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TWO EXAMPLES OF SUBSPACES IN L^{2l} SPANNED BY CHARACTERS OF FINITE ORDER

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Abstract. By a Fourier multiplier technique on Cantor-like Abelian groups with characters of finite order, the norms from L^2 into L^{2l} of certain embeddings of character sums are computed. It turns out that the orders of the characters are immaterial as soon as they are at least four.

Introduction. For a long time Khinchin's inequality has proven its value in analysis. There is a proof for one of its parts that can be performed with the aid of thin sets from commutative harmonic analysis. It has the benefit of providing a more general statement of how ±1-valued characters can be used to embed L^2 into L^p for $p > 2$. This method served as the motivation for the present work. The result alluded to is as follows:

Let $\{r_j\}_{j=1}^{\infty}$ be a realization of the Rademacher functions. Consider the family $\mathcal{C}_k$ consisting of all $\sigma = \prod_{n=1}^{k} r_{j_n}$, where $1 \le j_1 < \ldots < j_k$. Then for all exponents $p \ge 2$ and coefficient functions $a : \mathcal{C}_k \to \mathbb{C}$,

$$\Big\| \sum_{\sigma\in\mathcal{C}_k} a(\sigma)\sigma \Big\|_p \le (p-1)^{k/2} \Big\| \sum_{\sigma\in\mathcal{C}_k} a(\sigma)\sigma \Big\|_2.$$

In fact, the base $(p-1)^{1/2}$ is optimal in the sense that no exponential function ϱ^k with $\varrho < \sqrt{p-1}$ can replace the numerical factor in the displayed inequality without invalidating the conclusion for some k and suitable a.

A proof can be found in Bonami [B]. Instead of independent characters on a binary Cantor group, that is, the Rademacher functions, one could imagine other systems of characters. When the compact Abelian group $\mathbb{Z}_2^{\infty}$

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is replaced by the infinite torsion group $\mathbb{T}^\infty$, the characters $\exp(ix_j)$ will substitute the functions r_j. (On the product group we number the independent variables as x_1, x_2, and so on.) This program has been initiated in Bonami [B] and continued in Hare [H]. We arrive at a setting where conjugation produces new characters, so two results are the proper generalization of the previous proposition. Consider therefore two classes of characters $\mathcal{A}_k$ and $\mathcal{B}_k$. A character $\sigma = \prod_{j=1}^\infty \exp(in_jx_j)$ belongs to $\mathcal{B}_k$ when all $n_j \in \{0, \pm 1\}$ and $\sum |n_j| = k$, whereas membership in $\mathcal{A}_k$ means $n_j \in \{0, 1\}$ with the same sum.

To wit, the outcome in this setting is:

For the respective coefficient functions and all $p \geq 2$, two embeddings of L^2 into L^p hold:

$$\Big\| \sum_{\sigma \in \mathcal{A}_k} a(\sigma)\sigma \Big\|_p \leq (p/2)^{k/2} \Big\| \sum_{\sigma \in \mathcal{A}_k} a(\sigma)\sigma \Big\|_2,$$
$$\Big\| \sum_{\sigma \in \mathcal{B}_k} a(\sigma)\sigma \Big\|_p \leq (p-1)^{k/2} \Big\| \sum_{\sigma \in \mathcal{B}_k} a(\sigma)\sigma \Big\|_2.$$

Each numerical factor displays the optimal exponential rate of growth.

An alternative proof of this result of Bonami for the case of even integers $p = 2l$ will be an easy consequence of the method of the present paper.

Our aim now is to understand what happens if the characters used are of arbitrary finite order; in this note we specialize to the case $p = 2l$ at all times. It turns out that for general order, of at least four, the outcome is formally identical with the last stated result. It is obvious from the sample results above that order two behaves differently, and the reason why order three is exceptional will also become apparent in due time. The first part of this paper contains the translation of the program into the language of $\Lambda(p)$-sets. The final section, the bulk of the paper, establishes the exact multiplier inequalities which yield the norm bounds of the two embeddings.

1. Embeddings generated by $\Lambda(p)$-sets. We fix once and for all a sequence $\underline{n} = \{n_j\}_{j=1}^\infty \subseteq \{n \in \mathbb{N} : n \geq 4\} \cup \{\infty\}$. Our main object is the compact Abelian group $G_{\underline{n}}$ with the product topology, described by

$$G_{\underline{n}} = \prod_{j=1}^\infty C_{n_j}, \quad \text{where} \quad C_{n_j} \simeq \begin{cases} \mathbb{T} & \text{if } n_j = \infty, \\ \mathbb{Z}_{n_j} & \text{if } n_j < \infty. \end{cases}$$

In each of the factors C_{n_j} we also fix one character χ_j with maximal order, that is, χ_j has order n_j. We extend each χ_j to $G_{\underline{n}}$ by setting $\chi_j(\theta) = \chi_j(\theta_j)$, where $\theta = (\theta_j)_{j=1}^\infty \in G_{\underline{n}}$. No notational change is upheld. This means that $\{\chi_j\}_{j=1}^\infty$ is a sequence of independent characters on $G_{\underline{n}}$ with prescribed orders $n_j \geq 4$.

For the embedding of character sums we need the following result on multipliers. In the proof we make a reference to the last section of this paper, which is acceptable since those later considerations are independent of the arguments in this section.

THEOREM 1.1. *The generalized Riesz product* $m_r = \prod_{j=1}^{\infty}(1+r\chi_j+\overline{r\chi_j})$ *is a multiplier from* $L^2(G_{\underline{n}})$ *to* $L^{2l}(G_{\underline{n}})$ *exactly when* $r \in \mathbb{C}$ *satisfies* $|r| \le 1/\sqrt{2l-1}$. *The one-sided product* $\phi_r = \prod_{j=1}^{\infty}(1+r\chi_j)$ *is a multiplier in the same sense precisely when* $|r| \le 1/\sqrt{l}$. *In both cases, the multiplier norm is* 1 *whenever the stated condition on* r *holds.*

Proof. The necessity of $|r| \le 1/\sqrt{2l-1}$ and $|r| \le 1/\sqrt{l}$, respectively, for boundedness is a consequence of Hare [H], Proposition 1.4. Here it is decisive that the same number r appears in each factor. According to our Corollary 2.9 each factor $1+r\chi_j+\overline{r\chi_j}$ has (L^2, L^{2l})-multiplier norm 1 when $|r| \le 1/\sqrt{2l-1}$, and likewise for $1+r\chi_j$ when $|r| \le 1/\sqrt{l}$. By Bonami–Segal's lemma [B], Chapitre III, Lemme 1, it follows that m_r and ϕ_r have norm 1 for their indicated parameter sets. ∎

Recall now the two classes of characters that interest us at present:

$$\mathcal{A}_k = \Big\{\chi \in \widehat{G}_{\underline{n}} : \chi = \prod \chi_j^{\varepsilon_j},\ \varepsilon_j \in \{0,1\},\ \sum |\varepsilon_j| = k\Big\},$$
$$\mathcal{B}_k = \Big\{\chi \in \widehat{G}_{\underline{n}} : \chi = \prod \chi_j^{\varepsilon_j},\ \varepsilon_j \in \{0,\pm1\},\ \sum |\varepsilon_j| = k\Big\}.$$

The properties of synthesis for these two classes are relevant at the moment.

In classical theory, a subset $E \subset \widehat{G}$, where G is a compact Abelian group, is said to be a $\Lambda(p)$*-set* for $p > 2$ if there exists a constant C such that for every E-polynomial f we have $\|f\|_p \le C\|f\|_2$. The least possible value of C is denoted by $\Lambda(p,E)$. Here "E-polynomial" simply means that $\operatorname{supp}\widehat{f} \subseteq E$ is finite.

THEOREM 1.2. *For all positive integers k and l the sets $\mathcal{A}_k$ and $\mathcal{B}_k$ are $\Lambda(2l)$-sets with respect to the group $G_{\underline{n}}$. More precisely,*

$$\Lambda(2l,\mathcal{A}_k) \le l^{k/2} \quad \textit{and} \quad \Lambda(2l,\mathcal{B}_k) \le (2l-1)^{k/2}.$$

In addition,

$$\varlimsup_{k\to\infty} \Lambda(2l,\mathcal{A}_k)^{1/k} = \sqrt{l} \quad \textit{and} \quad \varlimsup_{k\to\infty} \Lambda(2l,\mathcal{B}_k)^{1/k} = \sqrt{2l-1}.$$

Proof. We will use the multipliers m_r and ϕ_r from the last theorem. Fix an integer k and take any $\mathcal{B}_k$-polynomial f. It is plain that real r give $m_r * f = r^k f$. It follows that for all $0 \le r \le 1/\sqrt{2l-1}$,

$$\|f\|_{2l} = r^{-k}\|m_r * f\|_{2l} \le r^{-k}\|f\|_2.$$

Since f does not depend on r we have $\Lambda(2l, \mathcal{B}_k) \leq (2l-1)^{k/2}$. A similar argument with ϕ_r provides $\Lambda(2l, \mathcal{A}_k) \leq l^{k/2}$.

This first step also tells us that the upper limits in the statement of the theorem do not exceed $\sqrt{l}$ and $\sqrt{2l-1}$, respectively. We need to prove the reverse inequalities.

Take any $\varrho > 0$ such that

$$\varlimsup_{k\to\infty} \Lambda(2l, \mathcal{B}_k)^{1/k} < 1/\varrho.$$

It follows that

$$\sum_{k=0}^{\infty} \varrho^k \Lambda(2l, \mathcal{B}_k) = C < \infty.$$

We claim that m_ϱ must be an (L^2, L^{2l})-multiplier of norm 1. Consider any polynomial $g \in L^2(G_{\underline{n}})$. We may decompose $g = \sum g_k$, where the sum is finite and each g_k is a $\mathcal{B}_k$-polynomial. Obviously, $\|g_k\|_2 \leq \|g\|_2$ and $m_\varrho * g_k = \varrho^k g_k$, whence

$$\begin{aligned} \|m_\varrho * g\|_{2l} &\leq \sum \|m_\varrho * g_k\|_{2l} = \sum \varrho^k \|g_k\|_{2l} \\ &\leq \sum \varrho^k \Lambda(2l, \mathcal{B}_k)\|g_k\|_2 \leq C\|g\|_2. \end{aligned}$$

Consequently, m_ϱ is a bounded multiplier from L^2 to L^{2l}. By Theorem 1.1 the multiplier norm must even be 1. The choice of ϱ thus allows the conclusion that

$$0 \leq \varrho < [\varlimsup \Lambda(2l, \mathcal{B}_k)^{1/k}]^{-1} \quad \text{implies} \quad \|m_\varrho\|_{L^2 \to L^{2l}} = 1.$$

On the other hand Theorem 1.1 states the exact condition on ϱ that allows the norm 1 on the right-hand side. In consequence,

$$\varlimsup \Lambda(2l, \mathcal{B}_k)^{1/k} \geq \sqrt{2l-1}.$$

An analogous line of argument with $\mathcal{A}_k$ and ϕ_r determines the upper limit for $\mathcal{A}_k$. ∎

For clarity let us now duplicate the preceding result so as to exhibit the embedding properties of character sums.

Proposition 1.3. *Let $\{\chi_j\}_{j=1}^{\infty}$ be independent characters of finite orders $n_j \geq 4$ or of infinite order on a compact Abelian group. Take $\mathcal{A}_k$ to consist of all products of at most k different χ_j (no duplication) and take the elements in $\mathcal{B}_k$ to be products of at most k factors in $\{\chi_j\} \cup \{\overline{\chi}_j\}$, still without duplication. For any coefficient functions $a : \mathcal{A}_k \to \mathbb{C}$ and $a : \mathcal{B}_k \to \mathbb{C}$*

and any integer $l \geq 1$ we have

$$\Big\| \sum_{\sigma\in\mathcal{A}_k} a(\sigma)\sigma \Big\|_{2l} \leq l^{k/2} \Big\| \sum_{\sigma\in\mathcal{A}_k} a(\sigma)\sigma \Big\|_2,$$
$$\Big\| \sum_{\sigma\in\mathcal{B}_k} a(\sigma)\sigma \Big\|_{2l} \leq (2l-1)^{k/2} \Big\| \sum_{\sigma\in\mathcal{B}_k} a(\sigma)\sigma \Big\|_2.$$

Each numerical factor displays the optimal exponential rate of growth.

The statement is intentionally made for "at most k factors", which may be founded on the inequality

$$\Big\| \sum_{\sigma\in\mathcal{A}_k} a(\sigma)\sigma \Big\|_{2l} \leq \Big\| \sum_{\sigma\in\mathcal{A}_k} l^{|\sigma|/2} a(\sigma)\sigma \Big\|_2 \leq l^{k/2} \Big\| \sum_{\sigma\in\mathcal{A}_k} a(\sigma)\sigma \Big\|_2$$

and similarly for $\mathcal{B}_k$. Here the length $|\sigma|$ is the number of different χ_j and $\overline{\chi}_j$ appearing in σ.

The generalization to other $p > 2$ replacing $2l$ requires other methods and will have to await future developments. Instead, we continue to describe the precise multiplier actions alluded to in the proof of Theorem 1.1.

2. The multiplier inequalities. Consider a compact Abelian group G and its dual $\widehat{G}$. Normalized Haar measure on G will henceforth be denoted by μ. We will deal with a general torsion character $\chi \in \widehat{G}$, $\chi \not\equiv 1$, of order n, that is, n is the least positive integer such that $\chi^n \equiv 1$. In consequence, $\int \chi^k \, d\mu = 0$ or 1 according as $n \nmid k$ or $n \mid k$.

In the technical calculations below we need a quantity $q_n(k)$. Let $A_n(k)$ denote the set $\{j \in [0,k] : j \in \mathbb{N},\ k - 2j \equiv 0 \pmod{n}\}$. We define

$$q_n(k) = \sum_{j=0}^{k} \binom{k}{j} \int \chi^{k-2j} \, d\mu = \sum \left\{ \binom{k}{j} : j \in A_n(k) \right\}.$$

Observe next that q_n can be reinterpreted as

$$\int_G (\chi + \overline{\chi})^k \, d\mu = \sum_{j=0}^{k} \binom{k}{j} \int_G \chi^{k-2j} \, d\mu = q_n(k)$$

and it is via this integral that q_n enters the proof. An elementary but important lemma captures two useful properties.

Lemma 2.1. *For all $n \geq 3$ and $k \geq 1$ we have*

(1) $0 \leq q_n(k) \leq 2^{k-1}$, *and*
(2) $q_{n+2}(k) \leq q_n(k)$.

Proof. Since $A_n(k)$ contains at most every second integer, the elementary fact that the sum of all odd (or even) numbered binomial coefficients equals 2^{k-1} leads to (1).

Membership in $A_{n+2}(k)$, on the other hand, hinges on $k-2j\equiv 0 \bmod n+2$, which may be separated into two cases:

(A1) $j=k/2\in A_{n+2}(k)$ gives $j+r\in A_n(k)$ with $r=0$.
(A2) $j\neq k/2$, $j\in A_{n+2}(k)$ give $j=\frac{1}{2}[k+r(n+2)]$ for an integer $r\neq 0$. Then $j-r=\frac{1}{2}[k+rn]\in A_n(k)$.

Clearly the map $j\mapsto r$ is an injection from $A_{n+2}(k)$ into the integer set $\{m\in\mathbb{Z} : |m|\leq k/(2n+4)\}$. The two cases thus produce an injection $A_{n+2}(k)\to A_n(k)$ via $j\mapsto j-r$, with the additional property $\binom{k}{j}\leq\binom{k}{j-r}$ since $j\leq j-r\leq k/2$ for $0\leq j\leq k/2$, and for the remaining case one has $j>j-r>k/2$. From this, claim (2) follows. ∎

For any complex number b the expression $1+b\chi+\overline{b\chi}$ is real on G, whence

$$\begin{aligned}\|1+b\chi+\overline{b\chi}\|_{2l}^{2l} &= \sum_{k=0}^{2l}\binom{2l}{k}\int_G (b\chi+\overline{b\chi}\,)^k\,d\mu\\ &= \sum_{k=0}^{2l}\binom{2l}{k}\sum_{j=0}^{k}\binom{k}{j} b^{k-j}\,\overline{b}^j\int\chi^{k-2j}\,d\mu\\ &\leq \sum_{k=0}^{2l}\binom{2l}{k}|b|^k\sum_{j=0}^{k}\binom{k}{j}\int\chi^{k-2j}\,d\mu = \sum_{k=0}^{2l}\binom{2l}{k}|b|^k q_n(k).\end{aligned}$$

The inequality holds since the integrals are either 0 or 1. Observe also that the Hilbert norm $\|1+b\chi+\overline{b\chi}\|_2$ is independent of the order n.

Lemma 2.2. *Denote by $S_{2l}^{(n)}(x)$ the polynomial $\sum_{k=0}^{2l}\binom{2l}{k}q_n(k)x^k$. When $b\in\mathbb{C}$, $l\geq 1$, and $n\geq 3$, we have $\|1+b\chi+\overline{b\chi}\|_{2l}^{2l}\leq S_{2l}^{(n)}(|b|)$. In addition, $S_{2l}^{(n+2)}(x)\leq S_{2l}^{(n)}(x)$ for all $x\geq 0$.*

Proof. The domination by $S_{2l}^{(n)}(|b|)$ has already been established. Next, Lemma 2.1 proves that the coefficients could only increase when $n+2$ is changed to n. Thus both claims hold true. ∎

The idea of the ensuing computations is the following. Suppose we could guarantee $S_{2l}^{(n)}(x/\sqrt{2l-1})\leq(1+2x^2)^l$ for $x\geq 0$. Repeated use of Lemma 2.2 would then give the same inequality for $n+2$, $n+4$, and so forth. As a consequence we would be able to derive

$$\left\|1+\frac{b\chi+\overline{b\chi}}{\sqrt{2l-1}}\right\|_{2l}\leq\|1+b\chi+\overline{b\chi}\|_2$$

for all orders n, $n+2$, $n+4$, and so on.

A similar program, but more laborious, also works for the one-sided case, at the cost of a coefficient alteration:

$$\begin{aligned}\|1+b\chi\|_{2l}^{2l} &= \int [1+|b|^2+b\chi+\overline{b\chi}\,]^l\,d\mu \\ &= \sum_{k=0}^{l}\binom{l}{k}(1+|b|^2)^{l-k}\int (b\chi+\overline{b\chi}\,)^k\,d\mu \\ &\le \sum_{k=0}^{l}\binom{l}{k}q_n(k)\sum_{m=0}^{l-k}\binom{l-k}{m}|b|^{k+2m} \\ &= \sum_{p=0}^{l}|b|^{2p}\sum_{k=0}^{p}\binom{l}{2k}\binom{l-2k}{p-k}q_n(2k) \\ &\quad+\sum_{p=2}^{l}|b|^{2p-1}\sum_{k=2}^{p}\binom{l}{2k-1}\binom{l-2k+1}{p-k}q_n(2k-1).\end{aligned}$$

Here $q_n(1)=0$ is responsible for the lower limit “$p=2$” in the odd part. Two elementary identities are useful (the second one is used for $k\ge 1$ only):

$$\binom{l}{2k}\binom{l-2k}{p-k}=\binom{l}{p}\binom{l-p}{k}\binom{p}{k}\binom{2k}{k}^{-1},$$

$$\binom{l}{2k-1}\binom{l-2k+1}{p-k}=\binom{l}{p}\binom{l-p}{k-1}\binom{p}{k}\binom{2k-1}{k}^{-1}.$$

Motivated by the above calculation we introduce the polynomials

$$U_{2l}^{(n)}(x)=\sum_{p=0}^{l}\binom{l}{p}x^{2p}\sum_{k=0}^{p}\binom{l-p}{k}\binom{p}{k}\binom{2k}{k}^{-1}q_n(2k),$$

$$V_{2l}^{(n)}(x)=\sum_{p=2}^{l}\binom{l}{p}x^{2p-1}\sum_{k=2}^{p}\binom{l-p}{k-1}\binom{p}{k}\binom{2k-1}{k}^{-1}q_n(2k-1).$$

LEMMA 2.3. *Any character χ of order $n\ge 2$ and integer $l\ge 1$ provide*

$$\|1+b\chi\|_{2l}^{2l}\le U_{2l}^{(n)}(|b|)+V_{2l}^{(n)}(|b|).$$

For even orders n this is the same as $\|1+b\chi\|_{2l}^{2l}\le U_{2l}^{(n)}(|b|)$. Also, for all $x\ge 0$,

$$U_{2l}^{(n+2)}(x)\le U_{2l}^{(n)}(x)\quad\textit{and}\quad V_{2l}^{(n+2)}(x)\le V_{2l}^{(n)}(x).$$

Proof. The first claim is contained in the motivating calculation above. For even n we have $q_n(2k-1)=0$ and hence $V_{2l}^{(n)}\equiv 0$. Since generally $q_{n+2}(k)\le q_n(k)$, the last claims also follow. ∎

REMARKS 2.4. (i) The program sketched above is easy for even order. As an alternative, the first part of the following lemma could have been derived as a corollary of a construction in [BJJ]. The present calculation is, however, a good illustration of the technique chosen here, without the further difficulties arising when odd orders are treated.

(ii) The above line of reasoning is a kind of transference from small cyclic groups to larger ones. It is useful to notice for all k and n the inequality $0 \le \int (e^{i\theta} + e^{-i\theta})^k \, d\theta/(2\pi) = \varepsilon_k \binom{k}{k/2} \le q_n(k)$. Here $\varepsilon_k = 0$ or 1 according as k is odd or even. A result for any of the finite cyclic groups thus produces the same norm inequality for $\mathbb{T}$ as those for finite groups, to be derived later in this section. ∎

LEMMA 2.5. *$S_{2l}^{(4)}(x/\sqrt{2l-1}) \le (1+2x^2)^l$ and $U_{2l}^{(4)}(x/\sqrt{l}) \le (1+x^2)^l$.*

Proof. We have $q_4(k) = \sum \left\{ \binom{k}{j} : k - 2j \equiv 0 \pmod 4 \right\}$, whence $q_4(0) = 1$ and $q_4(2k-1) = 0$, $q_4(2k) = 2^{2k-1}$ for all $k \ge 1$. It is plain that

$$\begin{aligned} S_{2l}^{(4)}(x) &= 1 + \sum_{k=1}^{l} \binom{2l}{2k} 2^{2k-1} x^{2k} \\ &\le 1 + \sum_{k=1}^{l} \binom{l}{k} (2x^2)^k (2l-1)^k \, \frac{2^{2k-1} k!}{(2k)!} \\ &\le \sum_{k=0}^{l} \binom{l}{k} (2[2l-1]x^2)^k = (1 + 2(2l-1)x^2)^l, \end{aligned}$$

since $2^{2k-1} k!/(2k)! \le 1$ when $k \ge 1$. This proves the first claim.

For $U_{2l}^{(4)}$ the calculation is similar:

$$\begin{aligned} U_{2l}^{(4)}(x) &= 1 + \sum_{p=1}^{l} \binom{l}{p} x^{2p} \left\{ 1 + \sum_{k=1}^{p} \binom{l-p}{k} \binom{p}{k} \binom{2k}{k} 2^{2k-1} \right\} \\ &\le 1 + \sum_{p=1}^{l} \binom{l}{p} x^{2p} \left\{ 1 + \sum_{k=1}^{p} \binom{p}{k} (l-p)^k \, \frac{2^{2k-1} k!}{(2k)!} \right\} \\ &\le 1 + \sum_{p=1}^{l} \binom{l}{p} x^{2p} \left\{ 1 + \sum_{k=1}^{p} \binom{p}{k} (l-p)^k \right\} \\ &= 1 + \sum_{p=1}^{l} \binom{l}{p} x^{2p} (l-p+1)^p \le (1 + lx^2)^l. \ \blacksquare \end{aligned}$$

Observe that this establishes Theorem 2.7 below for even orders n. The treatment of order five demands a method of transforming the odd powers in $V_{2l}^{(5)}$ into even powers and this efficiently enough not to disturb $U_{2l}^{(5)}$.

LEMMA 2.6. *For all $x \ge 0$ the inequalities $S_{2l}^{(5)}(x/\sqrt{2l-1}) \le (1+2x^2)^l$ and $U_{2l}^{(5)}(x/\sqrt{l}) + V_{2l}^{(5)}(x/\sqrt{l}) \le (1+x^2)^l$ hold true.*

Proof. Consider the decomposition

$$S_{2l}^{(5)}(x) = \sum_{k=0}^{l} \binom{2l}{2k} q_5(2k) x^{2k} + \sum_{k=2}^{l-1} \binom{2l}{2k+1} q_5(2k+1) x^{2k+1}.$$

We need two elementary inequalities for $l \geq k \geq 1$, easily obtained by expanding into factors and applying $q_5(j) \leq 2^{j-1}$. For even orders we use

$$\binom{2l}{2k} q_5(2k) \leq 2^k \binom{l}{k} (2l-1)^k \frac{2^{k-1}}{(2k-1)!!}.$$

The corresponding result for odd orders is

$$\binom{2l}{2k+1} q_5(2k+1) \leq 2^k \binom{l}{k} (2l-1)^{k+1} \frac{2^k}{(2k+1)!!}.$$

In case $0 \leq x\sqrt{2l-1} \leq 1$ this yields

$$\begin{aligned} S_{2l}^{(5)}\left(\frac{x}{\sqrt{2l-1}}\right) &\leq 1 + \sum_{k=1}^{l} 2^k \binom{l}{k} x^{2k} \frac{2^{k-1}}{(2k-1)!!} \\ &\quad + \sum_{k=2}^{l-1} 2^k \binom{l}{k} \sqrt{2l-1} x^{2k+1} \frac{2^k}{(2k+1)!!} \\ &\leq 1 + 2lx^2 + \sum_{k=2}^{l-1} 2^k \binom{l}{k} x^{2k} \left\{ \frac{2^{k-1}}{(2k-1)!!} + \frac{2^k}{(2k+1)!!} \right\} \\ &\quad + 2^l x^{2l} \frac{2^{l-1}}{(2l-1)!!} \\ &\leq 1 + 2lx^2 + \sum_{k=2}^{l-1} \binom{l}{k} 2^k x^{2k} \cdot \frac{14}{15} + 2^l x^{2l} \leq (1+2x^2)^l. \end{aligned}$$

For the remaining case we first observe that $k \geq 1$ provides

$$\binom{2l}{2k-1} \leq \binom{l}{k} \frac{2k(2l-1)^{k-1}}{(2k-1)!!},$$

and hence

$$\binom{2l}{2k-1} q_5(2k-1) \leq 2^k \binom{l}{k} (2l-1)^{k-1} \frac{2^{k-1} k}{(2k-1)!!}.$$

Applying this to the decomposition above we find

$$\begin{aligned} S_{2l}^{(5)}(x) &\leq 1 + \sum_{k=1}^{l} \binom{l}{k} 2^k (2l-1)^k x^{2k} \frac{2^{k-1}}{(2k-1)!!} \\ &\quad + \sum_{k=3}^{l} \binom{l}{k} 2^k (2l-1)^{k-1} x^{2k-1} \frac{2^{k-1} k}{(2k-1)!!}, \end{aligned}$$

and hence the inequality $1 \le x\sqrt{2l-1}$ produces

$$\begin{aligned} S_{2l}^{(5)}\left(\frac{x}{\sqrt{2l-1}}\right) &\le 1 + \sum_{k=1}^{l} \binom{l}{k} 2^k x^{2k} \frac{2^{k-1}}{(2k-1)!!} \\ &\quad + \sum_{k=3}^{l} \binom{l}{k} 2^k x^{2k} \frac{1}{x\sqrt{2l-1}} \frac{2^{k-1}k}{(2k-1)!!} \\ &\le 1 + 2lx^2 + \binom{l}{2} 4x^4 \cdot \frac{2}{3} \\ &\quad + \sum_{k=3}^{l} \binom{l}{k} 2^k x^{2k} \frac{2^{k-1}(k+1)}{(2k-1)!!}. \end{aligned}$$

Except for the $k = 3$ term every member is at most the desired $\binom{l}{k} 2^k x^{2k}$. Consequently,

$$\begin{aligned} S_{2l}^{(5)}\left(\frac{x}{\sqrt{2l-1}}\right) &\le 1 + 2lx^2 + 4\binom{l}{2} x^4 + \binom{2l}{5} q_5(5) \frac{x^5}{(2l-1)^{5/2}} \\ &\quad + \binom{2l}{6} q_5(6) \frac{x^6}{(2l-1)^3} + \sum_{k=4}^{l} \binom{l}{k} 2^k x^{2k}. \end{aligned}$$

By using $1 \le x\sqrt{2l-1}$ it is straightforward to verify that the exact values $q_5(5) = 2$ and $q_5(6) = 6$ combine to make the sum of the fifth and sixth degree terms at most $\binom{l}{3} 8x^6$, as desired. Hence we have proven for all $x \ge 0$ that

$$S_{2l}^{(5)}\left(\frac{x}{\sqrt{2l-1}}\right) \le \sum_{k=0}^{l} \binom{l}{k} 2^k x^{2k} \le (1 + 2x^2)^l.$$

In the single-sided case, matters become more laborious. We must consider the two polynomials $U_{2l}^{(5)}$ and $V_{2l}^{(5)}$. The inner sum in their definition is related to the quantity

$$\sum_{k=0}^{p} \binom{l-p}{k} \binom{p}{k} = \sum_{k=0}^{p} \binom{l-p}{k} \binom{p}{p-k} = \binom{l}{p}.$$

The additional factor can be controlled by using

$$\binom{2k}{k}^{-1} q_5(2k) \begin{cases} = 1, & 0 \le k \le 4, \\ \le \dfrac{2^{k-1}k!}{(2k-1)!!} < \dfrac{1}{2}k, & k \ge 5. \end{cases}$$

For $0 \le p \le 4$ we get the obvious estimate

$$\sum_{k=0}^{p} \binom{l-p}{k} \binom{p}{k} \binom{2k}{k}^{-1} q_5(2k) = \binom{l}{p} \le \frac{l^p}{p!},$$

while for $p \geq 5$ the simple estimate $\binom{2k}{k}^{-1} q_5(2k) < p/2$, for all $0 \leq k \leq p$, leads to

$$\sum_{k=0}^{p} \binom{l-p}{k} \binom{p}{k} \binom{2k}{k}^{-1} q_5(2k) \leq \frac{p}{2} \binom{l}{p} \leq \frac{l^p}{2(p-1)!}.$$

Our basic estimate for the even part is thus

$$U_{2l}^{(5)}(x) \leq \sum_{p=0}^{l} \binom{l}{p} l^p x^{2p} \gamma_p, \quad \text{where} \quad \gamma_p = \begin{cases} p!^{-1}, & 0 \leq p \leq 4, \\ \frac{1}{2}(p-1)!^{-1}, & p \geq 5. \end{cases}$$

When $3 \leq k \leq 7$ one finds

$$q_5(2k-1) = 2\binom{2k-1}{k-3}, \quad \text{so} \quad \binom{2k-1}{k}^{-1} q_5(2k-1) < 1.$$

For $k \geq 8$ one observes on the other hand that

$$\binom{2k-1}{k}^{-1} q_5(2k-1) \leq \frac{2^{2k-2} k!(k-1)!}{(2k-1)!} < \frac{k}{2}.$$

In consequence, $p \geq 8$ yields

$$\begin{aligned} \sum_{k=3}^{p} \binom{l-p}{k-1} \binom{p}{k} \binom{2k-1}{k}^{-1} q_5(2k-1) &< \frac{p}{2} \sum_{k=1}^{p} \binom{l-p}{k-1} \binom{p}{k} \\ &= \frac{p}{2} \binom{l}{p-1} < \frac{p\, l^{p-1}}{2(p-1)!}. \end{aligned}$$

For $3 \leq p \leq 7$ the factor $p/2$ may be removed from the right-hand side.

We have thus established for $x \geq 0$ the estimate

$$V_{2l}^{(5)}(x) \leq \sum_{p=1}^{l} \binom{l}{p} x^{2p-1} l^{p-1} \beta_p, \quad \text{where}$$

$$\beta_p = \begin{cases} 0, & p \leq 2, \\ (p-1)!^{-1}, & 3 \leq p \leq 7, \\ \frac{1}{2} p(p-1)!^{-1}, & p \geq 8. \end{cases}$$

Consider next the two possibilities (i) $lx \geq 1$, and (ii) $0 \leq lx \leq 1$. In the first case

$$V_{2l}^{(5)}(x) \leq \sum_{p=0}^{l} \binom{l}{p} l^p x^{2p} \beta_p,$$

whence

$$U_{2l}^{(5)}(x) + V_{2l}^{(5)}(x) \leq \sum_{p=0}^{l} \binom{l}{p} l^p x^{2p} (\gamma_p + \beta_p).$$

For the second possibility we observe

$$V_{2l}^{(5)}(x) \le \sum_{p=3}^{l} \binom{l}{p} x^{2p-1} l^{p-1} \beta_p \le \sum_{p=3}^{l} \binom{l}{p-1} l^{p-1} x^{2p-2} p^{-1} \beta_p \, lx$$
$$\le \sum_{p=2}^{l-1} \binom{l}{p} l^p x^{2p} \frac{\beta_{p+1}}{p+1}.$$

This implies that in the case $0 \le lx \le 1$,

$$U_{2l}^{(5)}(x) + V_{2l}^{(5)}(x) \le \sum_{p=0}^{l} \binom{l}{p} l^p x^{2p} \left(\gamma_p + \frac{\beta_{p+1}}{p+1} \right).$$

Since neither $\gamma_p + \beta_p$ nor $\gamma_p + \beta_{p+1}(p+1)^{-1}$ exceeds 1, we have established, for all $x \ge 0$,

$$U_{2l}^{(5)}(x) + V_{2l}^{(5)}(x) \le \sum_{p=0}^{l} \binom{l}{p} l^p x^{2p} = (1 + lx^2)^l. \ \blacksquare$$

Combination of the four lemmata now demonstrates the central result.

THEOREM 2.7. *Let a character $\chi \in \widehat{G}$ have order $n \ge 4$. For all $b \in \mathbb{C}$ and integers $l \ge 1$ the following two norm inequalities hold*:

$$\|1 + b\chi/\sqrt{l}\,\|_{2l} \le \|1 + b\chi\|_2,$$
$$\|1 + (b\chi + \overline{b\chi})/\sqrt{2l-1}\,\|_{2l} \le \|1 + b\chi + \overline{b\chi}\|_2.$$

Inspection of the calculations above shows that equality holds only for the choice $b = 0$ or the trivial $l = 1$.

REMARK. Consider the exceptional order $n = 3$. The quantity $q_3(3) = 2$ is decisive for the norm inequalities. For small real b and $p > 2$ it is not difficult to establish $\|1+(b\chi+\overline{b\chi})/\sqrt{p-1}\,\|_p^p = 1+pb^2+p(p-2)b^3/3+\mathcal{O}(b^4)$, which can be made to exceed $\|1+b\chi+\overline{b\chi}\,\|_2^p = 1+pb^2+\mathcal{O}(b^4)$ by choosing b suitably. Hence the second part of Theorem 2.7 fails for $n = 3$. It is difficult to calculate the correct number to replace $(2l-1)^{-1/2}$. With some elementary considerations $l = 2$ can be shown to demand exactly $(1 + 3/\sqrt{2}\,)^{-1/2}$. For $l \ge 3$ the exact value is not known.

Concerning the one-sided case with $n = 3$, a combinatorial argument establishes the particular case $\|1+b\chi/\sqrt{2}\,\|_4 \le \|1+b\chi\|_2$. However, for $n = 3$ this is the only inequality of a kind similar to the result established for higher orders. This singularity follows from the expressions $\|1 + b\chi/\sqrt{p/2}\,\|_p^p = 1 + pb^2/2 + p(p-2)(p-4)b^3/24 + \mathcal{O}(b^4)$ and $\|1 + b\chi\|_2^p = 1 + pb^2/2 + \mathcal{O}(b^4)$, valid for small real b. $\blacksquare$

We need a slightly more general statement than the second part of the preceding theorem.

THEOREM 2.8. *If the order of* χ *is at least* 4, *then for all* $a, b \in \mathbb{C}$ *and* $l \in \mathbb{Z}_+$,

$$\|1 + (a\chi + b\overline{\chi})/\sqrt{2l-1}\,\|_{2l} \le \|1 + a\chi + b\overline{\chi}\|_2.$$

Proof. The function $k_l = 1 + (\chi + \overline{\chi})/\sqrt{2l-1}$ is positive when $l \ge 3$ and of L^1-norm 1. Consequently, $\|k_l * f\|_p \le \|k_l * |f|\,\|_p \le \|\,|f|\,\|_p$ for all $p \ge 1$.

Consider $f = 1 + a\chi + b\overline{\chi}$, not constant. Then $|f|$ is positive, whence for suitable $c \in \mathbb{C}$ and $t > 0$ we find $k_l * |f| = t\big[1 + (c\chi + \overline{c\chi})/\sqrt{2l-1}\,\big]$. Theorem 2.7 now justifies the calculation

$$\begin{aligned}\left\|1 + \frac{a\chi + b\overline{\chi}}{\sqrt{2l-1}}\right\|_{2l} &\le \|k_l * |f|\,\|_{2l} \le t\,\|1 + c\chi + \overline{c\chi}\,\|_2 = \|k_l * |f|\,\|_2 \\ &\le \|\,|f|\,\|_2 = \|f\|_2 = \|1 + a\chi + b\overline{\chi}\,\|_2.\end{aligned}$$

The case $l = 2$ does not involve positivity and a calculation is necessary. We use at one instance below the trivial inequality $|a + \overline{b}|^2 \le 2|a|^2 + 2|b|^2$. The case of order $n \ge 5$ is dealt with in the following manner:

$$\begin{aligned}\|1 + (a\chi + b\overline{\chi})/\sqrt{3}\,\|_4^4 &= \big(1 + \tfrac{1}{3}|a|^2 + \tfrac{1}{3}|b|^2\big)^2 + \tfrac{2}{3}|a + \overline{b}|^2 + \tfrac{2}{9}|a|^2|b|^2 \\ &\le 1 + 2|a|^2 + 2|b|^2 + \tfrac{4}{9}|a|^2|b|^2 + \tfrac{1}{9}|a|^4 + \tfrac{1}{9}|b|^4 \\ &\le (1 + |a|^2 + |b|^2)^2 = \|1 + a\chi + b\overline{\chi}\,\|_2^4.\end{aligned}$$

With order $n = 4$ we observe $\chi^2 = \overline{\chi}^2$ and therefore

$$\begin{aligned}\|1 + (a\chi + b\overline{\chi})/\sqrt{3}\,\|_4^4 &= \big(1 + \tfrac{1}{3}|a|^2 + \tfrac{1}{3}|b|^2\big)^2 + \tfrac{2}{3}|a + \overline{b}|^2 + \tfrac{1}{9}(a\overline{b} + \overline{a}b)^2 \\ &\le 1 + 2|a|^2 + 2|b|^2 + \tfrac{2}{3}|a|^2|b|^2 + \tfrac{1}{9}|a|^4 + \tfrac{1}{9}|b|^4 \\ &\le \|1 + a\chi + b\overline{\chi}\,\|_2^4. \ \blacksquare\end{aligned}$$

COROLLARY 2.9. (1) $\|1 + rb\chi\|_{2l} \le \|1 + b\chi\|_2$ for any $r \in \mathbb{C}$ with $|r| \le 1/\sqrt{l}$.

(2) $\|1 + sa\chi + tb\overline{\chi}\|_{2l} \le \|1 + a\chi + b\overline{\chi}\|_2$ when $|s|, |t| \le 1/\sqrt{2l-1}$.

Proof. By Theorem 2.7 the first condition on r yields

$$\|1 + rb\chi\|_{2l} \le \|1 + b\chi r\sqrt{l}\,\|_2 \le \|1 + b\chi\|_2.$$

The second condition, when applied after Theorem 2.8, secures

$$\|1 + sa\chi + tb\overline{\chi}\,\|_{2l} \le \|1 + (sa\chi + tb\overline{\chi})\sqrt{2l-1}\,\|_2 \le \|1 + a\chi + b\overline{\chi}\,\|_2. \ \blacksquare$$

As a concluding remark it should be noted that some of the results remain true for general $p > 2$ replacing $2l$. However, technical obstructions arise, which will be dealt with elsewhere.

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