimal and we get a contradiction. Therefore if $X \neq \emptyset$ then $X = X_l$, for some $l \geq 1$, is one of the sets associated with I in the definition of $\mathcal{X}^I$. We note that only the following cases are possible:

(a*) $i \preceq_J *$ and $i \npreceq_J +$ for all $i \in X_l \cup \{c\}$,

(b*) $i \preceq_J *$ for all $i \in X_l \cup \{c\}$, $c \npreceq_J +$, and there exists $j \in X_l$ such that $j \preceq_J +$,

(a$^+$) $i \preceq_J +$ and $i \npreceq_J *$ for all $i \in X_l \cup \{c\}$,

(b$^+$) $i \preceq_J +$ for all $i \in X_l \cup \{c\}$, $c \npreceq_J *$, and there exists $j \in X_l$ such that $j \preceq_J *$,

(c) $c \preceq_J *$ and $c \preceq_J +$.

First assume $X = \emptyset$. Note that in case (a*) (resp. (a$^+$)) we have $J = I^*$ (resp. $J = I^+$). The remaining cases do not occur. Indeed, in (c) the subposet $*^\nabla \cap +^\nabla$ of J contains two incomparable elements, because $I = J \setminus \{c\}$ is connected and c is a maximal element of J^-. Then J is of infinite prinjective type by [21, Theorem 3.1], which contradicts our assumptions. Cases (b^*) and (b^+) are also impossible, because $X = X_l = \emptyset$.

Let now $X \neq \emptyset$. Note that $J = I^l$ in cases (a^*), (a^+) and (c), $J = I^{l,*}$ and $J = I^{l,+}$ in cases (b^*) and (b^+), respectively. Hence $J \in \mathcal{X}^I$. ∎

Using the following algorithm we will be able to construct all posets which are min- or max-dominated by a given poset I.

Algorithm 6.5.

Input: A sincere two-peak poset I of finite prinjective type.

Output: All sincere posets of finite prinjective type which are min- or max-dominated by I.

Description of the algorithm:

Step 1. Form the sets $X_1, \ldots, X_k$ defined above.

Step 2. Set $\mathcal{X}(I) = \mathcal{X}_I \cup \mathcal{X}^I$.

Step 3. Using Remark 4.15 form the set $\mathcal{SR}^+_{q_I}$ of all sincere positive roots of q_I. This set is non-empty and finite.

Step 4. Form the set $\mathcal{Y}(I)$ of all posets J in $\mathcal{X}(I)$ satisfying:

(a) J is of finite prinjective type.

(b) There exists $z \in \mathcal{SR}^+_{q_I}$ such that $D_c q_J(\widetilde{z}) = 1$, where $\{c\} = J \setminus I$ and $\widetilde{z} \in \mathbb{N}^J$ is defined by $\widetilde{z}_i = z_i$ for $i \in I$ and $\widetilde{z}_c = 1$. ∎

Theorem 6.6. *Let I be a sincere two-peak poset of finite prinjective type and let $\mathcal{Y}(I)$ be defined in Algorithm* 6.5. *A poset J belongs to $\mathcal{Y}(I)$ if and only if J is a two-peak sincere poset of finite prinjective type which is min- or max-dominated by I.*

Proof. Let J be a sincere two-peak poset of finite prinjective type which is min- or max-dominated by I and let $J \setminus I = \{c\}$. From Lemmata 6.3 and 6.4 it follows that $J \in \mathcal{X}(I)$, where $\mathcal{X}(I)$ is defined in Step 2 of Algorithm 6.5. Definition 6.1 yields a sincere positive root z of q_I such that $\widetilde{z}$ is a sincere positive root of q_J, where $\widetilde{z}_j = z_j$ for $j \in I$ and $\widetilde{z}_c = 1$. From Corollary 4.10 it follows that $D_c q_J(\widetilde{z}) = 1$. This implies that $J \in \mathcal{Y}(I)$. Conversely, every poset $J \in \mathcal{X}(I)$ of finite prinjective type satisfying condition (b) of Step 4 is min- or max-dominated by I (cf. Definition 6.1). ∎

Definition 6.7. We call a poset J *iterated dominated* by a poset I if the following two conditions are satisfied:

(a) the Tits quadratic forms q_I, q_J are weakly positive and sincere, or equivalently, I, J are sincere posets of finite prinjective type,

(b) there exists a chain of posets $J_{(0)}, \ldots, J_{(m)}$ such that $J_{(0)} = I$, $I_{(m)} = J$ and the poset $J_{(j)}$ is min- or max-dominated by $J_{(j-1)}$ for $j = 1, \ldots, m$.

Algorithm 6.8.

Input: The poset

$$\mathcal{F}_1^{(2)} = \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ * & & + \end{tikzcd}.$$

Output: All posets which are iterated dominated by $\mathcal{F}_1^{(2)}$.

Description of the algorithm:

Step 1. Apply Algorithm 6.5 to $\mathcal{F}_1^{(2)}$ to get $\mathcal{Y}_{(0)} := \mathcal{Y}(\mathcal{F}_1^{(2)})$.

Step 2. For $i \in \mathbb{N}$ define inductively $\mathcal{Y}_{(i)}$ in the following way. If $\mathcal{Y}_{(n)}$ is defined for some $n \in \mathbb{N}$, apply Algorithm 6.5 to any poset $I \in \mathcal{Y}_{(n)}$ to get $\mathcal{Y}(I)$. Set $\mathcal{Y}_{(n+1)} = \bigcup_{I \in \mathcal{Y}_{(n)}} \mathcal{Y}(I)$.

Step 3. Set $\mathcal{Y} = \bigcup_{n \in \mathbb{N}} \mathcal{Y}_{(n)}$. ∎

Theorem 6.9. *The set $\mathcal{Y}$ defined above is finite and consists of all two-peak posets of finite prinjective type which are iterated dominated by $\mathcal{F}_1^{(2)}$.*

Proof. Since $\mathcal{Y}$ contains only sincere two-peak posets of finite prinjective type, it is finite by Lemma 5.1.

The rest is a simple consequence of the definition of posets iterated dominated by $\mathcal{F}_1^{(2)}$, Algorithms 6.5, 6.8 and Theorem 6.6. ∎

Example 6.10. We apply Algorithm 6.5 to the poset

$$I: \quad \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd}$$

to find all posets min- or max-dominated by I.

First we note that

$$q_I(x) = x_1^2 + x_2^2 + x_3^2 + x_*^2 + x_+^2 - x_1x_* - x_1x_+ - x_2x_+ - x_3x_+,$$

where $x = (x_1, x_2, x_3, x_*, x_+)$.

Step 1. The sets $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{1,2\}$, $X_5 = \{1,3\}$, $X_6 = \{2,3\}$, $X_7 = \{1,2,3\}$ are all pairwise different sets consisting of pairwise incomparable elements of I^-.

Step 2. The set $\mathcal{X}(I)$ consists of the following posets:

$$I_1: \begin{tikzcd}[column sep=small, row sep=small] & c \arrow[d] & & \\ & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd} \quad I_2: \begin{tikzcd}[column sep=small, row sep=small] & & c \arrow[d] & \\ & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd} \quad I_4: \begin{tikzcd}[column sep=small, row sep=small] & c \arrow[d] \arrow[dr] & & \\ & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd} \quad I_6: \begin{tikzcd}[column sep=small, row sep=small] & & c \arrow[d] \arrow[dr] & \\ & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd}$$

$$I_7: \begin{tikzcd}[column sep=small, row sep=small] & & c \arrow[dl] \arrow[d] \arrow[dr] & \\ & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd} \quad I_*: \begin{tikzcd}[column sep=small, row sep=small] c \arrow[d] & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & \end{tikzcd} \quad I_+: \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ * & & + & c \arrow[l] \end{tikzcd} \quad I^4: \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[d] & 2 \arrow[dl] & \\ * & c \arrow[dr] & & 3 \arrow[dl] \\ & & + & \end{tikzcd}$$

$$I^6: \begin{tikzcd}[row sep=small, column sep=small] & & 2 \arrow[d] & 3 \arrow[dl] \\ & 1 \arrow[dl] \arrow[dr] & c \arrow[d] & \\ * & & + & \end{tikzcd} \qquad I^7: \begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[d] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ + & c \arrow[d] & \\ & * & \end{tikzcd} \qquad I^{1,+}: \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl] \arrow[d] & & \\ * & c \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ & & + & \end{tikzcd} \qquad I^{1,*}: \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl] \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ c \arrow[d] & & + & \\ * & & & \end{tikzcd}$$

because $I^1 = I_1$, $I^2 = I_2 = I^3 = I_3$, $I^4 = I^5$, $I_2 = I_3$ and $I_4 = I_5$.

STEP 3. Using Remark 4.15 we get $\mathcal{SR}^+_{q_I} = \{(1,1,1,1,1), (1,1,1,1,2), (2,1,1,1,2)\}$, where the coordinates of the roots $z = (z_1, z_2, z_3, z_*, z_+)$ are indexed by elements of I.

STEP 4. By [21, Theorem 3.1] the posets I_1 and I_+ are of infinite prinjective type. Now it is easy to see that the only posets satisfying conditions (a) and (b) of Step 4 of Algorithm 6.5 are I_2, I_4, I_* and $I^{1,+}$ and they are all min- or max-dominated by I. Moreover $I_2 = \mathcal{F}^{(2)}_{12}$, $I_4 = \mathcal{F}^{(2)}_{11}$, $I_* = \mathcal{F}^{(2)}_{10}$ and $I^{1,+} = \mathcal{F}^{(2)}_{46}$ (see Tables 3.2).

7. Proof of the main result. The proof of Theorem 3.1 will be preceded by two helpful lemmata. Denote by $\mathcal{PW}^{(n)}$ the set of all two-peak sincere posets of finite prinjective type of cardinality n. Moreover let $\mathcal{S}^{(n)}$ be the set of all sincere posets J of finite prinjective type which are min- or max-dominated by some poset in $\mathcal{PW}^{(n-1)}$.

LEMMA 7.1. *Let $m \in \mathbb{N}$ be such that the posets from $\mathcal{PW}^{(m)}$ do not contain the posets* (5.8), (5.9) *as peak subposets. Then $\mathcal{S}^{(m+1)} = \mathcal{PW}^{(m+1)}$.*

Proof. We note that $\mathcal{S}^{(m+1)} \subseteq \mathcal{PW}^{(m+1)}$, because any poset J from $\mathcal{S}^{(m+1)}$ is sincere of finite prinjective type and obviously $|J| = m + 1$. Let $J \in \mathcal{PW}^{(m+1)}$. Let z be a sincere positive root of q_J satisfying (4.8) for each $i \in J$. Let $i \in J$ be such that $D_i q_J(z) = 1$ and let $L = J \setminus \{i\}$ be a peak subposet of J. Clearly, J is dominated by L. Since L contains neither (5.8) nor (5.9), Lemma 5.7 yields that the assumptions of Lemma 5.3 or of Lemma 5.5 are satisfied (for J, z and some $j \in J$ such that $D_j q_J(z) = 1$). In the first (resp. second) case $D_c q_J(z) = 1$, where $j^\nabla = \{c \prec \ldots \prec j\}$ (resp. $j^\Delta = \{j \prec \ldots \prec c\}$). Note that c is a minimal (resp. maximal) element of J^-. Therefore J is min- (resp. max-) dominated by $I = J \setminus \{c\}$ and $J \in \mathcal{S}^{(m+1)}$. ■

LEMMA 7.2. *Let K be a field, I be one of the posets of Tables* 3.2 *and M be one of the KI-modules in Tables* 8.1 *of Section* 8. *Then*

(a) $M \in \operatorname{prin}(KI) \cap \operatorname{mod}_{\mathrm{sp}}(KI)$,
(b) *M is indecomposable, and*
(c) *M is sincere.*

Proof. The statements (a) and (c) follow immediately by a case by case inspection of the K-diagrams (shown in Tables 8.1) of the modules M.

(b) We show that $\mathrm{End}(M) \simeq K$. A detailed proof will be presented only in two cases. The proof in the remaining cases is left to the reader.

Case $M = M_{17}^{(5)}$. Assume that the vertices of $\mathcal{F}_{17}^{(2)}$ (see Tables 3.2) are numbered as follows:

$$\mathcal{F}_{17}^{(2)}: \quad \begin{tikzcd}[row sep=small, column sep=small] & & 2 \arrow[dl] \arrow[d] & \\ 1 \arrow[d] & 4 \arrow[dl] \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\ * & & + & \end{tikzcd}$$

and let

$$M_{17}^{(5)} = (M(1), \dots, M(5), M(*), M(+), {}_j\varphi_i : M(i) \to M(j))$$

be the $K\mathcal{F}_{17}^{(2)}$-module $M_{17}^{(5)}$ given in Tables 8.1. Let $f = (f_1, \dots, f_5, f_*, f_+) : M_{17}^{(5)} \to M_{17}^{(5)}$ be a $K\mathcal{F}_{17}^{(2)}$-endomorphism of $M_{17}^{(5)}$, that is, $f_j : M(j) \to M(j)$ is a K-linear map such that $f_j \cdot {}_j\varphi_i = {}_j\varphi_i \cdot f_i$ for all $i \preceq j \in \mathcal{F}_{17}^{(2)}$. If we identify f_j with its matrix with respect to the standard basis of $M(j)$ then the equalities $f_3 \cdot {}_3\varphi_2 = {}_3\varphi_2 \cdot f_2$, $f_4 \cdot {}_4\varphi_2 = {}_4\varphi_2 \cdot f_2$, $f_* \cdot {}_*\varphi_1 = {}_*\varphi_1 \cdot f_1$ and $f_+ \cdot {}_+\varphi_3 = {}_+\varphi_3 \cdot f_3$ yield

$$f_2 = a_{11}, \qquad f_3 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \qquad f_4 = \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix},$$

$$f_1 = c_{22}, \qquad f_* = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}, \qquad f_5 = d_{11}, \qquad f_+ = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}.$$

Moreover the equality $f_+ \cdot {}_+\varphi_5 = {}_+\varphi_5 \cdot f_5$ implies that $a_{22} = d_{11}$ and $a_{11} + a_{12} = d_{11}$. From $f_* \cdot {}_*\varphi_4 = {}_*\varphi_4 \cdot f_4$ it follows that $b_{12} = b_{13} = b_{23} = c_{21} = 0$, $b_{22} = c_{22}$ and $a_{11} = c_{11}$. Further, since $f_+ \cdot {}_+\varphi_4 = {}_+\varphi_4 \cdot f_4$ we have $b_{32} = a_{12} = 0$, $a_{11} = b_{22}$ and $a_{22} = b_{33}$. From this we easily conclude that $a_{ii} = b_{jj} = c_{ll} = d_{kk}$ and $a_{ij} = b_{ij} = c_{ij} = d_{ij} = 0$ for $i \neq j$. It follows that $f_1, \dots, f_5, f_*, f_+$ are diagonal matrices with a_{11} on the diagonal. This shows that $f \mapsto a_{11}$ defines a K-algebra isomorphism $\mathrm{End}(M_{17}^{(5)}) \simeq K$. In particular the module $M_{17}^{(5)}$ is indecomposable.

Case $M = M_{13}^{(3)}$. Assume that the vertices of $\mathcal{F}_{13}^{(2)}$ are numbered as follows:

$$\mathcal{F}_{13}^{(2)}: \quad \begin{tikzcd}[row sep=small, column sep=small] & 2 \arrow[d] \\ & 3 \arrow[d] \\ & 4 \arrow[d] \\ 1 \arrow[d] & 5 \arrow[dl] \arrow[d] \\ * & + \end{tikzcd}$$

We consider the $K\mathcal{F}_{13}^{(2)}$-module

$$M_{13}^{(3)} = (M(1), \dots, M(5), M(*), M(+), {}_j\varphi_i : M(i) \to M(j))$$

given in Tables 8.1. Let $f = (f_1, \dots, f_5, f_*, f_+) : M_{13}^{(3)} \to M_{13}^{(3)}$ be a $K\mathcal{F}_{13}^{(2)}$-endomorphism of $M_{13}^{(3)}$, that is, $f_j : M(j) \to M(j)$ is a K-linear map such that $f_j \cdot {}_j\varphi_i = {}_j\varphi_i \cdot f_i$ for $i \preceq j \in \mathcal{F}_{13}^{(2)}$. If we identify f_j with its matrix with respect to the standard basis of $M(j)$ then the equalities $f_3 \cdot {}_3\varphi_2 = {}_3\varphi_2 \cdot f_2$, $f_4 \cdot {}_4\varphi_3 = {}_4\varphi_3 \cdot f_3$, $f_5 \cdot {}_5\varphi_4 = {}_5\varphi_4 \cdot f_4$ and $f_* \cdot {}_*\varphi_1 = {}_*\varphi_1 \cdot f_1$ yield

$$f_3 = \begin{bmatrix} a_{11} & 0 \\ a_{14} & a_{44} \end{bmatrix}, \quad f_4 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{41} & a_{42} & a_{44} \end{bmatrix}, \quad f_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$f_2 = a_{44}, \quad f_1 = b_{11}, \quad f_* = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}.$$

The equality $f_* \cdot {}_*\varphi_5 = {}_*\varphi_5 \cdot f_5$ implies that $a_{12} = a_{13} = a_{41} = a_{42} = a_{43} = b_{12} = 0$, $a_{11} = b_{11}$ and $a_{44} = b_{22}$. If f_+ has the form

$$f_+ = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

then the equality $f_+ \cdot {}_+\varphi_5 = {}_+\varphi_5 \cdot f_5$ yields $a_{ii} = b_{jj}$ for $i = 1, \dots, 4$, $j = 1, 2$, and $a_{ij} = b_{ij} = 0$ for $i \neq j$. It follows that $f_1, \dots, f_5, f_*, f_+$ are diagonal matrices with a_{11} on the diagonal. Hence $M_{13}^{(3)}$ is indecomposable as in the previous case. ∎

Now let us present a method of construction of sincere positive roots.

Remark 7.3. Let I be a sincere poset of finite prinjective type such that $|I| = n$, and let q_I be the Tits quadratic form of I. Denote by $J^{(1)}, \dots, J^{(k)}$ all posets which min- or max-dominate I. Set $I \setminus J^{(i)} = \{c_i\}$ for $i = 1, \dots, k$. Let $v_1^{(i)}, \dots, v_{l_i}^{(i)}$, for $i = 1, \dots, k$, be all sincere positive roots of $q_{J^{(i)}}$ such that $\widetilde{v}_1^{(i)}, \dots, \widetilde{v}_{l_i}^{(i)}$ are roots of q_I, where $\widetilde{v}_j^{(i)}(l) = v_j^{(i)}(l)$ for $l \in J^{(i)}$ and $\widetilde{v}_j^{(i)}(c_i) = 1$. We call $\widetilde{v}_j^{(i)}$ the *starting roots* of q_I and we denote by $\mathcal{S}(I)$ the set of all starting roots of q_I. Then for each $\widetilde{v} \in \mathcal{S}(I)$, as in Remark 4.15, we construct all chains of the form (4.12) satisfying conditions (a), (b), (c) of Proposition 4.11. The elements of these chains without the elements $x^{(m)}$ form the set $\mathcal{SR}_{q_I}^+$ of sincere positive roots of q_I.

Proof of Theorem 3.1. First we prove statement (c), and then (a) and (b) together by a case by case inspection of the posets of Tables 3.2.

By Lemma 7.2 any $K\mathcal{F}_i^{(2)}$-module $M_i^{(j)}$ in Tables 8.1 is indecomposable prinjective and $\mathbf{cdn}\, M_i^{(j)} = z_i^j$, where z_i^j is the vector given in Tables 3.3. Conversely, let X be an indecomposable prinjective $K\mathcal{F}_i^{(2)}$-module satisfying $\mathbf{cdn}\, X = z_i^j$. Theorem 2.3(f) yields $X \simeq M_i^{(j)}$ and (c) is proved.

Now we turns to the proof of (a) and (b).

According to [21, Theorem 3.1] any poset $\mathcal{F}_i^{(2)}$ in Tables 3.2 is a two-peak poset of finite prinjective type. Moreover the prinjective $\mathcal{F}_i^{(2)}$-modules $M_i^{(j)}$ in Tables 8.1 satisfy $(\mathbf{cdn}\, M_i^{(j)})(k) \neq 0$ for all $k \in \mathcal{F}_i^{(2)}$ and therefore $\mathcal{F}_i^{(2)}$ is sincere.

We can easily see that any vector z_i^j in Tables 3.3 is a sincere positive vector such that $z_i^j \in \mathbb{N}^{\mathcal{F}_i^{(2)}}$.

Conversely, let I be a two-peak sincere poset of finite prinjective type and let z be a sincere positive root of q_I. We show that I is one of the posets of Tables 3.2 and z is one of the vectors of Tables 3.3. We use the following notations:

- $X_1, \ldots, X_k$ are pairwise different sets consisting of pairwise incomparable elements of I^-.
- I_i, I^i, I_*, I_+, $I^{i,*}$, $I^{i,+}$, $i = 1, \ldots, k$, are the posets which are dominated by I and defined in Section 6.
- We denote by c the element $c \notin I$ such that $I \cup \{c\} = I_i$ (resp. I^j, I_*, I_+, $I^{i,*}$, $I^{i,+}$).
- Let $\mathcal{X}(I) = \mathcal{X}_I \cup \mathcal{X}^I$ (see Section 6, Algorithm 6.5).
- Let $\mathcal{Y}(I)$ denote the output set of Algorithm 6.5 with input I.
- Let $C_I = *^\nabla \cap +^\nabla \subseteq I$.
- We denote by $\mathcal{S}(I)$ (resp. $\mathcal{SR}_{q_I}^+$) the set of all starting roots (resp. sincere positive roots) of q_I defined in Remark 7.3.

Note that, if we delete from $\mathcal{X}(I)$ any poset of infinite prinjective type, then $\mathcal{Y}(I)$ remains unchanged.

Since I is of finite prinjective type, according to [21, Theorem 3.1] the set C_I is linearly ordered. We denote by p (resp. q) the unique minimal (resp. maximal) element of C_I. Let $i \in \{1, \ldots, k\}$. Note that, if there exists $d \in C_I \cap X_i$ such that $d \neq p$, then C_{I_i} is not linearly ordered and therefore I_i is not of finite prinjective type. Similarly, if $X_i = \{d\}$ where $d \in C_I$ and $d \neq q$, then C_{I^i} is not linearly ordered and therefore I^i is not of finite prinjective type. Therefore instead of $\mathcal{X}(I)$ we can consider $\mathcal{X}(I)' = \mathcal{X}(I) \setminus (A \cup B)$, where $A = \{I_i : i \in \{1, \ldots, k\}$ and there exists $d \in C_I \cap X_i$ with $d \neq p\}$ and $B = \{I^i : i \in \{1, \ldots, k\}, X_i = \{d\}, d \in C_I$ and $d \neq q\}$.

We make one more reduction. For given i, let $D_i = \bigcap_{x \in X_i} \{s \in \max I : x \prec s\}$. If $D_i = \emptyset$, then C_{I_i} is not linearly ordered and therefore I_i is not of finite prinjective type. We replace $\mathcal{X}(I)'$ by $\overline{\mathcal{X}(I)} = \mathcal{X}(I)' \setminus \{I_i : i \in \{1, \ldots, k\}, D_i = \emptyset\}$. It is clear that if $\mathcal{X}(I)$ is replaced by $\overline{\mathcal{X}(I)}$ then $\mathcal{Y}(I)$ is unchanged.

For simplicity we will work with $\overline{\mathcal{X}(I)}$ instead of $\mathcal{X}(I)$.

Now we are ready to apply Algorithm 6.8 to produce all two-peak sincere posets of finite prinjective type which are iterated dominated by the poset $\mathcal{F}_1^{(2)}$. For any $i \in \{1, \ldots, 60\}$, applying Algorithm 6.5, we will construct the set $\mathcal{Y}(\mathcal{F}_i^{(2)})$ of all posets min- or max-dominated by $\mathcal{F}_i^{(2)}$. We will see that the posets $\mathcal{F}_j^{(2)}$ having 9 elements satisfy $\mathcal{Y}(\mathcal{F}_j^{(2)}) = \emptyset$. The reader can easily check that $\bigcup_{i=1}^{60} \mathcal{Y}(\mathcal{F}_i^{(2)})$ is the set of all posets given in Tables 3.2. Moreover using Remark 7.3, for $i = 1, \ldots, 60$, we will construct the set $\mathcal{SR}^+_{q_{\mathcal{F}_i^{(2)}}}$ of all sincere positive roots of the quadratic form $q_{\mathcal{F}_i^{(2)}}$. The reader can easily check that $\bigcup_{i=1}^{60} \mathcal{SR}^+_{q_{\mathcal{F}_i^{(2)}}}$ is the set of all vectors given in Tables 3.3. Thus (b) will be proved.

From the constructions given below and the remarks above it follows that the posets of Tables 3.2 form the set of all two-peak sincere posets of finite prinjective type iterated dominated by $\mathcal{F}_1^{(2)}$. We note that no poset of Tables 3.2 contains the posets (5.8), (5.9) as peak subposets. Therefore if we construct all sets $\mathcal{Y}(\mathcal{F}_i^{(2)})$, $i = 1, \ldots, 60$, then from Lemma 7.1, by a simple induction, it will follow that Tables 3.2 give all two-peak sincere posets of finite prinjective type. Thus (a) will be proved. In our considerations below we will use [21, Theorem 3.1] to decide whether a given poset is of finite prinjective type or not.

Our procedure is as follows:

1) We take the poset $I = \mathcal{F}_1^{(2)}$ (see Tables 3.2), and use Algorithm 6.5 to construct all two-peak posets of finite prinjective type which are min- or max-dominated by I.

(1.1) It is easy to see that $X_1 = \{1\}$ is the only set required in Step 1 of Algorithm 6.5.

(1.2) We note that $\overline{\mathcal{X}(I)}$ consists of the posets I_1, $I^{1,*}$ and I^*.

(1.3) It is easy to see that $(1, 1, 1)$ is the only sincere positive root of q_I. Hence $\mathcal{SR}^+_{q_I} = \{z_1^1 = (1, 1, 1)\}$ (see Tables 3.3).

(1.4) First we choose from $\overline{\mathcal{X}(I)}$ all posets which satisfy condition 4(b) of Algorithm 6.5. For this we note that

$$D_c q_{I_1}(z_1^1) = 2 + 1 - 2 = 1,$$
$$D_c q_{I^{1,*}}(z_1^1) = 2 + 1 - 1 = 2,$$
$$D_c q_{I^*}(z_1^1) = 2 - 1 = 1.$$

Only the posets I_1 and I^* satisfy condition 4(b). They are of finite prinjective type, and therefore form a complete list of posets min- or max-dominated by I. It follows that $\mathcal{Y}(\mathcal{F}_1^{(2)}) = \{I_1, I^*\}$. We note that $I_1 = \mathcal{F}_3^{(2)}$ and $I^* = \mathcal{F}_2^{(2)}$ (see Tables 3.2).

In the same way we proceed in the remaining cases. We omit simple calculations in Step 4.

2) Let $I = \mathcal{F}_2^{(2)}$.

(2.1) The sets required in Step 2 of our algorithm are: $X_1 = \{1\}$, $X_2 = \{2\}$ and $X_3 = \{1, 2\}$.

(2.2) $\overline{\mathcal{X}(I)} = \{I_1, I^{1,*}, I^{1,+}, I_2, I_3, I^3, I^*, I^+\}$.

(2.3) $\mathcal{S}(I) = \{z_2^1 = (1,1,1,1)\}$, because only $\mathcal{F}_1^{(2)}$ dominates I and the form $q_{\mathcal{F}_1^{(2)}}$ has only one sincere positive root. It is easy to see that $D_i q_I(z_2^1) \geq 0$ for all $i \in I$, hence $\mathcal{SR}^+_{q_I} = \mathcal{S}(I)$.

(2.4) It is easy to check that $D_c q_{I_1}(z_2^1) = 1$, $D_c q_{I^*}(z_2^1) = 1$, $D_c q_{I^+}(z_2^1) = 1$ and $D_c q_J(z_2^1) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$. Since I_1, I^* and I^+ are of finite prinjective type, $\mathcal{Y}(\mathcal{F}_2^{(2)}) = \{I_1 = \mathcal{F}_4^{(2)}, I^* = \mathcal{F}_6^{(2)}, I^+ = \mathcal{F}_5^{(2)}\}$ (see Tables 3.2).

3) $I = \mathcal{F}_3^{(2)}$.

(3.1) $X_1 = \{1\}$, $X_2 = \{2\}$.

(3.2) $\overline{\mathcal{X}(I)} = \{I_1,\ I^{1,*},\ I^{2,*},\ I^*\}$.

(3.3) $\mathcal{S}(I) = \mathcal{SR}^+_{q_I} = \{z_3^1 = (1,1,1,1)\}$ by the same arguments as in case 2).

(3.4) $D_c q_{I^*}(z_3^1) = 1$ and $D_c q_J(z_3^1) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$. Since I^* is of finite prinjective type, $\mathcal{Y}(\mathcal{F}_3^{(2)}) = \{I^* = \mathcal{F}_4^{(2)}\}$.

4) $I = \mathcal{F}_4^{(2)}$.

(4.1) $X_i = \{i\}$, $i = 1, 2, 3$, $X_4 = \{1, 3\}$, $X_5 = \{2, 3\}$.

(4.2) $\overline{\mathcal{X}(I)} = \{I_1,\ I^{1,*},\ I^{1,+},\ I^{2,*},\ I^{2,+},\ I_3,\ I_4,\ I^4,\ I^5,\ I^*,\ I^+\}$.

(4.3) I is dominated by $\mathcal{F}_2^{(2)}$ and $\mathcal{F}_3^{(2)}$, the respective Tits quadratic forms have only one sincere positive root each (see 2) and 3)). Hence $\mathcal{S}(I) = \{z_4^1 = (1,1,1,1,1)\}$. Moreover $D_i q_I(z_4^1) = -1$ if and only if $i = +$, and therefore $z_4^2 = (1,1,1,1,2)$ is the sincere positive root of q_I. Note that $D_i q_I(z_4^2) > -1$ for all $i \in I$. Finally, $\mathcal{SR}^+_{q_I} = \{z_4^1,\ z_4^2\}$ (cf. Tables 3.3).

(4.4) $D_c q_{I_1}(z_4^2) = 1$, $D_c q_{I^{1,+}}(z_4^2) = 1$, $D_c q_{I_3}(z_4^2) = 1$, $D_c q_{I^*}(z_4^1) = 1$, $D_c q_{I^+}(z_4^1) = 1$ and $D_c q_J(z_4^i) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$ and $i = 1, 2$. The result is $\mathcal{Y}(\mathcal{F}_4^{(2)}) = \{I_1 = \mathcal{F}_7^{(2)},\ I^{1,+} = \mathcal{F}_{11}^{(2)},\ I_3 = \mathcal{F}_8^{(2)},\ I^* = \mathcal{F}_9^{(2)}\}$, because I^+ is of infinite prinjective type.

In the remaining cases we omit the calculations in Step 3. We only give the sets $\mathcal{S}(I)$ and $\mathcal{SR}^+_{q_I}$. The reader can easily check the details.

5) $I = \mathcal{F}_5^{(2)}$.

(5.1) $X_i = \{i\}$, $i = 1,2,3$, $X_4 = \{1,2\}$, $X_5 = \{1,3\}$, $X_6 = \{2,3\}$, $X_7 = \{1,2,3\}$.

(5.2) $\overline{\mathcal{X}(I)} = \{I_1,\, I^{1,*},\, I_2,\, I_4,\, I^4,\, I_7,\, I^*\}$.

(5.3) $\mathcal{S}(I) = \mathcal{SR}_{q_I}^+ = \{z_5^1 = (1,1,1,1,1)\}$.

(5.4) $D_c q_{I_1}(z_5^1) = 1$, $D_c q_{I^*}(z_5^1) = 1$ and $D_c q_J(z_5^1) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$. Hence $\mathcal{Y}(\mathcal{F}_5^{(2)}) = \{I_1 = \mathcal{F}_9^{(2)},\, I^* = \mathcal{F}_{10}^{(2)}\}$.

6) The case $I = \mathcal{F}_6^{(2)}$ is discussed in Example 6.10. The result is $\mathcal{Y}(\mathcal{F}_6^{(2)}) = \{I_2 = \mathcal{F}_{12}^{(2)},\, I_4 = \mathcal{F}_{11}^{(2)},\, I^* = \mathcal{F}_{10}^{(2)},\, I^{1,+} = \mathcal{F}_{46}^{(2)}\}$.

7) $I = \mathcal{F}_7^{(2)}$.

(7.1) $X_i = \{i\}$, $i = 1,\ldots,4$, $X_5 = \{1,4\}$, $X_6 = \{2,4\}$, $X_7 = \{3,4\}$.

(7.2) $\overline{\mathcal{X}(I)} = \{I_1, I^{1,*}, I^{1,+}, I^{2,*}, I^{2,+}, I^{3,*}, I^{3,+}, I_4, I_5, I^5, I^6, I^7, I^*, I^+\}$.

(7.3) $\mathcal{S}(I) = \{z_7^1 = (1,1,1,1,1,2)\}$, $\mathcal{SR}_{q_I}^+ = \{z_7^1,\, z_7^2 = (1,1,1,1,2,2)\}$.

(7.4) $D_c q_{I_1}(z_7^2) = 1$, $D_c q_{I^{1,*}}(z_7^2) = 1$, $D_c q_{I^{1,+}}(z_7^1) = 1$, $D_c q_{I_4}(z_7^1) = 1$, $D_c q_{I^*}(z_7^1) = 1$ and $D_c q_J(z_7^i) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$ and $i = 1,2$. The result is $\mathcal{Y}(\mathcal{F}_7^{(2)}) = \{I_1 = \mathcal{F}_{13}^{(2)},\, I^{1,*} = \mathcal{F}_{16}^{(2)},\, I_4 = \mathcal{F}_{14}^{(2)}\}$, because $I^{1,+}$ and I^* are of infinite prinjective type.

8) $I = \mathcal{F}_8^{(2)}$.

(8.1) $X_i = \{i\}$, $i = 1,\ldots,4$, $X_5 = \{1,3\}$, $X_6 = \{1,4\}$, $X_7 = \{2,3\}$, $X_8 = \{2,4\}$.

(8.2) $\overline{\mathcal{X}(I)} = \{I_1, I^{1,*}, I^{1,+}, I^{2,*}, I^{2,+}, I_3, I^3, I_4, I_5, I^5, I_6, I^6, I^7, I^8, I^*, I^+\}$.

(8.3) $\mathcal{S}(I) = \mathcal{SR}_{q_I}^+ = \{z_8^1 = (1,1,1,1,1,2)\}$.

(8.4) $D_c q_{I_1}(z_8^1) = 1$, $D_c q_{I^{1,+}}(z_8^1) = 1$, $D_c q_{I^3}(z_8^1) = 1$, $D_c q_{I_4}(z_8^1) = 1$, $D_c q_{I^*}(z_8^1) = 1$ and $D_c q_J(z_8^1) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$. Since I^3 and I_4 are of infinite prinjective type, $\mathcal{Y}(\mathcal{F}_8^{(2)}) = \{I_1 = \mathcal{F}_{14}^{(2)},\, I^{1,+} = \mathcal{F}_{20}^{(2)},\, I^* = \mathcal{F}_{15}^{(2)}\}$.

9) $I = \mathcal{F}_9^{(2)}$.

(9.1) $X_i = \{i\}$, $i = 1,\ldots,4$, $X_5 = \{1,3\}$, $X_6 = \{1,4\}$, $X_7 = \{2,3\}$, $X_8 = \{2,4\}$, $X_9 = \{3,4\}$, $X_{10} = \{1,3,4\}$, $X_{11} = \{2,3,4\}$.

(9.2) $\overline{\mathcal{X}(I)} = \{I_1,\, I^{1,*},\, I^{2,*},\, I_3,\, I_5,\, I^5,\, I^7,\, I_{10},\, I^*\}$.

(9.3) $\mathcal{S}(I) = \{z_9^4,\, z_9^5\}$, $\mathcal{SR}_{q_I}^+ = \{z_9^1, \ldots, z_9^6\}$ (see Tables 3.3).

(9.4) $D_c q_{I_1}(z_9^5) = 1$, $D_c q_{I^{1,*}}(z_9^1) = 1$, $D_c q_{I_3}(z_9^1) = 1$, $D_c q_{I_5}(z_9^1) = 1$, $D_c q_{I^*}(z_9^4) = 1$ and $D_c q_J(z) \neq 1$ for the remaining $J \in \overline{\mathcal{X}(I)}$ and $z \in \mathcal{SR}_{q_I}^+$. Since I_1 and I^* are of infinite prinjective type, $\mathcal{Y}(\mathcal{F}_9^{(2)}) = \{I^{1,*} = \mathcal{F}_{17}^{(2)},\, I_3 = \mathcal{F}_{15}^{(2)},\, I_5 = \mathcal{F}_{16}^{(2)}\}$.

10) $I = \mathcal{F}_{10}^{(2)}$.

(10.1) $X_i = \{i\}$, $i = 1, \dots, 4$, $X_5 = \{1,2\}$, $X_6 = \{1,3\}$, $X_7 = \{1,4\}$, $X_8 = \{2,3\}$, $X_9 = \{2,4\}$, $X_{10} = \{3,4\}$, $X_{11} = \{1,2,3\}$, $X_{12} = \{1,2,4\}$, $X_{13} = \{1,3,4\}$, $X_{14} = \{2,3,4\}$.

(10.2) $\overline{\mathcal{X}(I)} = \{I_1, I^{1,*}, I^{1,+}, I_2, I_3, I_5, I^5, I_6, I^6, I_{10}, I^{10}, I_{11}, I_{13}, I^{13}, I^*, I^+\}$.

(10.3) $\mathcal{S}(I) = \{z_{10}^1,\ z_{10}^2,\ z_{10}^3\}$, $Z = \{z_{10}^1, \dots, z_{10}^4\}$ (see Tables 3.3).

(10.4) $\mathcal{Y}(\mathcal{F}_{10}^{(2)}) = \{I^{1,+} = \mathcal{F}_{48}^{(2)},\ I_2 = \mathcal{F}_{19}^{(2)},\ I_3 = \mathcal{F}_{18}^{(2)},\ I_6 = \mathcal{F}_{17}^{(2)}\}$, because $D_c q_J(z) \neq 1$ for $J \in \{I^{1,*},\ I^5,\ I^6,\ I_{10},\ I^{10},\ I_{11},\ I_{13},\ I^{13}\}$, $z \in \mathcal{SR}_{q_I}^+$, $D_c q_{I^{1,+}}(z_{10}^1) = 1$, $D_c q_{I_2}(z_{10}^3) = 1$, $D_c q_{I_3}(z_{10}^1) = 1$, $D_c q_{I_6}(z_{10}^1) = 1$ and I_1, I_5, I^*, I^+ are of infinite prinjective type.

The remaining cases are left to the reader. They are included in a complete version of the paper (available from the author on request). ∎

8. Tables of sincere prinjective modules. Below we present the set of canonical forms of sincere prinjective modules $M_i^{(j)}$ over the incidence algebra $K\mathcal{F}_j^{(2)}$. From Lemma 7.2 it follows that the $M_i^{(j)}$ are those modules in $\mathrm{prin}(K\mathcal{F}_i^{(2)})$ such that $\mathbf{cdn}\, M_i^{(j)} = z_i^j$ (see Tables 3.3). By Lemma 7.2 the modules $M_i^{(j)}$ are indecomposable.

In the tables below the K-linear maps $a, b, c, e, f, \dots$ in the definition of $M_i^{(j)}$ are defined by their matrices in the standard bases. Following [20] we denote by d the K-linear maps defined by the matrix $[1, 1, \dots, 1]$ (or by its transpose) in the standard bases. Moreover, K-linear maps denoted by $d_{i_1} \vee \dots \vee d_{i_n}$ are given by matrices which have i_jth standard vector as jth column (see [20]). The K-linear maps denoted by $\overset{\circ}{\downarrow}$, ${}^{\circ}\!\!\searrow$, $\swarrow\!\!{}^{\circ}$, $\circ\!\!\longrightarrow$, $\longleftarrow\!\!\circ$ are defined by matrices of the form

$$\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

or by their transposes. The remaining K-linear maps are defined by matrices of the form

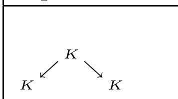
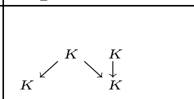
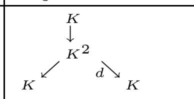
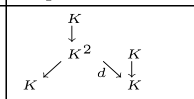
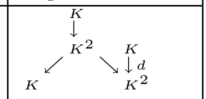
$$\begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

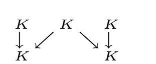
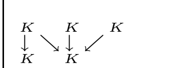
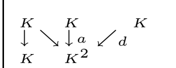
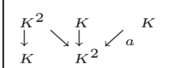
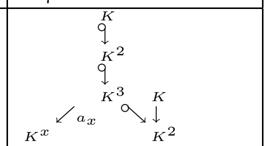
or by their transposes.

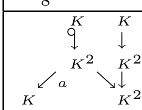
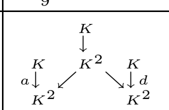
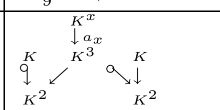
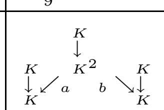
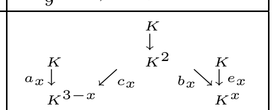
In the third row of the following tables the coordinate vectors z_i^j of the respective $K\mathcal{F}_i^{(2)}$-modules $M_i^{(j)}$ are given.

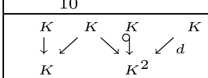
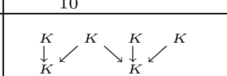
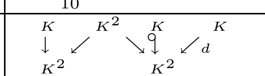
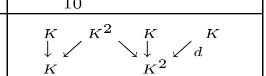
Tables 8.1

Sincere prinjective KI-modules, where I is a sincere two-peak poset of finite prinjective type

$M_1^{(1)}$	$M_2^{(1)}$	$M_3^{(1)}$	$M_4^{(1)}$	$M_4^{(2)}$
				
$\begin{smallmatrix}&1&\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}&1&1\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}&1&\\&1&\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}&1&\\&1&1\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}&1&\\&1&1\\1&&2\end{smallmatrix}$

$M_5^{(1)}$	$M_6^{(1)}$	$M_6^{(2)}$	$M_6^{(3)}$	$M_7^{(x)}$, $x=1,2$
				
$\begin{smallmatrix}1&1&1\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}1&1&1\\1&1&\end{smallmatrix}$	$\begin{smallmatrix}1&1&1\\1&2&\end{smallmatrix}$	$\begin{smallmatrix}2&1&1\\1&2&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\&1&\\&1&1\\x&&2\end{smallmatrix}$
		$a=[0\ 1]^{tr}$	$a=[1\ 1]^{tr}$	$a_1=[\,1\ 1\ 1\,]\ a_2=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

$M_8^{(1)}$	$M_9^{(1)}$	$M_9^{(x+1)}$, $x=1,2$	$M_9^{(4)}$	$M_9^{(x+4)}$, $x=1,2$
				
$\begin{smallmatrix}&1&1\\&1&1\\1&&2\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&1&1\\2&&2\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&2&1\\2&&2\end{smallmatrix}\ \begin{smallmatrix}&2&\\1&1&1\\2&&2\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&1&1\\1&&1\end{smallmatrix}$	$\begin{smallmatrix}&&1&\\1&&1&1\\3-x&&&x\end{smallmatrix}$
$a=[1\ 1]$	$a=[0\ 1]^{tr}$	$a_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\ a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a=[1\ 0]\ b=[1\ 1]$	$a_1=e_2=[0\ 1]^{tr}$ $b_1=c_2=[1\ 1]$

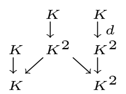
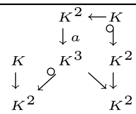
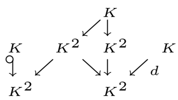
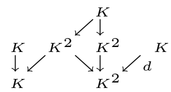
$M_{10}^{(1)}$	$M_{10}^{(2)}$	$M_{10}^{(3)}$	$M_{10}^{(4)}$
			
$\begin{smallmatrix}1&1&1&1\\1&&2&\end{smallmatrix}$	$\begin{smallmatrix}1&1&1&1\\1&&1&\end{smallmatrix}$	$\begin{smallmatrix}1&2&1&1\\2&&2&\end{smallmatrix}$	$\begin{smallmatrix}1&2&1&1\\1&&2&\end{smallmatrix}$

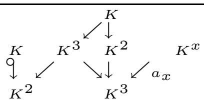
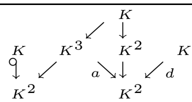
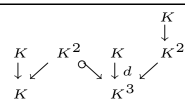
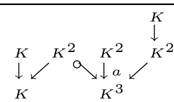
$M_{11}^{(1)}$	$M_{12}^{(1)}$	$M_{12}^{(x+1)},\ x{=}1,2$	$M_{12}^{(4)}$
$\begin{smallmatrix} & & K & & \\ & \swarrow & \downarrow & & \\ K^2 & & K^2 & & K \\ \downarrow & \searrow & \downarrow & \swarrow d & \\ K & & K^2 & & \end{smallmatrix}$	$\begin{smallmatrix} & & & & K \\ & & & & \downarrow d \\ K & & K & & K^2 \\ \downarrow & \searrow & \overset{\circ}{\downarrow} & \swarrow & \\ K & & K^2 & & \end{smallmatrix}$	$\begin{smallmatrix} & & & & K \\ & & & & \downarrow \\ K^2 & & K^x & & K^2 \\ \downarrow & \circ\!\!\searrow & \downarrow a_x & \swarrow & \\ K & & K^3 & & \end{smallmatrix}$	$\begin{smallmatrix} & & & & K \\ & & & & \downarrow d \\ K^2 & & K & & K^2 \\ \downarrow & \searrow & \downarrow & \swarrow & \\ K & & K^2 & & \end{smallmatrix}$
$\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 1 & 2 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ 1 & 1 & 1 \\ 1 & 2 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ 2 & x & 1 \\ 1 & 3 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 \\ 2 & 1 & 1 \\ 1 & 2 & \end{smallmatrix}$
		$a_1{=}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $a_2{=}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	

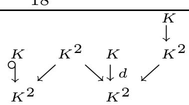
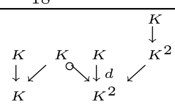
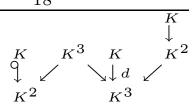
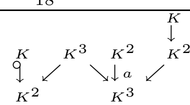
$M_{13}^{(1)}$	$M_{13}^{(2)}$	$M_{13}^{(3)}$	$M_{14}^{(1)}$
$\begin{smallmatrix} & & K \\ & & \overset{\circ}{\downarrow} \\ & & K^2 \\ & & \downarrow d_1\vee d_3 \\ & & K^3 \\ & & \downarrow d_1\vee d_2\vee d_4 \\ K & & K^4 \\ a\downarrow & \swarrow b & \downarrow c \\ K^3 & & K^2 \end{smallmatrix}$	$\begin{smallmatrix} & & K \\ & & \overset{\circ}{\downarrow} \\ & & K^2 \\ & & \downarrow d_1\vee d_3 \\ & & K^3 \\ & & \downarrow d_1\vee d_2\vee d_4 \\ K^2 & & K^4 \\ \downarrow & \swarrow a & \downarrow b \\ K^3 & & K^2 \end{smallmatrix}$	$\begin{smallmatrix} & & K \\ & & \overset{\circ}{\downarrow} \\ & & K^2 \\ & & \downarrow d_1\vee d_3 \\ & & K^3 \\ & & \downarrow d_1\vee d_2\vee d_4 \\ K & & K^4 \\ \downarrow & \swarrow a & \downarrow b \\ K^2 & & K^2 \end{smallmatrix}$	$\begin{smallmatrix} K & & \\ \overset{\circ}{\downarrow} & & \\ K^2 & & K \\ \overset{\circ}{\downarrow} & & \downarrow \\ K^3 & & K^2 \\ a\downarrow & \searrow & \downarrow \\ K^2 & & K^3 \end{smallmatrix}$
$\begin{smallmatrix} 1 & \\ 1 & \\ 1 & \\ 1 & 1 \\ 3 & 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & \\ 1 & \\ 2 & 1 \\ 3 & 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & \\ 1 & \\ 1 & 1 \\ 2 & 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & 1 \\ 1 & 1 \\ 2 & 3 \end{smallmatrix}$
$a{=}d_2$ $b{=}d_1\vee d_2\vee 0\vee d_3$ $c{=}\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a{=}d_1\vee d_2\vee 0\vee d_3$ $b{=}\begin{bmatrix}1&1&0&1\\1&0&1&0\end{bmatrix}$	$a{=}d_1\vee 0\vee 0\vee d_2$ $b{=}\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a{=}\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

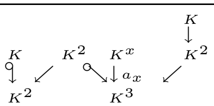
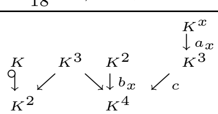
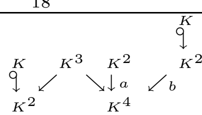
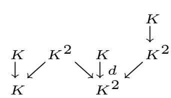
$M_{14}^{(x+1)}\ x{=}1,2$	$M_{14}^{(4)}$	$M_{14}^{(5)}$	$M_{14}^{(x+5)}\ x{=}1,2$	$M_{15}^{(x)}\ x{=}1,2$
$\begin{smallmatrix} K^x & & \\ \overset{\circ}{\downarrow} & & \\ K^3 & & K \\ \downarrow & & \downarrow \\ K^4 & & K^2 \\ a_x\downarrow & \searrow & \downarrow \\ K^2 & & K^3 \end{smallmatrix}$	$\begin{smallmatrix} K & & \\ \overset{\circ}{\downarrow} & & \\ K^2 & & K \\ \downarrow a & & \downarrow \\ K^4 & & K^2 \\ b\downarrow & \searrow & \downarrow \\ K^2 & & K^3 \end{smallmatrix}$	$\begin{smallmatrix} K & & \\ \overset{\circ}{\downarrow} & & \\ K^2 & & K \\ \overset{\circ}{\downarrow} & & \downarrow \\ K^3 & & K^2 \\ d\downarrow & \searrow & \downarrow \\ K & & K^3 \end{smallmatrix}$	$\begin{smallmatrix} K & & \\ \downarrow & & \\ K^2 & & K \\ \downarrow & & \overset{\circ}{\downarrow} \\ K^3 & & K^2 \\ a_x\downarrow & \searrow & \downarrow \\ K^x & & K^2 \end{smallmatrix}$	$\begin{smallmatrix} & & K^x & & K \\ & & \downarrow a_x & & \downarrow \\ K & & K^3 & & K^2 \\ \overset{\circ}{\downarrow} & \swarrow & & \searrow & \downarrow \\ K^2 & & & & K^3 \end{smallmatrix}$
$\begin{smallmatrix} x & & \\ 3-x & & 1 \\ 1 & & 1 \\ 2 & & 3 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & 1 \\ 2 & 1 \\ 2 & 3 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{smallmatrix}$	$\begin{smallmatrix} 1 & \\ 1 & 1 \\ 1 & 1 \\ x & 2 \end{smallmatrix}$	$\begin{smallmatrix} & 1 & 1 & & & 2 & 1 \\ 1 & 2 & 1 & & 1 & 1 & 1 \\ 2 & & 3 & & 2 & & 3 \end{smallmatrix}$
$a_1{=}\begin{bmatrix}1&1&0&1\\1&0&1&0\end{bmatrix}$ $a_2{=}\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a{=}d_2\vee d_3$ $b{=}\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$		$a_1{=}d$ $a_2{=}\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$	$a_1{=}d$ $a_2{=}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$

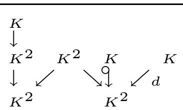
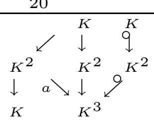
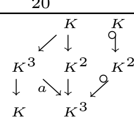
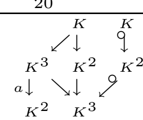
$M_{15}^{(x+2)}\ x{=}1,2$	$M_{15}^{(5)}$	$M_{15}^{(x+5)}\ x{=}1,2$	$M_{15}^{(8)}$
$\begin{smallmatrix} & & K^2 & & K \\ & & \downarrow a_x & & \downarrow \\ K^x & & K^4 & & K^2 \\ b_x\downarrow & \swarrow c & & \searrow & \downarrow \\ K^3 & & & & K^3 \end{smallmatrix}$	$\begin{smallmatrix} & & K^2 & & K \\ & & \downarrow a & & \downarrow \\ K & & K^4 & & K^2 \\ \downarrow & \swarrow b & & \searrow & \downarrow \\ K^2 & & & & K^3 \end{smallmatrix}$	$\begin{smallmatrix} & & K^x & & K \\ & & \downarrow a_x & & \overset{\circ}{\downarrow} \\ K & & K^3 & & K^2 \\ \downarrow & \circ\!\!\swarrow & & \searrow & \downarrow \\ K^2 & & & & K^2 \end{smallmatrix}$	$\begin{smallmatrix} & & K & & K \\ & & \downarrow & & \overset{\circ}{\downarrow} \\ K & & K^2 & & K^2 \\ d\downarrow & \swarrow & & \searrow & \downarrow \\ K^2 & & & & K^2 \end{smallmatrix}$
$\begin{smallmatrix} & 2 & 1 \\ x & 2 & 1 \\ 3 & & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 2 & 1 \\ 1 & 2 & 1 \\ 2 & & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 1 & 1 & & & 2 & 1 \\ 1 & 2 & 1 & & 1 & 1 & 1 \\ 2 & & 2 & & 2 & & 2 \end{smallmatrix}$	$\begin{smallmatrix} & 1 & 1 \\ 1 & 1 & 1 \\ 2 & & 2 \end{smallmatrix}$
$a_1{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $a_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $b_1{=}d_2$ $b_2{=}d_1\vee d_2$ $c{=}d_1\vee d_2\vee 0\vee d_3$	$a{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b{=}d_1\vee 0\vee 0\vee d_2$	$a_1{=}d$ $a_2{=}\begin{bmatrix}0&1\\1&0\\1&1\end{bmatrix}$	

$M_{15}^{(9)}$	$M_{16}^{(1)}$	$M_{17}^{(1)}$	$M_{17}^{(2)}$
			
11 111 1 2	11 111 2 2	1 1111 2 2	1 1111 1 2
	$a=\begin{bmatrix}0&1\\1&0\\1&1\end{bmatrix}$		

$M_{17}^{(x+2)}$ $x=1,2$	$M_{17}^{(5)}$	$M_{18}^{(1)}$	$M_{18}^{(2)}$
			
1 121x 2 3	1 1211 2 2	1 1211 1 3	1 1221 1 3
$c_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $c_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$		$a=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$

$M_{18}^{(3)}$	$M_{18}^{(4)}$	$M_{18}^{(5)}$	$M_{18}^{(6)}$
			
1 1211 2 2	1 1111 1 2	1 1311 2 3	1 1321 2 3
			$a=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$

$M_{18}^{(x+6)}$, $x=1,2$	$M_{18}^{(x+8)}$, $x=1,2$	$M_{18}^{(11)}$	$M_{18}^{(12)}$
			
1 12x1 2 3	1 132x 2 4	2 1321 2 4	1 1211 1 2
$a_1=d$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a_1=d_3$ $a_2=d_1\vee d_3$ $c=d_1\vee d_2\vee d_4$ $b_1=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $b=d_1\vee d_4$	

$M_{19}^{(1)}$	$M_{20}^{(1)}$	$M_{20}^{(2)}$	$M_{20}^{(3)}$
			
1 1211 2 2	11 111 13	11 211 13	11 211 23
	$a=\begin{bmatrix}1&1\\0&1\\0&1\end{bmatrix}$	$a=\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

$M_{20}^{(4)}$	$M_{21}^{(x)}$ $x{=}1,2$	$M_{21}^{(3)}$	$M_{21}^{(x+3)}$ $x{=}1,2$
$\begin{tikzcd} & K \arrow[dl] \arrow[d] & K \arrow[d] \\ K^2 \arrow[d, "d"'] \arrow[dr] & K^2 \arrow[d] & K^2 \arrow[dl] \\ K & K^2 & \end{tikzcd}$	$\begin{tikzcd} & & & K \arrow[d] \\ & & & K^2 \arrow[d] \\ & K^2 \arrow[dl] \arrow[dr] & K^x \arrow[d, "a_x"] & K^3 \arrow[dl] \\ K & & K^3 & \end{tikzcd}$	$\begin{tikzcd} & & & K \arrow[d] \\ & & & K^2 \arrow[d] \\ & K^2 \arrow[dl] \arrow[dr, "a"'] & K^2 \arrow[d, "b"] & K^3 \arrow[dl] \\ K & & K^4 & \end{tikzcd}$	$\begin{tikzcd} & & & K \arrow[d] \\ & & & K^2 \arrow[d] \\ & K^3 \arrow[dl, "a_x"] \arrow[dr, "b"'] & K^2 \arrow[d, "c_x"] & K^3 \arrow[dl] \\ K^x & & K^4 & \end{tikzcd}$
$\begin{smallmatrix} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & \end{smallmatrix}$	$\begin{smallmatrix} & & & 1 \\ & & & 1 \\ & 2 & x & 1 \\ 1 & & 3 & \end{smallmatrix}$	$\begin{smallmatrix} & & & 1 \\ & & & 1 \\ & 2 & 2 & 1 \\ 1 & & 4 & \end{smallmatrix}$	$\begin{smallmatrix} & & & 1 \\ & & & 1 \\ & 3 & 2 & 1 \\ x & & 4 & \end{smallmatrix}$
	$a_1{=}d$ $a_2{=}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a{=}d_1 \vee d_4$ $b{=}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$a_1{=}d_2$ $b{=}d_1 \vee d_2 \vee d_4$ $c_1{=}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c_2{=}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$M_{22}^{(1)}$	$M_{23}^{(1)}$	$M_{23}^{(2)}$	$M_{23}^{(3)}$
$\begin{tikzcd} & & K \arrow[d] \arrow[dr] & K \arrow[d, "d"] \\ & K^2 \arrow[dl] \arrow[dr] & K^2 \arrow[d] & K^3 \arrow[dl] \\ K & & K^3 & \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d] \\ & K^2 \arrow[d, "d_1 \vee d_3"] \\ & K^3 \arrow[d, "d_1 \vee d_2 \vee d_4"] \\ & K^4 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_5"] \\ K^2 \arrow[d, "a"'] & K^5 \arrow[dl, "b"] \arrow[d, "c"] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d] \\ & K^2 \arrow[d, "d_1 \vee d_3"] \\ & K^3 \arrow[d, "d_1 \vee d_2 \vee d_4"] \\ & K^4 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_6"] \\ K^2 \arrow[d, "a"'] & K^6 \arrow[dl, "b"] \arrow[d, "c"] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d] \\ & K^2 \arrow[d, "d_1 \vee d_3"] \\ & K^3 \arrow[d, "d_1 \vee d_2 \vee d_5"] \\ & K^5 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_4 \vee d_6"] \\ K^2 \arrow[d, "a"'] & K^6 \arrow[dl, "b"] \arrow[d, "c"] \\ K^4 & K^3 \end{tikzcd}$
$\begin{smallmatrix} & & 1 & 1 \\ & 2 & 1 & 1 \\ 1 & & 3 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 \\ & 1 \\ & 1 \\ & 1 \\ 2 & 1 \\ 4 & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 1 \\ & 1 \\ & 1 \\ & 1 \\ 2 & 2 \\ 4 & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 1 \\ & 1 \\ & 1 \\ & 2 \\ 2 & 1 \\ 4 & 3 \end{smallmatrix}$
	$a{=}d_2 \vee d_3$ $b{=}d_1 \vee d_2 \vee d_3 \vee 0 \vee d_4$ $c{=}\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$	$a{=}d_2 \vee d_3$ $b{=}d_1 \vee d_2 \vee d_3 \vee 0 \vee 0 \vee d_4$ $c{=}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$	$a{=}d_2 \vee d_3$ $b{=}d_1 \vee d_2 \vee d_3 \vee 0 \vee 0 \vee d_4$ $c{=}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$

$M_{23}^{(4)}$	$M_{23}^{(5)}$	$M_{23}^{(6)}$
$\begin{tikzcd} & K \arrow[d] \\ & K^2 \arrow[d, "d_1 \vee d_4"] \\ & K^4 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_5"] \\ & K^5 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_4 \vee d_6"] \\ K^2 \arrow[d, "a"'] & K^6 \arrow[dl, "b"] \arrow[d, "c"] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d] \\ & K^3 \arrow[d, "d_1 \vee d_2 \vee d_4"] \\ & K^4 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_5"] \\ & K^5 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_4 \vee d_6"] \\ K^2 \arrow[d, "a"'] & K^6 \arrow[dl, "b"] \arrow[d, "c"] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} & K^2 \arrow[d] \\ & K^3 \arrow[d, "d_1 \vee d_3 \vee d_4"] \\ & K^4 \arrow[d, "d_1 \vee d_2 \vee d_4 \vee d_5"] \\ & K^5 \arrow[d, "d_1 \vee d_2 \vee d_3 \vee d_5 \vee d_6"] \\ K^2 \arrow[d] & K^6 \arrow[dl, "a"] \arrow[d, "b"] \\ K^4 & K^3 \end{tikzcd}$
$\begin{smallmatrix} & 1 \\ & 1 \\ & 2 \\ & 1 \\ 2 & 1 \\ 4 & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 1 \\ & 2 \\ & 1 \\ & 1 \\ 2 & 1 \\ 4 & 3 \end{smallmatrix}$	$\begin{smallmatrix} & 2 \\ & 1 \\ & 1 \\ & 1 \\ 2 & 1 \\ 4 & 3 \end{smallmatrix}$
$a{=}d_2 \vee d_3$ $b{=}d_1 \vee d_2 \vee d_3 \vee 0 \vee 0 \vee d_4$ $c{=}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$	$a{=}d_2 \vee d_3$ $b{=}d_1 \vee d_2 \vee d_3 \vee 0 \vee 0 \vee d_4$ $c{=}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$a{=}d_1 \vee d_2 \vee 0 \vee 0 \vee d_3 \vee d_4$ $b{=}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

$M_{23}^{(7)}$	$M_{23}^{(x+7)}$, $x=1,2$	$M_{23}^{(10)}$
$\begin{tikzcd} & K \arrow[d, o->] \\ & K^2 \arrow[d, "d_1\vee d_3"] \\ & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ & K^4 \arrow[d, "d_1\vee d_2\vee d_3\vee d_5"] \\ K^2 \arrow[d] & K^5 \arrow[dl, "a"'] \arrow[d, "b"] \\ K^3 & K^2 \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d, o->] \\ & K^2 \arrow[d, "d_1\vee d_3"] \\ & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ & K^4 \arrow[d, "d_1\vee d_2\vee d_3\vee d_5"] \\ K^x \arrow[d, "a_x"'] & K^5 \arrow[dl, "b"'] \arrow[d, "c_x"] \\ K^3 & K^3 \end{tikzcd}$	$\begin{tikzcd} & K \arrow[d, o->] \\ & K^2 \arrow[d, "d_1\vee d_3"] \\ & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ & K^4 \arrow[d, "d_1\vee d_2\vee d_3\vee d_5"] \\ K^2 \arrow[d, "a"'] & K^5 \arrow[dl, "b"'] \arrow[d, "c"] \\ K^4 & K^2 \end{tikzcd}$
1 1 1 1 21 32	1 1 1 1 x1 33	1 1 1 1 21 42
$a=d_1\vee d_2\vee 0\vee 0\vee d_3$ $b=\begin{bmatrix}0&1&1&0&1\\1&0&1&1&0\end{bmatrix}$	$a_1=d_2$ $b=d_1\vee d_2\vee 0\vee 0\vee d_3$ $c_1=\begin{bmatrix}0&0&0&1&1\\1&0&1&0&0\\0&1&1&1&0\end{bmatrix}$ $c_2=\begin{bmatrix}1&0&0&0&1\\0&1&1&1&0\\1&1&0&1&0\end{bmatrix}$	$a=d_2\vee d_3$ $b=d_1\vee d_2\vee d_3\vee 0\vee d_4$ $c=\begin{bmatrix}0&1&1&0&1\\1&1&0&1&0\end{bmatrix}$

$M_{23}^{(11)}$	$M_{24}^{(1)}$	$M_{25}^{(x)}$, $x=1,2$
$\begin{tikzcd} & K \arrow[d, o->] \\ & K^2 \arrow[d, "d_1\vee d_3"] \\ & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ & K^4 \arrow[d, "d_1\vee d_2\vee d_3\vee d_5"] \\ K \arrow[d, "a"'] & K^5 \arrow[dl, "b"'] \arrow[d, "c"] \\ K^3 & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, o->] \arrow[dddr, "d_3"] & \\ K^2 \arrow[d, "a"'] & \\ K^3 \arrow[d, "b"'] & K \arrow[d, "d_2"] \\ K^4 \arrow[d, "c"'] \arrow[dr, "e"'] & K^3 \arrow[d] \\ K^2 & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K \arrow[d, o->] \\ K^2 \arrow[d] & K^2 \arrow[d, o->] \\ K^3 \arrow[d] & K^3 \arrow[dl] \arrow[d, "a_x"] \\ K^3 & K^x \end{tikzcd}$
1 1 1 1 11 32	1 1 11 11 23	11 11 11 3x
$a=d_2$ $b=d_1\vee d_2\vee 0\vee 0\vee d_3$ $c=\begin{bmatrix}0&1&0&1&1\\1&1&1&0&0\end{bmatrix}$	$a=d_1\vee d_3$ $b=d_1\vee d_2\vee d_4$ $c=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$ $e=d_1\vee d_2\vee 0\vee d_3$	$a_1=d$ $a_2=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

$M_{25}^{(3)}$	$M_{25}^{(x+3)}$, $x=1,2$	$M_{25}^{(x+5)}$, $x=1,2$	$M_{25}^{(8)}$
$\begin{tikzcd} K \arrow[d, o->] & K \arrow[d, o->] \\ K^2 \arrow[d, "a"] & K^2 \arrow[d, "d_2\vee d_3"] \\ K^3 \arrow[d, "b"'] & K^4 \arrow[dl] \arrow[d, "c"] \\ K^4 & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, o->] & K^x \arrow[d, o->] \\ K^2 \arrow[d, "a"] & K^3 \arrow[d] \\ K^3 \arrow[d, "b"'] & K^4 \arrow[dl] \arrow[d, "c_x"] \\ K^4 & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^x \arrow[d, o->] \\ K^2 \arrow[d] & K^3 \arrow[d, "d_2\vee d_3\vee d_4"] \\ K^3 \arrow[d] & K^5 \arrow[dl] \arrow[d, "a_x"] \\ K^4 & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, o->] \\ K^2 \arrow[d] & K^4 \arrow[d] \\ K^3 \arrow[d] & K^5 \arrow[dl] \arrow[d, "a"] \\ K^4 & K^2 \end{tikzcd}$
11 11 12 42	11 12 12 11 11 11 42 42	12 11 11 12 12 12 42 42	12 12 11 42
$a=d_1\vee d_3$ $b=d_1\vee d_2\vee d_4$ $c=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a=d_1\vee d_3$ $b=d_1\vee d_2\vee d_4$ $c_1=\begin{bmatrix}1&1&0&1\\1&0&1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a_1=\begin{bmatrix}0&1&1&0&1\\1&1&0&1&0\end{bmatrix}$ $a_2=\begin{bmatrix}0&1&0&1&1\\1&1&1&0&0\end{bmatrix}$	$a=\begin{bmatrix}0&1&1&0&1\\1&0&1&1&0\end{bmatrix}$

$M_{25}^{(x+8)},\ x{=}1{,}2$	$M_{25}^{(11)}$	$M_{25}^{(x+11)},\ x{=}1{,}2$	$M_{25}^{(14)}$
$\begin{tikzcd} K \arrow[d] & K^x \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^3 \arrow[d, "d_2\vee d_3\vee d_4"] \\ K^3 \arrow[d] & K^5 \arrow[d, "a_x"] \arrow[dl] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^4 \arrow[d] \\ K^3 \arrow[d] & K^5 \arrow[d, "a"] \arrow[dl] \\ K^4 & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^x \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^3 \arrow[d] \\ K^3 \arrow[d] & K^4 \arrow[d, "a_x"] \arrow[dl] \\ K^3 & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^2 \arrow[d, "d_2\vee d_3"] \\ K^3 \arrow[d] & K^4 \arrow[d, "a"] \arrow[dl] \\ K^3 & K^2 \end{tikzcd}$
1 x 1 $3{-}x$ 1 2 4 3	12 12 11 43	11 12 12 11 11 11 32 32	11 11 12 32
$a_1=\begin{bmatrix}0&0&0&1&1\\1&1&0&0&0\\0&1&1&1&0\end{bmatrix}$ $a_2=\begin{bmatrix}0&0&0&1&1\\1&0&1&0&0\\0&1&1&1&0\end{bmatrix}$	$a=\begin{bmatrix}1&0&0&0&1\\0&1&1&1&0\\1&1&0&1&0\end{bmatrix}$	$a_1=\begin{bmatrix}1&1&0&1\\1&0&1&0\end{bmatrix}$ $a_2=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$	$a=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$

$M_{25}^{(15)}$	$M_{25}^{(x+15)},\ x{=}1{,}2$	$M_{25}^{(18)}$	$M_{25}^{(19)}$
$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^4 \arrow[d, "d_2\vee d_3\vee d_4\vee d_5"] \\ K^3 \arrow[d] & K^6 \arrow[d, "a"] \arrow[dl] \\ K^5 & K^3 \end{tikzcd}$	$\begin{tikzcd} K^x \arrow[d] & K^2 \arrow[d, "\circ" very near start] \\ K^{x+1} \arrow[d] & K^4 \arrow[d, "d_2\vee d_3\vee d_4\vee d_5"] \\ K^4 \arrow[d] & K^6 \arrow[d, "a"] \arrow[dl] \\ K^5 & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, "\circ" very near start] \\ K^3 \arrow[d] & K^4 \arrow[d, "d_2\vee d_3\vee d_4\vee d_5"] \\ K^4 \arrow[d] & K^6 \arrow[d, "a"] \arrow[dl] \\ K^5 & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^4 \arrow[d] \\ K^3 \arrow[d] & K^6 \arrow[d, "a"] \arrow[dl] \\ K^4 & K^3 \end{tikzcd}$
12 12 12 53	12 22 12 12 22 12 53 53	12 22 12 53	12 12 12 43
$c=\begin{bmatrix}0&0&0&1&0&1\\1&0&1&0&1&0\\1&1&0&1&0&0\end{bmatrix}$	$a=\begin{bmatrix}0&0&0&1&0&1\\1&1&0&0&1&0\\0&1&1&1&0&0\end{bmatrix}$	$a=\begin{bmatrix}0&0&0&1&0&1\\1&0&1&1&0&0\\1&1&0&0&1&0\end{bmatrix}$	$a=\begin{bmatrix}1&0&0&0&0&1\\0&1&0&1&1&0\\1&1&1&0&0&0\end{bmatrix}$

$M_{26}^{(1)}$	$M_{26}^{(x+1)},\ x{=}1{,}2$	$M_{26}^{(4)}$
$\begin{tikzcd} K \arrow[d] & K^3 \arrow[d] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^2 \arrow[d, "a"'] & K^5 \arrow[dl] \arrow[dr, "b"'] & K^2 \arrow[d] \\ K^4 & & K^3 \end{tikzcd}$	$\begin{tikzcd} K^x \arrow[d, "a_x"'] & K^3 \arrow[d] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^3 \arrow[d, "b"'] & K^5 \arrow[dl] \arrow[dr, "c"'] & K^2 \arrow[d] \\ K^4 & & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^3 \arrow[d] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^5 \arrow[dl] \arrow[dr, "a"'] & K^2 \arrow[d] \\ K^3 & & K^3 \end{tikzcd}$
121 121 4 3	$\overset{x}{3{-}x}$ 21 4 21 3	121 121 3 3
$a=\begin{bmatrix}1&0\\0&1\\0&0\\1&1\end{bmatrix}$ $b=\begin{bmatrix}0&1&0&1&0\\1&1&1&0&0\\1&0&0&0&1\end{bmatrix}$	$a=\begin{bmatrix}1\\0\\1\end{bmatrix}$ $b=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\\1&1&0\end{bmatrix}$ $c=\begin{bmatrix}0&1&0&1&0\\0&1&1&0&0\\1&0&0&0&1\end{bmatrix}$	$a=\begin{bmatrix}0&1&0&1&0\\1&1&1&0&0\\1&0&0&0&1\end{bmatrix}$

$M_{26}^{(x+4)},\ x{=}1{,}2$	$M_{26}^{(x+6)},\ x{=}1{,}2$	$M_{26}^{(9)}$
$\begin{tikzcd} K \arrow[d] & K^{x+1} \arrow[d, "a_x"] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^4 \arrow[dl] \arrow[dr, "b_x"'] & K^2 \arrow[d, "\circ" very near start] \\ K^3 & & K^3 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^{x+1} \arrow[d, "a_x"] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^4 \arrow[dl] \arrow[dr, "b_x"'] & K^2 \arrow[d] \\ K^3 & & K^2 \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d, "\circ" very near start] & K \arrow[l, "\circ" very near start] \arrow[d, "\circ" very near start] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr, "a"'] & K^2 \arrow[d] \\ K^3 & & K^2 \end{tikzcd}$
1 3 $-x$ x 1 2 1 3 3	1 3 $-x$ x 1 2 1 3 2	111 111 3 2
$a_1{=}d_2\vee d_3\ \ a_2{=}d_1\vee d_2\vee d_3$ $b_1=\begin{bmatrix}0&1&0&1\\1&1&0&0\\1&0&1&0\end{bmatrix}$ $b_2=\begin{bmatrix}1&0&0&1\\0&1&0&1\\1&0&1&0\end{bmatrix}$	$a_1{=}d_2\vee d_3\ \ a_2{=}d_1\vee d_2\vee d_3$ $b_1=\begin{bmatrix}0&1&0&1\\1&1&1&0\end{bmatrix}$ $b_a=\begin{bmatrix}1&1&0&1\\1&0&1&0\end{bmatrix}$	$a=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

$M_{26}^{(10)}$	$M_{27}^{(x)}$, $x=1,2$	$M_{27}^{(3)}$
$\begin{tikzcd} K \arrow[d] & K^2 \arrow[d] & K \arrow[l] \arrow[d, "d"] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr, "a"'] & K^2 \arrow[d] \\ K^2 & & K^2 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^x \arrow[d, "a_x"] & K^2 \arrow[d] \\ K \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^3 \arrow[d] \\ K^2 & & K^3 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^2 \arrow[d, "a"] & K^2 \arrow[d] \\ K \arrow[d] & K^4 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^2 & & K^3 \end{tikzcd}$
$\begin{array}{ccc} 1&1&1\\ 1&1&1\\ 2&&2 \end{array}$	$\begin{array}{ccc} &&1\\ &x&1\\ 1&3-x&1\\ 2&&3 \end{array}$	$\begin{array}{ccc} &&1\\ &2&1\\ 1&2&1\\ 2&&3 \end{array}$
$a=\begin{bmatrix} 1&1&0\\ 1&0&1 \end{bmatrix}$	$a_1=d$ $a_2=\begin{bmatrix} 1&1\\ 1&0\\ 0&1 \end{bmatrix}$	$a=\begin{bmatrix} 1&1\\ 1&0\\ 0&1\\ 1&0 \end{bmatrix}$ $b=d_1\vee 0\vee 0\vee d_2$

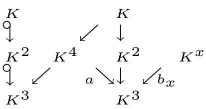
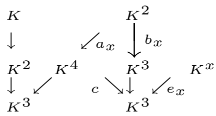
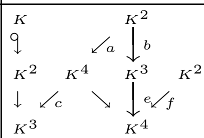
$M_{27}^{(4)}$	$M_{27}^{(5)}$	$M_{27}^{(6)}$
$\begin{tikzcd} & & K \arrow[d] \\ & K^2 \arrow[d, "a"] & K^2 \arrow[d] \\ K \arrow[d, "d_2"'] & K^4 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^3 & & K^3 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^2 \arrow[d, "a"] & K^2 \arrow[d] \\ K^2 \arrow[d] & K^4 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^3 & & K^3 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^2 \arrow[d, "a"] & K^2 \arrow[d, "d_1\vee d_3"] \\ K \arrow[d] & K^4 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ K^2 & & K^4 \end{tikzcd}$
$\begin{array}{ccc} &&1\\ &2&1\\ 1&2&1\\ 3&&3 \end{array}$	$\begin{array}{ccc} &&1\\ &2&1\\ 2&2&1\\ 3&&3 \end{array}$	$\begin{array}{ccc} &&1\\ &2&1\\ 1&2&1\\ 2&&4 \end{array}$
$a=\begin{bmatrix} 1&1\\ 1&0\\ 0&1\\ 1&0 \end{bmatrix}$ $b=d_1\vee d_2\vee 0\vee d_3$	$a=\begin{bmatrix} 0&1\\ 1&1\\ 0&1\\ 1&0 \end{bmatrix}$ $b=d_1\vee d_2\vee 0\vee d_3$	$a=\begin{bmatrix} 1&1\\ 1&0\\ 0&1\\ 1&0 \end{bmatrix}$ $b=d_1\vee 0\vee 0\vee d_2$

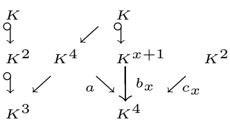
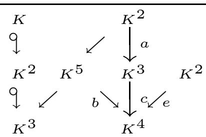
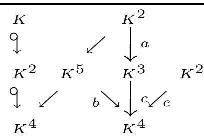
$M_{27}^{(x+6)}$, $x=1,2$	$M_{27}^{(x+8)}$, $x=1,2$	$M_{27}^{(x+10)}$, $x=1,2$
$\begin{tikzcd} & & K \arrow[d] \\ & K^2 \arrow[d, "a_x"] & K^2 \arrow[d, "d_1\vee d_3"] \\ K^x \arrow[d, "b_x"'] & K^4 \arrow[dl, "c"] \arrow[dr] & K^3 \arrow[d, "d_1\vee d_2\vee d_4"] \\ K^3 & & K^4 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^{x+1} \arrow[d, "a_x"] & K^2 \arrow[d] \\ K^2 \arrow[d] & K^5 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^3 & & K^4 \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d] \\ & K^{x+1} \arrow[d, "a_x"] & K^2 \arrow[d] \\ K \arrow[d, "d_2"'] & K^5 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^3 & & K^4 \end{tikzcd}$
$\begin{array}{ccc} &&1\\ &2&1\\ x&2&1\\ 3&&4 \end{array}$	$\begin{array}{ccc} &&1\\ &x+1&1\\ 2&4-x&1\\ 3&&4 \end{array}$	$\begin{array}{ccc} &&1\\ &x+1&1\\ 1&4-x&1\\ 3&&4 \end{array}$
$a_1=\begin{bmatrix} 1&1\\ 1&0\\ 0&1\\ 1&0 \end{bmatrix}$ $a_2=\begin{bmatrix} 0&1\\ 1&1\\ 0&1\\ 1&0 \end{bmatrix}$ $b_1=d_2$ $c=d_1\vee d_2\vee 0\vee d_3$	$a_1=\begin{bmatrix} 1&1\\ 1&0\\ 1&1\\ 0&1\\ 1&1 \end{bmatrix}$ $a_2=\begin{bmatrix} 0&1&1\\ 0&1&0\\ 1&1&1\\ 1&0&0\\ 1&0&1 \end{bmatrix}$ $b=d_1\vee d_2\vee 0\vee 0\vee d_3$	$a_1=\begin{bmatrix} 1&0\\ 1&1\\ 1&0\\ 0&1\\ 0&1 \end{bmatrix}$ $a_2=\begin{bmatrix} 1&0&0\\ 0&1&1\\ 1&1&1\\ 1&1&0\\ 1&0&1 \end{bmatrix}$ $b=d_1\vee d_2\vee 0\vee 0\vee d_3$

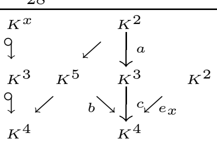
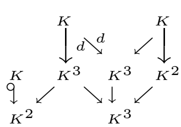
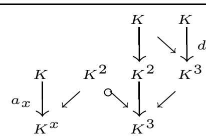
$M_{27}^{(x+12)}$, $x{=}1{,}2$	$M_{27}^{(15)}$	$M_{27}^{(16)}$	$M_{27}^{(17)}$
$\begin{tikzcd}[row sep=small] & & K \arrow[d] \\ & K^{x+1} \arrow[d, "a_x"] & K^2 \arrow[d] \\ K^2 \arrow[d, "b"'] & K^5 \arrow[dl, "c"] \arrow[dr] & K^3 \arrow[d] \\ K^4 & & K^4 \end{tikzcd}$	$\begin{tikzcd}[row sep=small] & & K \arrow[d] \\ & K^3 \arrow[d, "a"] & K^2 \arrow[d] \\ K^2 \arrow[d] & K^6 \arrow[dl, "b"] \arrow[dr] & K^3 \arrow[d] \\ K^4 & & K^4 \end{tikzcd}$	$\begin{tikzcd}[row sep=small] & & K \arrow[d] \\ & K^3 \arrow[d, "a"] & K^2 \arrow[d] \\ K^2 \arrow[d, "b"'] & K^6 \arrow[dl, "c"] \arrow[dr] & K^3 \arrow[d] \\ K^4 & & K^5 \end{tikzcd}$	$\begin{tikzcd}[row sep=small] & & K \arrow[d] \\ & K^3 \arrow[d, "a"] & K^3 \arrow[d] \\ K^2 \arrow[d, "b"'] & K^6 \arrow[dl, "c"] \arrow[dr] & K^4 \arrow[d] \\ K^4 & & K^5 \end{tikzcd}$
1 x+11 24−x1 4 4	1 31 231 4 4	1 31 231 4 5	1 32 231 4 5
$a_1{=}\begin{bmatrix} 1&0\\1&0\\1&1\\0&1\\0&1 \end{bmatrix}$ $a_2{=}\begin{bmatrix} 1&0&0\\1&1&1\\0&1&0\\0&1&1\\1&0&1 \end{bmatrix}$ $b{=}d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_3\vee 0\vee d_4$	$a{=}\begin{bmatrix} 1&0&0\\1&1&0\\0&1&0\\1&1&1\\0&0&1\\1&0&0 \end{bmatrix}$ $b{=}d_1\vee d_2\vee 0\vee 0\vee d_3\vee d_4$	$a{=}\begin{bmatrix} 1&1&0\\0&1&1\\0&1&0\\1&1&1\\1&0&0\\1&1&1 \end{bmatrix}$ $b{=}d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_3\vee 0\vee 0\vee d_4$	$a{=}\begin{bmatrix} 1&1&0\\1&0&0\\1&0&1\\0&1&1\\0&1&0\\0&1&1 \end{bmatrix}$ $b{=}d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_3\vee 0\vee 0\vee d_4$

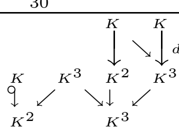
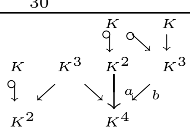
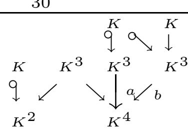
$M_{27}^{(x+17)}$, $x{=}1{,}2$	$M_{28}^{(1)}$	$M_{28}^{(2)}$
$\begin{tikzcd}[row sep=small] & & K^x \arrow[d] \\ & K^3 \arrow[d, "a"] & K^{x+1} \arrow[d] \\ K^2 \arrow[d, "b"'] & K^6 \arrow[dl, "c"] \arrow[dr] & K^4 \arrow[d] \\ K^4 & & K^5 \end{tikzcd}$	$\begin{tikzcd}[row sep=small] K \arrow[d] & & K \arrow[dl, {Circle[open]->}] \arrow[d, {Circle[open]->}] & \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr, "a"'] & K^2 \arrow[d] & K \arrow[dl, "d"] \\ K^3 & & K^2 & \end{tikzcd}$	$\begin{tikzcd}[row sep=small] K \arrow[d, {Circle[open]->}] & & K \arrow[dl] \arrow[d] & \\ K^2 \arrow[d] & K^2 \arrow[dl] \arrow[dr] & K^2 \arrow[d] & K \arrow[dl, "d"] \\ K^2 & & K^2 & \end{tikzcd}$
1 2 31 31 232 231 4 5 4 5	1 1 1211 3 2	1 1 1111 2 2
$a{=}\begin{bmatrix} 1&1&0\\1&0&0\\1&1&1\\0&1&1\\0&1&0\\0&1&1 \end{bmatrix}$ $b{=}d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_3\vee 0\vee 0\vee d_4$	$a{=}\begin{bmatrix} 1&0&0\\0&1&1 \end{bmatrix}$	

$M_{28}^{(3)}$	$M_{28}^{(x+3)}$, $x{=}1{,}2$	$M_{28}^{(x+5)}$, $x{=}1{,}2$
$\begin{tikzcd}[row sep=small] K \arrow[d, {Circle[open]->}] & & K \arrow[dl, "d"] \arrow[d, "d"] & \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr, {Circle[open]->}] & K^2 \arrow[d] & K \arrow[dl] \\ K^2 & & K^2 & \end{tikzcd}$	$\begin{tikzcd}[row sep=small] K \arrow[d, {Circle[open]->}] & & K \arrow[dl] \arrow[d] & \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d] & K^x \arrow[dl, "a_x"] \\ K^2 & & K^3 & \end{tikzcd}$	$\begin{tikzcd}[row sep=small] K \arrow[d] & & K \arrow[dl, {Circle[open]->}] \arrow[d, {Circle[open]->}] & \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, {Circle[open]->}] & K^x \arrow[dl, "a_x"] \\ K^3 & & K^3 & \end{tikzcd}$
1 1 1211 2 2	1 1 121x 2 3	1 1 121x 3 3
	$a_1{=}d$ $a_2{=}\begin{bmatrix} 1&1\\1&0\\0&1 \end{bmatrix}$	$a_1{=}d$ $a_2{=}\begin{bmatrix} 1&1\\1&0\\0&1 \end{bmatrix}$

$M_{28}^{(x+7)}$, x=1,2	$M_{28}^{(x+9)}$, x=1,2	$M_{28}^{(12)}$
		
1 1 131x 3 3	1 2 121x 3 3	1 2 1212 3 4
$a{=}d_1\vee d_1\vee d_2\vee d_3$ $b_1{=}d$ $b_2{=}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a_1{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $a_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $b_1{=}\begin{bmatrix}1&1\\1&0\\1&0\end{bmatrix}$ $b_2{=}\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}$ $c{=}d_1\vee d_2\vee 0\vee d_3$ $e_1{=}d_2$ $e_2{=}d_1\vee d_2$	$a{=}d_1\vee d_4$ $b{=}d_1\vee d_3$ $c{=}d_1\vee d_2\vee 0\vee d_3$ $e{=}d_1\vee d_2\vee d_4$ $f{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$

$M_{28}^{(x+12)}$, x=1,2	$M_{28}^{(15)}$	$M_{28}^{(16)}$
		
1 1 13x2 3 4	1 2 1312 3 4	1 2 1312 4 4
$a{=}d_4\vee d_1\vee d_2\vee d_3$ $b_2{=}d_1\vee d_2\vee d_4$ $b_1{=}d_1\vee d_4$ $c_1{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $c_2{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$	$a{=}d_1\vee d_3$ $b{=}d_1\vee d_4\vee d_1\vee d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_4$ $e{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$	$a{=}d_1\vee d_3$ $b{=}d_1\vee d_4\vee d_1\vee d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_4$ $e{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$

$M_{28}^{(x+16)}$, x=1,2	$M_{29}^{(1)}$	$M_{30}^{(x)}$, x=1,2
		
x 2 3−x312 4 4	1 1 1211 2 3	11 1211 x 3
$a{=}d_1\vee d_3$ $b{=}d_1\vee d_4\vee d_1\vee d_2\vee d_3$ $c{=}d_1\vee d_2\vee d_4$ $e_1{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $e_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$		$a_2{=}d_2$

$M_{30}^{(3)}$	$M_{30}^{(4)}$	$M_{30}^{(5)}$
		
11 1311 2 3	11 1311 2 4	11 1321 2 4
	$a{=}\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$	$a{=}\begin{bmatrix}0&1&0\\1&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$

$M_{30}^{(6)}$	$M_{30}^{(7)}$	$M_{30}^{(8)}$
$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d, "a"] \\ K \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "b"] & K^4 \arrow[dl] \\ K^2 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d, "a"] \\ K \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "b"] & K^4 \arrow[dl] \\ K^2 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d, "d_1\vee d_3"] \\ K \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^2 & & K^5 & \end{tikzcd}$
$\begin{smallmatrix} & & 1 & 2 \\ 1 & 3 & 1 & 1 \\ 2 & & 4 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & & 4 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & & 5 & \end{smallmatrix}$
$a=\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}1&1&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$

$M_{30}^{(9)}$	$M_{30}^{(10)}$	$M_{30}^{(11)}$
$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d] \\ K \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^2 & & K^5 & \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d] \\ K \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^3 & & K^5 & \end{tikzcd}$	$\begin{tikzcd} & & K \arrow[d, hook] \arrow[dr, hook] & K^2 \arrow[d] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^3 & & K^5 & \end{tikzcd}$
$\begin{smallmatrix} & & 1 & 2 \\ 1 & 4 & 2 & 1 \\ 2 & & 5 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & 2 \\ 1 & 4 & 2 & 1 \\ 3 & & 5 & \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 3 & & 5 & \end{smallmatrix}$
$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$

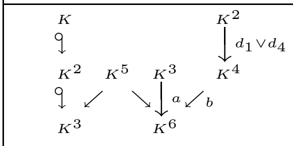
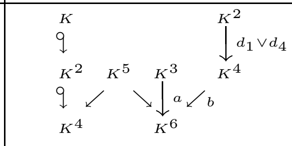
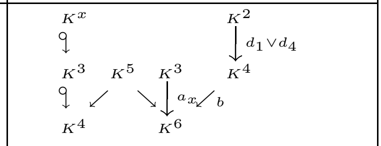
$M_{31}^{(x)}$, $x=1,2$	$M_{31}^{(3)}$	$M_{31}^{(4)}$	$M_{31}^{(5)}$
$\begin{tikzcd} K \arrow[d] & & & K \arrow[d, hook] \\ K^2 \arrow[d] & K^2 \arrow[dl] \arrow[dr] & K^x \arrow[d, "a_x"] & K^2 \arrow[dl, hook] \\ K^2 & & K^3 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d] & & & K \arrow[d, hook] \\ K^2 \arrow[d] & K^2 \arrow[dl] \arrow[dr] & K \arrow[d, "d"] & K^2 \arrow[dl] \\ K^2 & & K^2 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "a"] & K^2 \arrow[dl] \\ K^2 & & K^3 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K \arrow[d, "d"] & K^2 \arrow[dl] \\ K^2 & & K^3 & \end{tikzcd}$
$\begin{smallmatrix} 1 & & & 1 \\ 1 & 2 & x & 1 \\ 2 & & 3 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 2 & 1 & 1 \\ 2 & & 2 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 3 & 2 & 1 \\ 2 & & 3 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 3 & 1 & 1 \\ 2 & & 3 & \end{smallmatrix}$
$a_1=d$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$		$a=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	

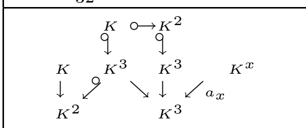
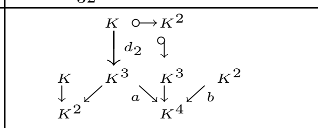
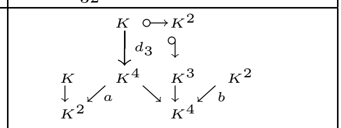
$M_{31}^{(x+5)}$, $x=1,2$	$M_{31}^{(8)}$	$M_{31}^{(x+8)}$, $x=1,2$
$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d] \\ K^2 \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^x \arrow[d, "a_x"] & K^2 \arrow[dl] \\ K^3 & & K^3 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d, hook] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "a"] & K^2 \arrow[dl, "b"] \\ K^2 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K^x \arrow[d, "a_x"] \\ K^2 \arrow[d] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "b_x"] & K^3 \arrow[dl, "c"] \\ K^2 & & K^4 & \end{tikzcd}$
$\begin{smallmatrix} 1 & & & 1 \\ 1 & 3 & x & 1 \\ 3 & & 3 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 3 & 2 & 1 \\ 2 & & 4 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & x \\ 1 & 3 & 2 & 3-x \\ 2 & & 4 & \end{smallmatrix}$
$a_1=d$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $b=d_1\vee d_4$	$a_1=d_3$ $a_2=d_1\vee d_3$ $c=d_1\vee d_2\vee d_4$ $b_1=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$

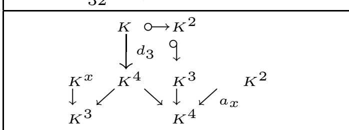
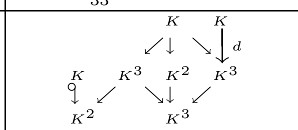
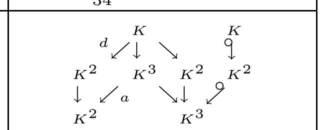
$M_{31}^{(x+10)}$, $x{=}1{,}2$	$M_{31}^{(13)}$	$M_{31}^{(14)}$
$\begin{tikzcd} K \arrow[d, hook] & & & K^x \arrow[d, "a_x"] \\ K^2 \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "b_x"] & K^3 \arrow[dl, "c"] \\ K^3 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d, hook] \\ K^2 \arrow[d, hook] & K^3 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "a"] & K^2 \arrow[dl, "b"] \\ K^3 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K \arrow[d, hook] \\ K^2 \arrow[d] & K^4 \arrow[dl, "a"] \arrow[dr] & K^2 \arrow[d, "b"] & K^2 \arrow[dl, "c"] \\ K^3 & & K^4 & \end{tikzcd}$
$\begin{smallmatrix} 1 & & & & x \\ 1 & 3 & 2 & & 3-x \\ 3 & & 4 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 3 & 2 & 1 \\ 3 & & 4 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 1 \\ 1 & 4 & 2 & 1 \\ 3 & & 4 & \end{smallmatrix}$
$a_1{=}d_3$ $a_2{=}d_1{\vee}d_3$ $c{=}d_1{\vee}d_2{\vee}d_4$ $b_1{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $b{=}d_1{\vee}d_4$	$b{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $a{=}d_1{\vee}d_2{\vee}0{\vee}d_3$ $c{=}d_1{\vee}d_4$

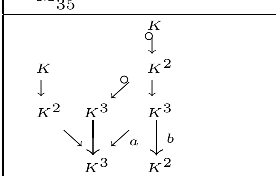
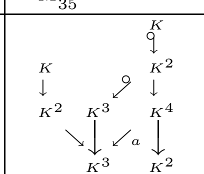
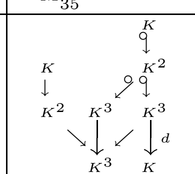
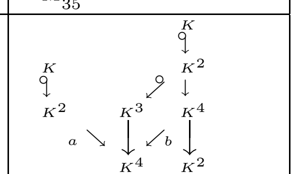
$M_{31}^{(x+14)}$, $x{=}1{,}2$	$M_{31}^{(x+16)}$, $x{=}1{,}2$	$M_{31}^{(x+18)}$, $x{=}1{,}2$
$\begin{tikzcd} K \arrow[d, hook] & & & K^x \arrow[d, "a_x"] \\ K^2 \arrow[d] & K^4 \arrow[dl, "b"] \arrow[dr] & K^2 \arrow[d, "c_x"] & K^3 \arrow[dl, "e"] \\ K^3 & & K^4 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K^x \arrow[d, "a_x"] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "b_x"] & K^3 \arrow[dl, "c"] \\ K^3 & & K^5 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K^x \arrow[d, "a_x"] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "b_x"] & K^3 \arrow[dl, "c"] \\ K^3 & & K^5 & \end{tikzcd}$
$\begin{smallmatrix} 1 & & & & x \\ 1 & 4 & 2 & & 3-x \\ 3 & & 4 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & & x \\ 1 & 4 & 2 & & 3-x \\ 3 & & 5 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & & x \\ 1 & 4 & 3 & & 3-x \\ 3 & & 5 & & \end{smallmatrix}$
$a_1{=}d_3$ $a_2{=}d_1{\vee}d_3$ $b{=}d_1{\vee}d_2{\vee}0{\vee}d_3$ $e{=}d_1{\vee}d_2{\vee}d_4$ $c_1{=}\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $c_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a_1{=}d_3$ $a_2{=}d_1{\vee}d_3$ $c{=}d_1{\vee}d_2{\vee}d_5$ $b_1{=}\begin{bmatrix}0&1\\1&0\\1&1\\0&1\\1&0\end{bmatrix}$ $b_2{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\\1&0\end{bmatrix}$	$a_1{=}d_3$ $a_2{=}d_1{\vee}d_3$ $c{=}d_1{\vee}d_2{\vee}d_5$ $b_1{=}\begin{bmatrix}1&0&1\\0&1&1\\0&1&0\\0&1&1\\1&0&0\end{bmatrix}$ $b_2{=}\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\\1&0&1\\1&0&0\end{bmatrix}$

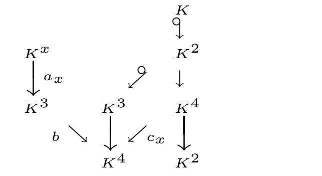
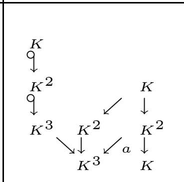
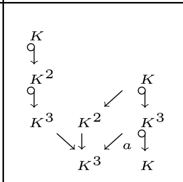
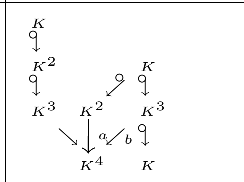
$M_{31}^{(21)}$	$M_{31}^{(22)}$	$M_{31}^{(23)}$
$\begin{tikzcd} K \arrow[d, hook] & & & K^2 \arrow[d, "d_1\vee d_4"] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^2 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^3 & & K^5 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K^2 \arrow[d, "d_1\vee d_4"] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^3 & & K^5 & \end{tikzcd}$	$\begin{tikzcd} K \arrow[d, hook] & & & K^2 \arrow[d, hook] \\ K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[dr] & K^3 \arrow[d, "a"] & K^4 \arrow[dl, "b"] \\ K^3 & & K^6 & \end{tikzcd}$
$\begin{smallmatrix} 1 & & & 2 \\ 1 & 4 & 2 & 2 \\ 3 & & 5 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 2 \\ 1 & 4 & 3 & 2 \\ 3 & & 5 & \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & 2 \\ 1 & 4 & 3 & 2 \\ 3 & & 6 & \end{smallmatrix}$
$a{=}\begin{bmatrix}0&1\\1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b{=}d_1{\vee}d_2{\vee}d_3{\vee}d_5$	$a{=}\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\1&0&0\end{bmatrix}$ $b{=}d_1{\vee}d_2{\vee}d_3{\vee}d_5$	$a{=}\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\\0&1&0\\0&1&0\\1&0&0\end{bmatrix}$ $b{=}d_1{\vee}d_2{\vee}d_5{\vee}d_6$

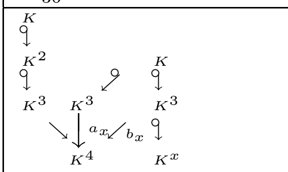
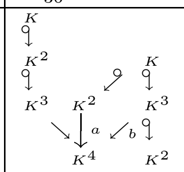
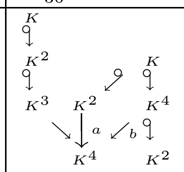
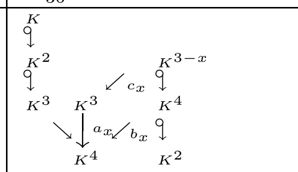
$M_{31}^{(24)}$	$M_{31}^{(25)}$	$M_{31}^{(x+25)}$, $x=1,2$
		
1 2 1532 3 6	1 2 1532 4 6	x 2 $3-x$ 532 4 6
$a=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $b=d_1\vee d_2\vee d_3\vee d_6$	$a=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $b=d_1\vee d_2\vee d_3\vee d_6$	$a_1=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&1&0\\0&0&1\\1&0&0\end{bmatrix}$ $a_2=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $b=d_1\vee d_2\vee d_3\vee d_6$

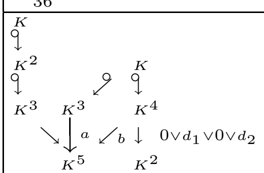
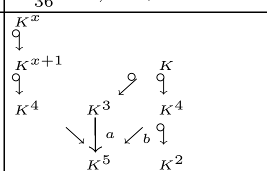
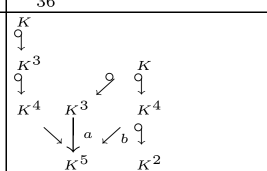
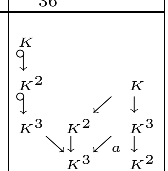
$M_{32}^{(x)}$, $x=1,2$	$M_{32}^{(3)}$	$M_{32}^{(4)}$
		
11 121x 2 3	11 1212 2 4	11 1312 2 4
$a_1=d$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a=d_1\vee d_3\vee d_4$ $b=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a=0\vee d_1\vee d_2\vee 0$ $b=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$

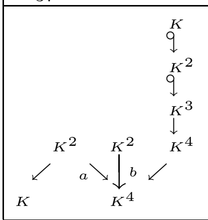
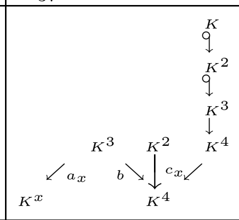
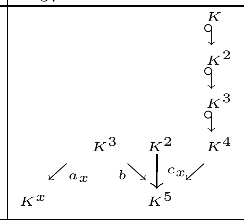
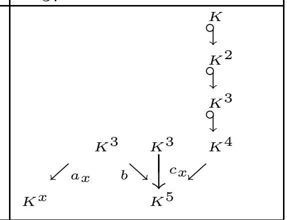
$M_{32}^{(x+4)}$, $x=1,2$	$M_{33}^{(1)}$	$M_{34}^{(1)}$
		
11 x312 3 4	11 1211 2 3	1 1 1211 2 3
$a_1=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$		$a=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$

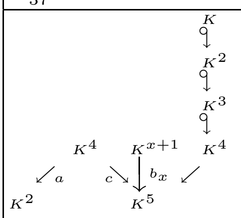
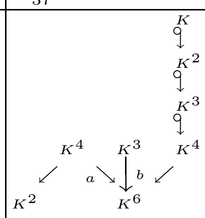
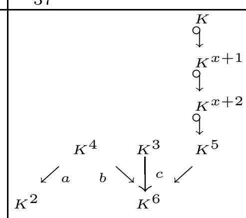
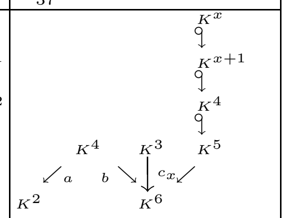
$M_{35}^{(1)}$	$M_{35}^{(2)}$	$M_{35}^{(3)}$	$M_{35}^{(4)}$
			
1 1 1 111 32	1 1 1 112 32	1 1 1 111 31	1 1 1 112 42
$a=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$ $b=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}$	$c=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$		$a=d_1\vee d_4$ $b=\begin{bmatrix}0&0&0&1\\1&0&1&1\\0&1&0&1\\0&0&1&0\end{bmatrix}$

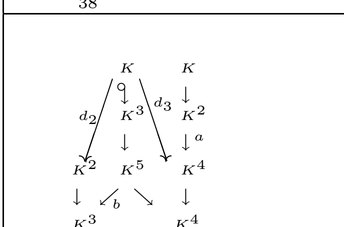
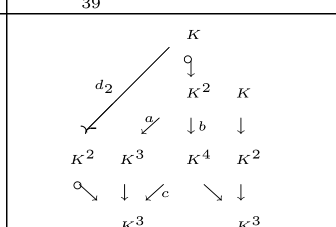
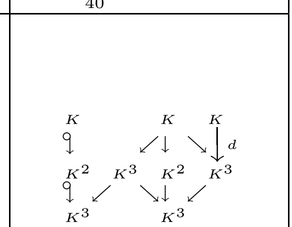
$M_{35}^{(x+4)}$, $x{=}1{,}2$	$M_{36}^{(1)}$	$M_{36}^{(2)}$	$M_{36}^{(3)}$
			
1 x 1 $3-x$ 12 42	1 1 1 111 31	1 1 1 112 31	1 1 1 112 41
$a_1{=}d_3$ $a_2{=}d_1\vee d_3$ $b{=}d_1\vee d_2\vee d_4$ $c_1=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&0&0&1\\1&0&1&1\\0&1&0&1\\0&0&1&0\end{bmatrix}$	$c=\begin{bmatrix}1&1\\0&1\\0&1\end{bmatrix}$	$a=\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\end{bmatrix}$	$a{=}d_1\vee d_4$ $b=\begin{bmatrix}0&1&0\\1&1&0\\0&1&0\\1&0&1\end{bmatrix}$

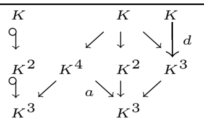
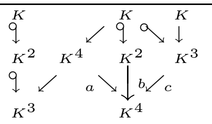
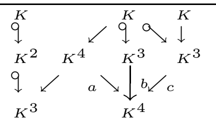
$M_{36}^{(x+3)}$, $x{=}1{,}2$	$M_{36}^{(6)}$	$M_{36}^{(7)}$	$M_{36}^{(x+7)}$, $x{=}1{,}2$
			
1 1 1 122 $4x$	1 1 1 112 42	1 1 1 113 42	1 1 $3-x$ $1x$ $x+1$ 4 2
$a_1{=}d_1\vee d_2\vee d_4$ $b_1=\begin{bmatrix}1&1&0\\1&0&0\\0&1&0\\1&0&1\end{bmatrix}$ $a_2=\begin{bmatrix}0&1&0\\1&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $b_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1\end{bmatrix}$	$a_1{=}d_1\vee d_2\vee d_4$ $b_1=\begin{bmatrix}0&1&1&0\\1&1&0&0\\0&1&0&0\\1&0&0&1\end{bmatrix}$ $a_2=\begin{bmatrix}1&1&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $b_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1\end{bmatrix}$ $c_1{=}d_1\vee d_3$ $c_2{=}d_3$

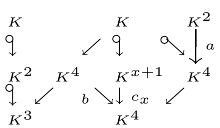
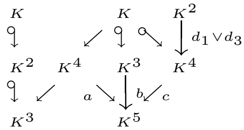
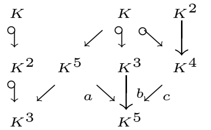
$M_{36}^{(10)}$	$M_{36}^{(x+10)}$, $x{=}1{,}2$	$M_{36}^{(13)}$	$M_{36}^{(14)}$
			
1 1 1 123 52	1 2 1 1 1 1 223 123 52 52	1 2 1 123 52	1 1 1 112 32
$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$	$a=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$

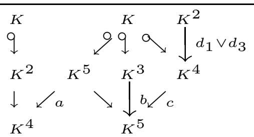
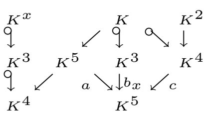
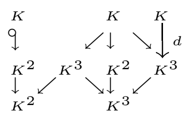
$M_{37}^{(1)}$	$M_{37}^{(x+1)}$, $x=1,2$	$M_{37}^{(x+3)}$, $x=1,2$	$M_{37}^{(x+5)}$, $x=1,2$
			
1 1 1 221 1 4	1 1 1 321 x 4	1 1 1 321 x 5	1 1 1 331 x 5
$a=d_1\vee d_4$ $b=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a_1=d_2$ $b=d_1\vee d_2\vee d_4$ $c_1=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$ $c_2=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$	$a_1=d_2$ $b=d_1\vee d_2\vee d_5$ $c_1=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\\1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1\\1&0\\1&1\\0&1\\1&0\end{bmatrix}$	$a_1=d_2$ $b=d_1\vee d_2\vee d_5$ $c_1=\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\\1&0&1\\1&0&0\end{bmatrix}$ $c_2=\begin{bmatrix}1&0&1\\0&1&1\\0&1&0\\0&1&1\\1&0&0\end{bmatrix}$

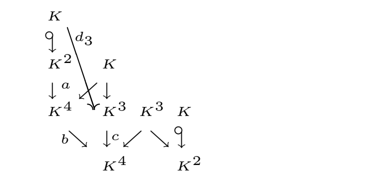
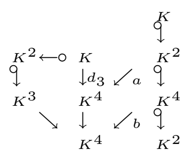
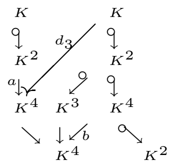
$M_{37}^{(x+7)}$, $x=1,2$	$M_{37}^{(10)}$	$M_{37}^{(x+10)}$, $x=1,2$	$M_{37}^{(x+12)}$, $x=1,2$
			
1 1 1 4 $x+1$ 1 2 5	1 1 1 431 2 6	1 1 1 2 1 1 432 431 2 6 2 6	1 2 1 1 2 1 432 431 2 6 2 6
$a=0\vee d_1\vee d_2\vee 0$ $b_1=\begin{bmatrix}0&1\\1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $b_2=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\1&0&0\end{bmatrix}$ $c=d_1\vee d_2\vee d_3\vee d_5$	$a=d_1\vee d_2\vee d_5\vee d_6$ $b=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\\0&1&0\\0&1&0\\1&0&0\end{bmatrix}$	$a=0\vee d_1\vee d_2\vee 0$ $b=d_1\vee d_2\vee d_3\vee d_6$ $c=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$a=0\vee d_1\vee d_2\vee 0$ $b=d_1\vee d_2\vee d_3\vee d_6$ $c_1=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&1&0\\0&0&1\\1&0&0\end{bmatrix}$ $c_1=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$

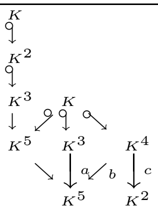
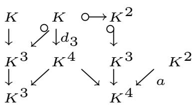
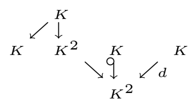
$M_{38}^{(1)}$	$M_{39}^{(1)}$	$M_{40}^{(1)}$
		
11 21 121 3 4	1 11 1121 3 3	1 11 1211 3 3
$a=\begin{bmatrix}1&0\\0&1\\0&0\\1&1\end{bmatrix}$ $b=\begin{bmatrix}0&1&0&1&0\\1&1&1&0&0\\1&0&0&0&1\end{bmatrix}$	$b=d_2\vee d_3$ $a=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$ $c=\begin{bmatrix}0&1&0&1\\1&1&0&0\\1&0&1&0\end{bmatrix}$	

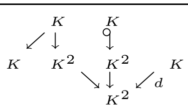
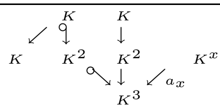
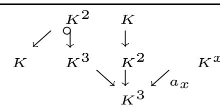
$M_{40}^{(2)}$	$M_{40}^{(3)}$	$M_{40}^{(4)}$
		
1 11 1311 3 3	1 11 1311 3 4	1 11 1321 3 4
$a{=}d_1\vee d_1\vee d_2\vee d_3$	$a{=}d_4\vee d_1\vee d_2\vee d_3$ $b{=}\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$	$a{=}d_4\vee d_1\vee d_2\vee d_3$ $b{=}\begin{bmatrix}0&1&0\\1&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$

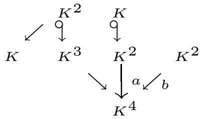
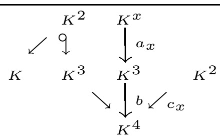
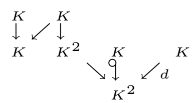
$M_{40}^{(x+4)}$, $x{=}1{,}2$	$M_{40}^{(7)}$	$M_{40}^{(8)}$
		
1 12 13x1 3 4	1 12 1321 3 5	1 12 1421 3 5
$a{=}\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$ $b{=}d_4\vee d_1\vee d_2\vee d_3$ $c_1{=}\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $c_2{=}\begin{bmatrix}1&1&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$a{=}d_5\vee d_1\vee d_2\vee d_3$ $b{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$	$a{=}d_5\vee d_1\vee d_2\vee d_3\vee d_4$ $b{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$

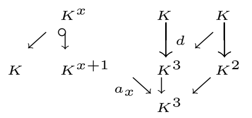
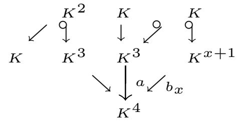
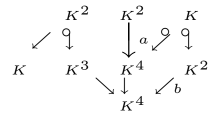
$M_{40}^{(9)}$	$M_{40}^{(x+9)}$, $x{=}1{,}2$	$M_{40}^{(12)}$
		
1 12 1421 4 5	x 12 $3{-}x$421 4 5	1 11 1211 2 3
$a{=}d_1\vee d_2\vee 0\vee d_3\vee d_4$ $b{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$	$a{=}d_5\vee d_1\vee d_2\vee d_3\vee d_4$ $b_1{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b_2{=}\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$	

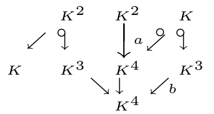
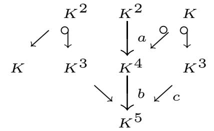
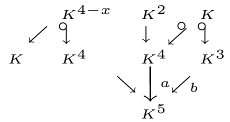
$M_{41}^{(1)}$	$M_{42}^{(1)}$	$M_{43}^{(1)}$
		
1 11 1131 4 2	1 111 112 42	1 1 1 1 112 4 2
$a=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$ $c=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$	$a=d_1\vee d_4$ $b=\begin{bmatrix}0&1&1&0\\1&1&0&0\\0&1&0&0\\1&0&0&1\end{bmatrix}$	$a=d_1\vee d_4$ $b=\begin{bmatrix}0&1&0&0\\1&1&1&0\\0&1&0&1\\1&0&0&0\end{bmatrix}$

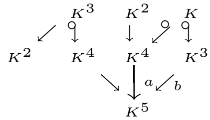
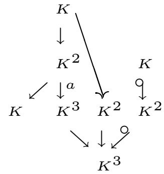
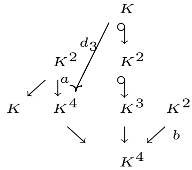
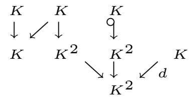
$M_{44}^{(1)}$	$M_{45}^{(1)}$	$M_{46}^{(1)}$
		
1 1 11 123 52	111 1312 3 4	1 1111 2
$a=\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$ $c=0\vee d_1\vee 0\vee d_2$	$a=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	

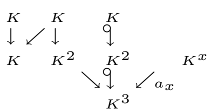
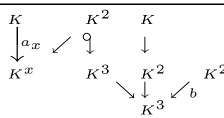
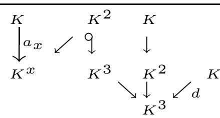
$M_{47}^{(1)}$	$M_{47}^{(x+1)}$, $x=1,2$	$M_{47}^{(x+3)}$, $x=1,2$
		
11 1111 2	11 111x 3	21 111x 3
	$a_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$

$M_{47}^{(6)}$	$M_{47}^{(x+6)}$, $x=1,2$	$M_{48}^{(1)}$
		
21 1112 4	2 x 11 $3-x$ 2 4	11 1111 2
$a=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}$ $b=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a_1=d_3$ $a_2=d_1\vee d_3$ $b=d_1\vee d_2\vee d_4$ $c_1=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	

$M_{49}^{(x)}$, $x{=}1{,}2$	$M_{49}^{(x+2)}$, $x{=}1{,}2$	$M_{49}^{(5)}$
		
$\begin{array}{c} x11\\ 1111\\ 3 \end{array}$	$\begin{array}{c} 211\\ 111x\\ 4 \end{array}$	$\begin{array}{c} 221\\ 1111\\ 4 \end{array}$
$a_1{=}d_2 \vee d_3$	$a{=}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$ $b_1{=}\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $b_2{=}\begin{bmatrix}0&1&0\\1&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$a{=}\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$

$M_{49}^{(6)}$	$M_{49}^{(7)}$	$M_{49}^{(x+7)}$, $x{=}1{,}2$
		
$\begin{array}{c} 221\\ 1112\\ 4 \end{array}$	$\begin{array}{c} 221\\ 1112\\ 5 \end{array}$	$\begin{array}{ccc} 4{-}x & & 21\\ 1 & x & 12\\ & & 5 \end{array}$
$a{=}\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&1&0\\0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	$a{=}d_1 \vee d_3$ $b{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$ $c{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$a{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$M_{49}^{(10)}$	$M_{50}^{(1)}$	$M_{51}^{(1)}$	$M_{52}^{(1)}$
			
$\begin{array}{c} 321\\ 2112\\ 5 \end{array}$	$\begin{array}{c} 1\\ 1\ \ 1\\ 1111\\ 3 \end{array}$	$\begin{array}{c} 1\\ 21\\ 1112\\ 4 \end{array}$	$\begin{array}{c} 111\\ 1111\\ 2 \end{array}$
$a{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}$ $b{=}\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&0\\0&0&1\end{bmatrix}$	$a{=}\begin{bmatrix}1&1\\0&1\\0&1\end{bmatrix}$	$a{=}d_1 \vee d_4$ $b{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	

$M_{52}^{(x+1)}$, $x{=}1{,}2$	$M_{52}^{(x+3)}$, $x{=}1{,}2$	$M_{52}^{(x+5)}$, $x{=}1{,}2$
		
$\begin{array}{c} 111\\ 111x\\ 3 \end{array}$	$\begin{array}{c} 121\\ x112\\ 3 \end{array}$	$\begin{array}{c} 121\\ x111\\ 3 \end{array}$
$a_1{=}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $a_2{=}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a_2{=}d_2$ $b{=}\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$	$a_2{=}d_2$

$M_{52}^{(x+7)},\ x{=}1,2$	$M_{52}^{(x+9)},\ x{=}1,2$	$M_{52}^{(x+11)},\ x{=}1,2$
$\begin{array}{cccc} K & K^2 & K & \\ \downarrow{\scriptstyle a_x}\ \swarrow & \overset{\circ}{\downarrow} & \overset{\circ}{\downarrow} & \\ K^x & K^3 & K^2 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c} & \\ & & K^4 & \end{array}$	$\begin{array}{cccc} K & K^2 & K^x & \\ \downarrow\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a_x} & \\ K & K^3 & K^3 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c_x} & \\ & & K^4 & \end{array}$	$\begin{array}{cccc} K & K^2 & K^x & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a_x} & \\ K^2 & K^3 & K^3 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c_x} & \\ & & K^4 & \end{array}$
$\begin{array}{l} 121 \\ x112 \\ \ \ 4 \end{array}$	$\begin{array}{l} 12\ \ \ x \\ 11\ 3{-}x\ 2 \\ \quad\ 4 \end{array}$	$\begin{array}{l} 12\ \ \ x \\ 21\ 3{-}x\ 2 \\ \quad\ 4 \end{array}$
$a_2{=}d_2\quad b{=}d_1{\vee}d_4\quad c{=}\begin{bmatrix} 0&1\\1&1\\0&1\\1&0 \end{bmatrix}$	$a_1{=}d_3\quad a_2{=}d_1{\vee}d_3\quad b{=}d_1{\vee}d_2{\vee}d_4$ $c_1{=}\begin{bmatrix} 1&1\\1&0\\0&1\\1&0 \end{bmatrix}\quad c_2{=}\begin{bmatrix} 0&1\\1&1\\0&1\\1&0 \end{bmatrix}$	$a_1{=}d_3\quad a_2{=}d_1{\vee}d_3\quad b{=}d_1{\vee}d_2{\vee}d_4$ $c_1{=}\begin{bmatrix} 1&1\\1&0\\0&1\\1&0 \end{bmatrix}\quad c_2{=}\begin{bmatrix} 0&1\\1&1\\0&1\\1&0 \end{bmatrix}$

$M_{52}^{(x+13)},\ x{=}1,2$	$M_{52}^{(x+15)},\ x{=}1,2$	$M_{52}^{(x+17)},\ x{=}1,2$
$\begin{array}{cccc} K & K^3 & K^x & \\ \overset{\circ}{\downarrow} & \swarrow\downarrow & \downarrow{\scriptstyle a_x} & \\ K^2 & K^4 & K^3 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c_x} & \\ & & K^4 & \end{array}$	$\begin{array}{cccc} K & K^3 & K^x & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a_x} & \\ K^2 & K^4 & K^3 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c_x} & \\ & & K^5 & \end{array}$	$\begin{array}{cccc} K & K^3 & K^x & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a_x} & \\ K^2 & K^4 & K^3 & K^3 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c_x} & \\ & & K^5 & \end{array}$
$\begin{array}{l} 13\ \ \ x \\ 21\ 3{-}x\ 2 \\ \quad\ 4 \end{array}$	$\begin{array}{l} 13\ \ \ x \\ 21\ 3{-}x\ 2 \\ \quad\ 5 \end{array}$	$\begin{array}{l} 13\ \ \ x \\ 21\ 3{-}x\ 3 \\ \quad\ 5 \end{array}$
$a_1{=}d_3\quad a_2{=}d_1{\vee}d_3\quad b{=}d_1{\vee}d_2{\vee}d_4$ $c_1{=}\begin{bmatrix} 1&1\\1&0\\0&1\\1&0 \end{bmatrix}\quad c_2{=}\begin{bmatrix} 0&1\\1&1\\0&1\\1&0 \end{bmatrix}$	$a_1{=}d_3\quad a_2{=}d_1{\vee}d_3\quad b{=}d_1{\vee}d_2{\vee}d_5$ $c_1{=}\begin{bmatrix} 0&1\\1&0\\1&1\\0&1\\1&0 \end{bmatrix}\quad c_2{=}\begin{bmatrix} 0&1\\1&1\\0&1\\1&0\\1&0 \end{bmatrix}$	$a_1{=}d_3\quad a_2{=}d_1{\vee}d_3\quad b{=}d_1{\vee}d_2{\vee}d_5$ $c_1{=}\begin{bmatrix} 1&0&1\\0&1&1\\0&1&0\\0&1&1\\1&0&0 \end{bmatrix}\quad c_2{=}\begin{bmatrix} 0&1&0\\0&0&1\\0&1&1\\1&0&1\\1&0&0 \end{bmatrix}$

$M_{52}^{(20)}$	$M_{52}^{(21)}$	$M_{52}^{(22)}$
$\begin{array}{cccc} K & K^3 & K^2 & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a} & \\ K^2 & K^4 & K^4 & K^3 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c} & \\ & & K^5 & \end{array}$	$\begin{array}{cccc} K & K^3 & K^2 & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \downarrow{\scriptstyle a} & \\ K^2 & K^4 & K^4 & K^2 \\ & & \searrow\downarrow{\scriptstyle b}\swarrow{\scriptstyle c} & \\ & & K^5 & \end{array}$	$\begin{array}{cccc} K & K^3 & K^2 & \\ \overset{\circ}{\downarrow}\ \swarrow & \overset{\circ}{\downarrow} & \overset{\circ}{\downarrow} & \\ K^2 & K^4 & K^4 & K^3 \\ & & \searrow\downarrow{\scriptstyle a}\swarrow{\scriptstyle b} & \\ & & K^6 & \end{array}$
$\begin{array}{l} 132 \\ 2123 \\ \ \ 5 \end{array}$	$\begin{array}{l} 132 \\ 2122 \\ \ \ 5 \end{array}$	$\begin{array}{l} 132 \\ 2123 \\ \ \ 6 \end{array}$
$a{=}d_1{\vee}d_4\quad b{=}d_1{\vee}d_2{\vee}d_3{\vee}d_5$ $c{=}\begin{bmatrix} 0&1&0\\0&1&1\\0&0&1\\1&0&1\\1&0&0 \end{bmatrix}$	$a{=}d_1{\vee}d_4\quad b{=}d_1{\vee}d_2{\vee}d_3{\vee}d_5$ $c{=}\begin{bmatrix} 0&1\\1&1\\1&0\\0&1\\1&0 \end{bmatrix}$	$a{=}d_1{\vee}d_2{\vee}d_5{\vee}d_6\quad b{=}\begin{bmatrix} 1&0&1\\0&1&1\\0&0&1\\0&1&0\\0&1&0\\1&0&0 \end{bmatrix}$

<table>
<tr><td>$M_{52}^{(23)}$</td><td>$M_{52}^{(24)}$</td><td>$M_{52}^{(25)}$</td></tr>
<tr>
<td>$$\begin{tikzcd} K \arrow[d, hook] & K^3 \arrow[dl] \arrow[d, hook] & K^2 \arrow[d, "a"] & \\ K^2 & K^5 \arrow[dr] & K^4 \arrow[d, "b"] & K^3 \arrow[dl, "c"] \\ & & K^6 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} K \arrow[d, hook] & K^4 \arrow[dl] \arrow[d, hook] & K^2 \arrow[d, "a"] & \\ K^2 & K^5 \arrow[dr] & K^4 \arrow[d, "b"] & K^3 \arrow[dl, "c"] \\ & & K^6 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} K \arrow[d, hook] & K^4 \arrow[dl] \arrow[d, hook] & K^2 \arrow[d, "a"] & \\ K^3 & K^5 \arrow[dr] & K^4 \arrow[d, "b"] & K^3 \arrow[dl, "c"] \\ & & K^6 & \end{tikzcd}$$</td>
</tr>
<tr>
<td>$\begin{array}{c}132\\2223\\6\end{array}$</td>
<td>$\begin{array}{c}142\\2123\\6\end{array}$</td>
<td>$\begin{array}{c}142\\3123\\6\end{array}$</td>
</tr>
<tr>
<td>$a=d_1\vee d_4$ $\quad b=d_1\vee d_2\vee d_3\vee d_6$
$c=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$</td>
<td>$a=d_1\vee d_4$ $\quad b=d_1\vee d_2\vee d_3\vee d_6$
$c=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$</td>
<td>$a=d_1\vee d_4$ $\quad b=d_1\vee d_2\vee d_3\vee d_6$
$c=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&1&0\\0&0&1\\1&0&0\end{bmatrix}$</td>
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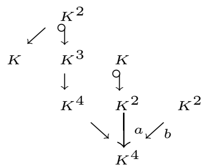
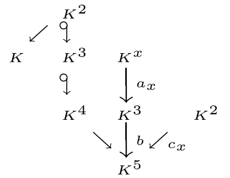
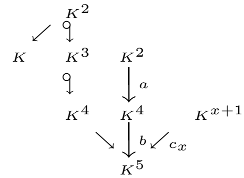
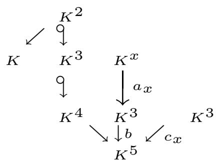
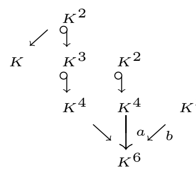
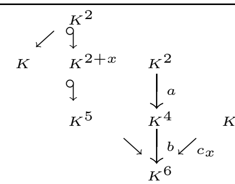
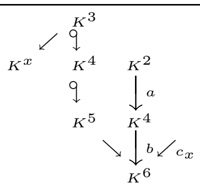
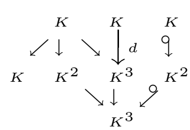
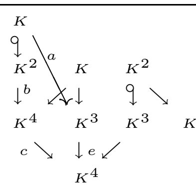
</table>

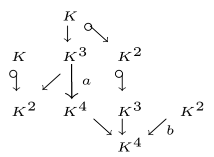
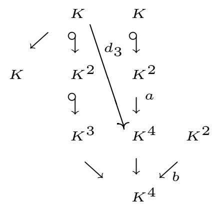
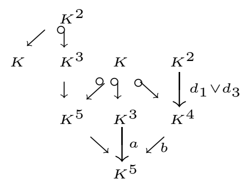
<table>
<tr><td>$M_{52}^{(26)}$</td><td>$M_{52}^{(27)}$</td><td>$M_{53}^{(x)}$, $x=1,2$</td></tr>
<tr>
<td>$$\begin{tikzcd} K^2 \arrow[d, hook] & K^4 \arrow[dl] \arrow[d, hook] & K^2 \arrow[d, "a"] & \\ K^3 & K^5 \arrow[dr] & K^4 \arrow[d, "b"] & K^3 \arrow[dl, "c"] \\ & & K^6 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} K \arrow[d, hook] & K^3 \arrow[dl] \arrow[d] & K \arrow[d, hook] & \\ K^2 & K^4 \arrow[dr] & K^2 \arrow[d, "a"] & K^2 \arrow[dl, "b"] \\ & & K^4 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} & K \arrow[dl] \arrow[d, hook] & & \\ K & K^2 \arrow[d, hook] & K \arrow[d] & \\ & K^3 \arrow[dr] & K^2 \arrow[d] & K^x \arrow[dl, "a_x"] \\ & & K^3 & \end{tikzcd}$$</td>
</tr>
<tr>
<td>$\begin{array}{c}242\\3123\\6\end{array}$</td>
<td>$\begin{array}{c}131\\2112\\4\end{array}$</td>
<td>$\begin{array}{c}1\\111\\11x\\3\end{array}$</td>
</tr>
<tr>
<td>$a=d_1\vee d_4$
$b=d_1\vee d_2\vee d_3\vee d_6$ $\quad c=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$</td>
<td>$a=d_1\vee d_4$ $\quad b=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$</td>
<td>$a_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad a_2=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$</td>
</tr>
</table>

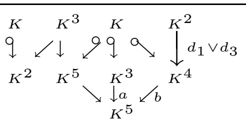
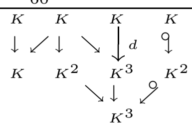
<table>
<tr><td>$M_{53}^{(3)}$</td><td>$M_{53}^{(x+3)}$, $x=1,2$</td><td>$M_{53}^{(x+5)}$, $x=1,2$</td></tr>
<tr>
<td>$$\begin{tikzcd} & K \arrow[dl] \arrow[d, hook] & & \\ K & K^2 \arrow[d, hook] & K \arrow[d, hook] & \\ & K^3 \arrow[dr] & K^2 \arrow[d, "a"] & K^2 \arrow[dl, "b"] \\ & & K^4 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} & K \arrow[dl] \arrow[d, hook] & & \\ K & K^2 \arrow[d, hook] & K^x \arrow[d, "a_x"] & \\ & K^3 \arrow[dr] & K^3 \arrow[d, "b"] & K^2 \arrow[dl, "c_x"] \\ & & K^4 & \end{tikzcd}$$</td>
<td>$$\begin{tikzcd} & K^2 \arrow[dl] \arrow[d, hook] & & \\ K & K^3 \arrow[d] & K^x \arrow[d, "a_x"] & \\ & K^4 \arrow[dr] & K^3 \arrow[d, "b"] & K^2 \arrow[dl, "c_x"] \\ & & K^4 & \end{tikzcd}$$</td>
</tr>
<tr>
<td>$\begin{array}{c}1\\111\\112\\4\end{array}$</td>
<td>$\begin{array}{c}1\\11\quad x\\1\ \ 3{-}x\ \ 2\\4\end{array}$</td>
<td>$\begin{array}{c}2\\11\quad x\\1\ \ 3{-}x\ \ 2\\4\end{array}$</td>
</tr>
<tr>
<td>$a=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}$ $\quad c=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$</td>
<td>$a_1=d_3$ $\quad a_2=d_1\vee d_3$ $\quad b=d_1\vee d_2\vee d_4$
$c_1=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $\quad c_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$</td>
<td>$a_1=d_3$ $\quad a_2=d_1\vee d_3$ $\quad b=d_1\vee d_2\vee d_4$
$c_1=\begin{bmatrix}1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $\quad c_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$</td>
</tr>
</table>

$M_{53}^{(8)}$	$M_{53}^{(x+8)}$, $x{=}1,2$	$M_{53}^{(x+10)}$, $x{=}1,2$
		
2 111 112 4	2 11 x 1 $3{-}x$ 2 5	2 112 12 $x{+}1$ 5
$a=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}$ $c=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a_1=d_3$ $a_2=d_1\vee d_3$ $b=d_1\vee d_2\vee d_5$ $c_1=\begin{bmatrix}0&1\\1&0\\1&1\\0&1\\1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1\\1&1\\0&1\\1&0\\1&0\end{bmatrix}$	$a=d_1\vee d_4$ $b=d_1\vee d_2\vee d_3\vee d_5$ $c_1=\begin{bmatrix}0&1\\1&1\\1&0\\0&1\\1&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\1&0&0\end{bmatrix}$

$M_{53}^{(x+12)}$, $x{=}1,2$	$M_{53}^{(15)}$	$M_{53}^{(x+15)}$, $x{=}1,2$
		
2 11 x 1 $3{-}x$ 3 5	2 112 123 6	2 2 1 x 2 $3{-}x$ 23 6
$a_1=d_3$ $a_2=d_1\vee d_3$ $b=d_1\vee d_2\vee d_5$ $c_1=\begin{bmatrix}1&0&1\\0&1&1\\0&1&0\\0&1&1\\1&0&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\\1&0&1\\1&0&0\end{bmatrix}$	$a=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $b=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\\0&1&0\\0&1&0\\1&0&0\end{bmatrix}$	$a=d_1\vee d_4$ $b=d_1\vee d_2\vee d_3\vee d_6$ $c_1=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&1&0\\0&0&1\\1&0&0\end{bmatrix}$

$M_{53}^{(x+17)}$, $x{=}1,2$	$M_{54}^{(1)}$	$M_{55}^{(1)}$
		
3 x12 123 6	111 1111 3	1 112 1111 4
$a=d_1\vee d_4$ $b=d_1\vee d_2\vee d_3\vee d_6$ $c_1=\begin{bmatrix}0&1&1\\0&0&1\\0&1&0\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $c_2=\begin{bmatrix}0&1&0\\0&1&1\\0&0&1\\1&0&1\\0&1&0\\1&0&0\end{bmatrix}$		$a=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $b=\begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ $c=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$ $e=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\1&0&1\end{bmatrix}$

$M_{56}^{(1)}$	$M_{57}^{(1)}$	$M_{58}^{(1)}$
		
1 121 2112 4	11 111 112 4	2 1112 121 5
$a{=}d_3{\vee}d_1{\vee}d_4 \quad b{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a{=}d_1{\vee}d_4 \quad b{=}\begin{bmatrix}0&1\\1&1\\0&1\\1&0\end{bmatrix}$	$a{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad b{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$

$M_{59}^{(1)}$	$M_{60}^{(1)}$
	
1312 2121 5	1111 1111 3
$a{=}\begin{bmatrix}1&0&0\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad b{=}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}$	

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