COLLOQUIUM MATHEMATICUM

VOL. 87 2001 NO. 1

A COMBINATORIAL CONSTRUCTION OF SETS WITH GOOD QUOTIENTS BY AN ACTION OF A REDUCTIVE GROUP

BY

JOANNA ŚWIĘCICKA (Warszawa)

Abstract. The aim of this paper is to construct open sets with good quotients by an action of a reductive group starting with a given family of sets with good quotients. In particular, in the case of a smooth projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$, we show that all open sets with good quotients that embed in a toric variety can be obtained from the family of open sets with projective good quotients. Our method applies in particular to the case of Grassmannians.

Introduction. Let X be an algebraic variety with an action of a reductive group G. One of the main problems of geometric invariant theory is to describe all open G-invariant subsets $U \subset X$ such that there exists a good quotient $U \to U/\!\!/G$. In the case of $X = \mathbb{P}^n$ this problem was solved in [5]. We give in 1.10 and 2.5 a new formulation of the results obtained in [5] for $G = T$ a torus and show that open T-invariant sets with good quotients are unions of collections of intersections of subsets with projective quotients. Moreover, we emphasize the fact that, for any T-maximal subset of $\mathbb{P}^n$, the quotient space embeds in a toric variety. In the present paper we investigate two possible ways of generalizing the results of [5].

First we assume that we are given a set of G-invariant open subvarieties of X with good quotients by the action of G, and we describe a procedure for constructing a larger class of G-invariant subsets of X that admit good quotients (Theorem 2.6). The new sets obtained by our method are unions of "good collections" of intersections of old ones. In this way we generalize the results of [5] concerning existence of good quotients to the case of an arbitrary reductive group G and any variety X.

Then we consider the action of a reductive group G on a projective smooth variety with $\mathrm{Pic}(X) = \mathbb{Z}$ and we study the problem of describing all open G-invariant subsets of X admitting quotients that embed in a toric

2000 *Mathematics Subject Classification*: Primary 14L24, 14L30.

Key words and phrases: group actions, orbit spaces, quotients.

This work was completed with the support of KBN Grant 2 PO3A 03808.

variety. We prove that any open G-invariant subset of X with this property is a saturated subset of a union of a "good collection" of intersections of open sets with projective quotients. In this way we generalize results of [5] to the case of actions of reductive groups on smooth projective varieties with $\mathrm{Pic}(X) = \mathbb{Z}$.

The outline of the paper is as follows. In Section 1 we recall useful terminology, definitions and theorems concerning the theory of good quotients. In particular, we give a new formulation of the results of [5] that will be needed in what follows.

In Section 2 we consider an algebraic G-variety X with a given set of subvarieties V_i, $i \in \mathcal{J}$, with good quotients by G. We consider open sets that are unions of finite intersections of sets V_i. We define a "good collection" of such intersections and in Theorem 2.6 we prove that there exists a good quotient of the union of any good collection.

In Section 3 we forget about the action of G and concentrate on algebraic varieties with the following property: for any n points $x_1, \ldots, x_n \in X$ there exists an affine open set $U \subset X$ such that $x_i \in U$ for $i = 1, \ldots, n$. We call them A_n-varieties. Any quasiprojective variety is an A_n-variety for every $n \in \mathbb{N}$ (see [7]). We obtain some results that are partially connected with results obtained in [11]. In Theorem 3.5 we prove that in any algebraic variety there are only finitely many maximal (with respect to inclusion) open sets that are A_n.

In Section 4 we consider the following problem: describe all open, G-invariant subsets of X that are maximal with respect to saturated inclusion in the set of all subvarieties for which there exists a good A_n-quotient, and we prove some preliminary facts.

In Section 5 we give the construction which provides all such open subvarieties in the case of a smooth projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$.

Finally, in Section 6 we still assume that X is smooth, projective with $\mathrm{Pic}(X) = \mathbb{Z}$ and we formulate corollaries of the results of Section 5. In particular we notice in Corollary 6.2 that for any open T-invariant $U \subset X$, maximal with respect to saturated inclusion in the set of all open subsets with good A_2-quotients, there exists an equivariant embedding $X \hookrightarrow \mathbb{P}^n$ and a T-maximal $V \subset \mathbb{P}^n$ such that $U = X \cap V$.

1. Notation and terminology. Let X be an algebraic variety with an action of a reductive group G, both defined over the field of complex numbers. We recall some useful facts and definitions from [5] and we fix the notation and terminology that we shall use in the paper:

DEFINITION 1.1. Let Y be an algebraic space (not necessarily an algebraic variety) with a trivial action of G. A G-morphism $q : X \to Y$ is said to be a *good quotient* if the following conditions are satisfied:

(i) q is affine,
(ii) $\mathcal{O}_Y = q_*(\mathcal{O}_X)^G$.

If $q : X \to Y$ is a good quotient then the space Y is called a *quotient space* of X by G and is denoted by $X/\!\!/G$.

EXAMPLE 1.2. Let X be an algebraic variety with an action of a reductive group G and let $\mathcal{L}$ be a G-linearized line bundle on X. In [8], Mumford defined the set $X^{\mathrm{ss}}(\mathcal{L})$ of semistable points. We recall that $x \in X^{\mathrm{ss}}$ if and only if there exists $n \in \mathbb{N}$ and a section $s \in \Gamma(X, \mathcal{L}^n)$ such that $s(x) \neq 0$ and $\{y \in X : s(y) \neq 0\}$ is affine. Then Mumford proved (Theorem 1.13 of [8]) that there exists a good quotient

$$X^{\mathrm{ss}}(\mathcal{L}) \to X^{\mathrm{ss}}(\mathcal{L})/\!\!/G$$

and the quotient space is quasiprojective. Moreover, for an ample line bundle $\mathcal{L}$, the quotient space is projective.

DEFINITION 1.3. Let U be an open G-invariant subset of X. Then U is G-*saturated* in X if, for any $x \in U$, the closures of the G-orbit $G \cdot x$ in U and in X coincide.

REMARK 1.4. Assume that U is G-saturated in X and there exists a good quotient $q : X \to X/\!\!/G$. Then there exists a good quotient $U \to U/\!\!/G$ and the induced morphism $U/\!\!/G \to X/\!\!/G$ is an open embedding.

DEFINITION 1.5. An open G-invariant subset U in X is called G-*maximal* if there exists a good quotient $U \to U/\!\!/G$ and if U is maximal in X with respect to saturated inclusion in the family of all open G-invariant subsets of X that admit good quotients with respect to the action of G.

According to Lemma 7.14 of [5], any open G-invariant subset of $\mathbb{P}^n$ with a quasiprojective good quotient is contained as a G-saturated subset in $(\mathbb{P}^n)^{\mathrm{ss}}(\mathcal{L})$ for some G-linearized ample line bundle $\mathcal{L}$. Open subsets of $\mathbb{P}^n$ with projective good quotients by an action of an algebraic torus have a nice combinatorial description (see Example 1.9).

Let $X = \mathbb{P}^n$ with homogeneous coordinates $(x_0, \ldots, x_n)$ and $G = T$ be an algebraic torus acting on X with characters $\chi_0, \ldots, \chi_n$ in homogeneous coordinates, i.e. for any $t \in T$ and $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$ let

$$t \cdot (x_0, \ldots, x_n) = (\chi_0(t)x_0, \ldots, \chi_n(t)x_n).$$

Let $X(T)$ be the $\mathbb{Z}$-module of characters of T and consider the real vector space $X_{\mathbb{R}}(T) = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

DEFINITION 1.6. Let T act on $\mathbb{P}^n$ as above. For any set $\{\chi_{i_1}, \ldots, \chi_{i_k}\} \subset \{\chi_i : i = 1, \ldots, n\}$ the polytope $\operatorname{conv}\{\chi_{i_1}, \ldots, \chi_{i_k}\} \subset X_{\mathbb{R}}(T)$ is called a *distinguished polytope*. For any $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$ we define the convex polytope

$$P(x) = \mathrm{conv}\{\chi_i : x_i \neq 0\}.$$

For any set $U \subset X$ we define the *combinatorial closure* $C(U)$:

$$x \in C(U) \Leftrightarrow \exists y \in U \text{ such that } P(x) = P(y).$$

A set $U \subset \mathbb{P}^n$ is called *combinatorially defined* (or *combinatorially closed*) if $C(U) = U$.

If P is a distinguished polytope, then there exists $x \in \mathbb{P}^n$ such that $P = P(x)$.

Definition 1.7. For any collection Δ of distinguished polytopes let

$$U(\Delta) = \{x \in \mathbb{P}^n : P(x) \in \Delta\}.$$

In particular $U(P) = \{x \in \mathbb{P}^n : P(x) = P\}$.

A set $U \subset \mathbb{P}^n$ is combinatorially defined if and only if there exists a collection Δ of distinguished polytopes such that $U = U(\Delta)$. Corollary 5.7 of [5] implies the following:

Proposition 1.8. *Let $\Delta = \{P_1, \ldots, P_k\}$ be a collection of distinguished polytopes. Then $U(\Delta)$ is open if and only if for any $i = 1, \ldots, k$ and any distinguished polytope Q,*

$$P_i \subset Q \Rightarrow Q \in \Delta.$$

Example 1.9. Let $p \in \mathrm{conv}\{\chi_0, \ldots, \chi_n\} \subset X_{\mathbb{R}}(T)$. Then

$$U(p) := \{x \in \mathbb{P}^n : p \in P(x)\}$$

has a good projective quotient and every open subset with a projective good quotient is obtained in this way. This follows from Example 7.12 and Proposition 7.13 of [5]. Therefore, for any $p \in \mathrm{conv}\{\chi_0, \ldots, \chi_n\}$, there exists a T-linearization of $\mathcal{L} = \mathcal{O}(m)$ (for some $m \in \mathbb{N}$) such that $(\mathbb{P}^n)^{\mathrm{ss}}(\mathcal{L}) = U(p)$ and for any linearization of $\mathcal{O}(m) = \mathcal{L}$ there exists $p \in X_{\mathbb{R}}(T)$ such that $(\mathbb{P}^n)^{\mathrm{ss}}(\mathcal{L}) = U(p)$. Notice that p is (in general) not uniquely defined by the choice of a linearization.

Remark 1.10. Let P be a distinguished polytope and $\Delta(P)$ be the collection of all distinguished polytopes containing P as a subset:

$$Q \in \Delta(P) \Leftrightarrow P \subset Q.$$

Then $U(\Delta(P))$ is open and there exists a finite set $\{p_1, \ldots, p_r\}$ such that

$$U(\Delta(P)) = \bigcap_{i=1}^{r} U(p_i).$$

It follows that there exist T-linearized ample line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on $\mathbb{P}^n$ such that

$$U(\Delta(P)) = \bigcap_{i=1}^{r} (\mathbb{P}^n)^{\mathrm{ss}}(\mathcal{L}_i).$$

Moreover, if $U = U(\Delta)$, $\Delta = \{P_1, \ldots, P_k\}$, is an open, combinatorially defined subset of $\mathbb{P}^n$ then

$$U = \bigcup_{i=1}^{k} U(\Delta(P_i)).$$

Therefore, every open combinatorially defined subset of $\mathbb{P}^n$ is a finite union of intersections of sets of semistable points corresponding to linearizations of $\mathcal{O}(m)$.

A description of those combinatorially defined open subsets of $\mathbb{P}^n$ that have good quotients was given in [5].

Definition 1.11. Let $\Delta = \{P_1, \ldots, P_k\}$ be a collection of distinguished polytopes. Δ is a *good collection* if it satisfies the following conditions:

(i) $P_i \cap P_j \neq \emptyset$ for every $i, j = 1, \ldots, k$,
(ii) if F is a face P_i and $F \subset P_i \cap P_j$ then $F \in \Delta$,
(iii) for any $i = 1, \ldots, k$ and a distinguished polytope Q,

$$P_i \subset Q \Rightarrow Q \in \Delta.$$

Theorem 1.12 (Thm. 7.8 of [5]). *Assume that an algebraic torus acts on $\mathbb{P}^n$ as above. Let $U(\Delta)$, $\Delta = \{P_1, \ldots, P_k\}$, be a combinatorially defined open subset of $\mathbb{P}^n$. Then there exists a good quotient of U by T if and only if Δ is a good collection of distinguished polytopes.*

By Theorem A of [5] any T-maximal subset of $\mathbb{P}^n$ is combinatorially defined, thus Theorem 1.12 implies that all T-maximal subsets of $\mathbb{P}^n$ will be of the form $U(\Delta)$ for some good collection Δ.

By the results of [5], every variety obtained as the quotient space of an open subset of $\mathbb{P}^n$ by an action of an algebraic torus, can be embedded in a toric variety. More exactly, let $\alpha : T \to \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n)$ be a homomorphism determined by the action of T on $\mathbb{P}^n$ and let S denote any maximal torus in $\mathrm{PGL}(n)$ containing $\alpha(T)$. Then $\mathbb{P}^n$ is a toric variety with respect to the action of S and we have:

Proposition 1.13 (Corollary 6.1 of [5]). *Let U be a combinatorially defined open subset of $\mathbb{P}^n$. Then U is a toric variety (with respect to the action of S). In particular U is S-invariant. If moreover $U/\!/T$ exists, then the quotient space is a toric variety with respect to the (induced) action of a quotient of S. In particular, if $U \subset \mathbb{P}^n$ is T-maximal, then both U and $U/\!/T$ are toric varieties.*

In this paper we shall use as an important tool the following

THEOREM 1.14 (Theorem C of [6]). *Let a reductive group G act on an algebraic variety X. Assume that, for any two points $x, y \in X$, there exists an open G-invariant neighborhood $V_{x,y}$ of x and y such that there exists a good quotient $V_{x,y} \to V_{x,y}/\!\!/G$. Then there exists a good quotient $X \to X/\!\!/G$. If all quotient spaces $V_{x,y}/\!\!/G$ are algebraic varieties, then $X/\!\!/G$ is also an algebraic variety.*

2. Good collections of cells. In this section we generalize the notion of a good collection of distinguished polytopes (previously defined in the case of an action of an algebraic torus on $\mathbb{P}^n$) to a good collection of intersections of open sets with good quotients in the case of an action of a reductive group G on an algebraic variety X. In particular, we prove that there exists a good quotient of the union of open sets of any good collection (2.6).

DEFINITION 2.1. Let $\mathcal{J}$ be a set and, for any $i \in \mathcal{J}$, let U_i be an open subset of X. Elements of $\mathcal{J}$ (and often the sets U_i, $i \in \mathcal{J}$) will be called *vertices*. Any finite subset $I \subset \mathcal{J}$ will be called a *cell* with vertices in $\mathcal{J}$. For any cell $I \subset \mathcal{J}$ let

$$U(I) = \bigcap_{i \in I} U_i$$

and, for any collection Π of cells, let

$$U(\Pi) = \bigcup_{I \in \Pi} U(I).$$

For any cell I, the set $U(I)$ is open.

From now on (with the exception of Section 3) we shall assume that all open sets U_i, $i \in \mathcal{J}$, are G-invariant.

DEFINITION 2.2. A *boundary vertex* of a cell I is any $i \in I$ such that $U(I)$ is not G-saturated in U_i. The *boundary* $\delta(I)$ of I is the set of all boundary vertices of I.

EXAMPLE 2.3. Consider an action of an algebraic torus T on $\mathbb{P}^n$ as in Example 1.9. We define an *elementary polytope* to be a convex polytope Q in $X_{\mathbb{R}}(T)$ satisfying the following conditions:

(i) Q is an intersection of distinguished polytopes;
(ii) for any distinguished polytope P,

$$P \cap Q^\circ \neq \emptyset \Rightarrow Q \subset P$$

where Q° denotes the relative interior of the polytope Q. Notice that, for any $p \in \operatorname{conv}\{\chi_0, \ldots, \chi_n\}$, there exists exactly one elementary polytope (denoted by) $Q(p)$ such that $p \in Q$. Moreover, for any $p_1, p_2 \in \operatorname{conv}\{\chi_0, \ldots, \chi_n\}$,

$$U(p_1) = U(p_2) \Leftrightarrow Q(p_1) = Q(p_2).$$

It follows from Example 1.9 that there is a one-to-one correspondence between elementary polytopes and sets of semistable points (corresponding to various linearizations of ample line bundles on $\mathbb{P}^n$). The set of semistable points corresponding to an elementary polytope Q will be denoted by $S(Q)$.

Let $\mathcal{J}$ be the set indexing the collection of all elementary polytopes in $X_{\mathbb{R}}(T)$ and, for any $i \in \mathcal{J}$, let Q_i be the corresponding elementary polytope. Let $U(i) = S(Q_i)$.

Assume now that a subset $I \subset \mathcal{J}$ is a cell with vertices in $\mathcal{J}$ (see 2.1). Then

$$U(I) = \bigcap_{i \in I} S(Q_i),$$

hence

$$x \in U(I) \Leftrightarrow \forall i \in I,\ Q_i \subset P(x).$$

Let now $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$ and let $I(x)$ be the cell defined by

$$I(x) = \{i \in \mathcal{J} : Q_i \subset P(x)\}.$$

It follows from 5.13–5.15 of [5] that the boundary of the cell $I(x)$ consists of all vertices corresponding to elementary polytopes contained in the boundary of $P(x)$.

Definition 2.4. Let $\mathcal{J}$ be a set and, for $i \in \mathcal{J}$, let U_i be a G-invariant open subset of X such that there exists a good quotient $U_i \to U_i /\!\!/ G$. Consider a collection $\Pi = \{I_1, \ldots, I_s\}$ of cells with vertices in $\mathcal{J}$. The collection Π is *good* if, for any cells $I_i, I_j \in \Pi$, the following two conditions are satisfied:

$$I_i \cap I_j \neq \emptyset, \tag{1}$$

$$I_i \cap I_j \subset \delta(I_i) \Rightarrow I_i \cap I_j \in \Pi. \tag{2}$$

Example 2.5. Consider as before an action of T on $\mathbb{P}^n$. Let $\mathcal{J}$ be a set of elementary polytopes as in 2.3. As noticed in 1.10, for any distinguished polytope P, there exists a finite set $\{p_1, \ldots, p_r\}$ such that

$$U(\Delta(P)) = \bigcap_{i=1}^{r} U(p_i).$$

Hence there exists a finite set $\{Q_1, \ldots, Q_r\}$ of elementary polytopes such that

$$U(\Delta(P)) = \bigcap_{i=1}^{r} S(Q_i).$$

It follows that to any distinguished polytope P, there corresponds a cell $I_P = \{Q_i : Q_i \subset P\}$ with vertices in $\mathcal{J}$ such that $U(\Delta(P)) = U(I_P)$. Let

$\mathit{\Delta}$ be a collection of distinguished polytopes. It can be deduced from 2.3 that $\mathit{\Delta}$ is a good collection of distinguished polytopes iff the corresponding collection of cells (with vertices in the set of elementary polytopes) is good.

THEOREM 2.6. *Let $\mathcal{J}$ be a set and for any $i \in \mathcal{J}$ let U_i be an open G-invariant subset of X such that there exists a good quotient $U_i \to U_i/\!\!/G$. Then for any good collection $\mathit{\Pi}$ of cells with vertices in $\mathcal{J}$ there exists a good quotient $U(\mathit{\Pi}) \to U(\mathit{\Pi})/\!\!/G$. Moreover if, for any $i \in \mathcal{J}$, the quotient space $U_i/\!\!/G$ is an algebraic variety then $U(\mathit{\Pi})/\!\!/G$ is also an algebraic variety.*

Proof. For any cell $I_i \in \mathit{\Pi}$ the set $U(I_i)$ is a finite intersection of open sets with good quotients. It follows from Proposition 1.1 of [4] that there exists a good quotient $U(I_i) \to U(I_i)/\!\!/G$. Assume that

$$x, y \in U(\mathit{\Pi}) = \bigcup_{I \in \mathit{\Pi}} U(I).$$

Let $x \in U(I_1)$ and $y \in U(I_2)$. Then $x, y \in U(I_1) \cup U(I_2)$. We shall prove that there exists a good quotient

$$U(I_1) \cup U(I_2) \to (U(I_1) \cup U(I_2))/\!\!/G.$$

Assume first that $I_1 \cap I_2 \not\subset \delta(I_i)$ for $i = 1, 2$. Then there exist vertices $j_1, j_2 \in I_1 \cap I_2$ such that $j_1 \notin \delta(I_1)$ and $j_2 \notin \delta(I_2)$. In this case for $i = 1, 2$ the set $U(I_i)$ is a saturated subset of $U_{j_1} \cap U_{j_2}$. Hence, $U(I_1) \cup U(I_2)$ is G-saturated in $U_{j_1} \cap U_{j_2}$ and by 1.4 there exists a good quotient

$$U(I_1) \cup U(I_2) \to (U(I_1) \cup U(I_2))/\!\!/G.$$

Consider now the case $I_1 \cap I_2 \subset \delta(I_1)$. According to condition (2) of 2.4,

$$J = I_1 \cap I_2 \in \mathit{\Pi},$$

hence, $U(J) \subset U(\mathit{\Pi})$, $U(J) \to U(J)/\!\!/G$ exists and $x, y \in U(J)$.

We have proved that for any two points $x, y \in U(\mathit{\Pi})$ there exists an open G-invariant neighborhood $V_{x,y}$ of x, y in $U(\mathit{\Pi})$ such that there exists a good quotient $V_{x,y} \to V_{x,y}/\!\!/G$. We use Theorem 1.14 to infer that there exists a good quotient $U(\mathit{\Pi}) \to U(\mathit{\Pi})/\!\!/G$.

LEMMA 2.7. *Let X, G and $\mathcal{J}$ be as in* 2.6, *and let $\mathit{\Pi}$ be a good collection of cells with vertices in $\mathcal{J}$. Assume that $I \in \mathit{\Pi}$ and I is minimal in $\mathit{\Pi}$. Then $U(I)$ is G-saturated in $U(\mathit{\Pi})$.*

Proof. Assume that $I \in \mathit{\Pi}$ and I is minimal in $\mathit{\Pi}$. Suppose that $U(I)$ is not saturated in $U(\mathit{\Pi})$. Then there exist $x \in U(I)$, $y \in U(\mathit{\Pi})$ such that $y \in \overline{G \cdot x}$ and $y \notin U(I)$. Let $I_1 \in \mathit{\Pi}$ be a cell such that $y \in U(I_1)$. Then $y \in U_j$ for any $j \in I_1$. It follows that $I \cap I_1 \subset \delta(I)$. We have assumed that $\mathit{\Pi}$ is a good collection of cells and therefore $\emptyset \neq I \cap I_1 \in \mathit{\Pi}$ and, since I is minimal, it follows that $I = I \cap I_1$. Hence $y \in U(I)$, giving a contradiction. ■

COROLLARY 2.8. *For any good collection Π of cells with vertices in $\mathcal{J}$, there exists a subcollection $\Pi' \subset \Pi$ such that $U(\Pi') = U(\Pi)$ and, for any $I \in \Pi'$, the set $U(I)$ is saturated in $U(\Pi') = U(\Pi)$.*

Proof. Indeed, define Π' to be the set of minimal cells in Π. Obviously $U(\Pi') \subset U(\Pi)$. For any $I \in \Pi$, there exists $J \in \Pi'$ such that $J \subset I$, hence $U(I) \subset U(J)$ and therefore $U(\Pi) = U(\Pi')$. The statement follows from Lemma 2.7. ∎

3. A_n-varieties. In this section we consider a class of algebraic varieties satisfying some additional condition. This class is larger than the class of projective varieties.

DEFINITION 3.1. Let X be an algebraic variety. We say that X is an A_n-*variety* if, for any $x_1, \ldots, x_n \in X$, there exists an open affine $U \subset X$ such that $x_1, \ldots, x_n \in U$. If X is an A_n-variety for every $n \in \mathbb{N}$ then we say that X is an A_∞-*variety.*

Thus, any algebraic variety is an A_1-variety. In [10] Włodarczyk proved that a normal variety X is an A_2-variety if and only if it can be embedded in a toric variety.

Any quasiprojective variety is an A_∞-variety and under some assumptions the converse is true. For smooth varieties this was proved by Kleiman [7] and, for a certain class of normal varieties, by Włodarczyk [11].

THEOREM 3.2 ([11], Theorem B). *Let X' be a normal variety for which there exists an open embedding $X' \subset X$ in a complete normal variety X such that $(\mathrm{Div}(X)/\mathrm{Car}(X))\otimes\mathbb{Q}$ is of finite dimension. Then X' is quasiprojective iff any finite subset is contained in some open affine subset.*

THEOREM 3.3 ([11], Theorem A). *Let X be a complete normal variety such that $(\mathrm{Div}(X)/\mathrm{Car}(X))\otimes\mathbb{Q}$ is finite-dimensional. Then X contains only finitely many maximal (in the sense of inclusion) open quasiprojective sets.*

We prove that any algebraic variety contains only finitely many maximal open A_n-subsets.

LEMMA 3.4. *Let $M \subset X^n = X \times \ldots \times X$ be defined as follows: $(x_1, \ldots, x_n) \in M$ if and only if there exists an affine $U \subset X$ such that $x_i \in U$ for $i = 1, \ldots, n$. Then M is open.*

Proof. $M = \bigcup U \times \ldots \times U$ where U runs over the set of all affine open subsets of X. ∎

THEOREM 3.5. *Let X be an algebraic variety. For every $n \in \mathbb{N}$, the number of open maximal A_n-subvarieties of X is finite.*

Proof. First, assume X is complete. Let $M \subset X^n$ be as in Lemma 3.4. Then $Y = X^n - M$ and $Z(n) = \mathrm{pr}_1(Y)$ are complete. Let $Y = Y_1 \cup \ldots \cup Y_k$ be the decomposition into irreducible components and let $Z_i = \mathrm{pr}_1(Y_i)$. Then $Z(n) = \bigcup_{i=1}^k Z_i$. Let U be any open A_n-subset of X. Equivalently, $U \subset X$ is open and $U^n \cap Y = \emptyset$. Consider the set $K = \{i : U \cap Z_i = \emptyset\}$. Then $S(U) = X - \bigcup_{i \in K} Z_i$ is open. We prove that $S(U)$ is an A_n-variety. Suppose that there exist $x_1, \ldots, x_n \in S(U)$ such that $(x_1, \ldots, x_n) \in Y$. Let $(x_1, \ldots, x_n) \in Y_{j_0}$ for some $j_0 \in \{1, \ldots k\}$. Let p_i be the projection onto the ith factor. Then $p_i(Y_{j_0}) = Z_{i_0}$ for some $i_0 \in \{1, \ldots, k\}$. Since $Z_{i_0} \cap S(U) \neq \emptyset$, it follows that $U \cap Z_{i_0} \neq \emptyset$. Therefore, for every $i = 1, \ldots, n$ we have $p_i^{-1}(U) \cap Y_{j_0} \neq \emptyset$. It follows that $Y \cap U^n \neq \emptyset$, contradicting the assumption that U is an A_n-variety. It follows that any maximal A_n-set is obtained by removing from X a finite collection of closed subsets. Hence there are only finitely many maximal open A_n-subsets of any complete variety X.

Let now X be arbitrary and let $X \hookrightarrow X'$ be an open embedding in a complete algebraic variety. Let $U_1, \ldots, U_k$ be the collection of all maximal open A_n-subsets of X'. Any open A_n-subvariety of X is contained in $X \cap U_i$ for some $i = 1, \ldots, k$. But, for any i, $X \cap U_i$ is an A_n-variety. Hence, the number of maximal A_n-subvarieties of X is finite. ∎

Theorems 3.2, 3.3 and 3.5 suggest the following

CONJECTURE 3.6. Let X be an algebraic normal variety. Then X contains only finitely many maximal (in the sense of inclusion) open A_∞-subsets.

4. Sets with A_n-quotients. Let X be an algebraic variety with an action of a reductive group G. We shall prove that there are only finitely many open subsets that are maximal with respect to saturated inclusion in the family of all subsets with an A_n-variety as a quotient space. The problem of finiteness of the set of G-maximal subsets was considered in [1].

THEOREM 4.1 (Main Theorem of [1]). *Let X be a normal variety with an action of a reductive group G. Then the number of G-maximal subsets of X is finite.*

As an easy corollary we get (see Introduction of [1])

COROLLARY 4.2. *Let X and G be as in* 4.1. *Then there are only finitely many maximal* (*with respect to saturated inclusion*) *subsets with a variety as a quotient.*

DEFINITION 4.3. Let $U \subset X$ be an open G-invariant subset with a good quotient. Assume that the quotient space $U /\!\!/ G$ is an A_n- or an A_∞-variety or is quasiprojective. We shall say that U is (G, n)-*maximal* (resp. (G, ∞)-*maximal*, *qp-G-maximal*) if U is maximal with respect to saturated inclusion in the set of all G-invariant open subsets of X that have a good A_n-quotient (resp. A_∞-quotient, quasiprojective quotient).

THEOREM 4.4. *Let X be a complex normal variety with an action of a reductive group G. Then, for any $n \in \mathbb{N}$, the number of (G,n)-maximal subsets of X is finite.*

Proof. According to 4.1 there are a finite number of G-maximal subsets of X. Any (G,n)-maximal subvariety of X is contained as a saturated subset in a G-maximal set. Theorem 3.5 implies that, for any G-maximal set $U \subset X$, the number of maximal A_n-subsets in the quotient space $U/\!\!/G$ is finite. This completes the proof. ∎

PROPOSITION 4.5. *Assume that X is a G-variety and let U be a G-invariant open subset of X such that there exists a good A_2-quotient. Then there exists a good collection $\Pi = \{I_1, \ldots, I_m\}$ of cells with vertices in the set $\mathcal{J}$ of G-invariant affine subsets U_j, $j \in \mathcal{J}$, of X such that $U = U(\Pi)$. Moreover, for any cell $I_j \in \Pi$ and any $i \in I_j$, the set $U(I_j)$ is saturated in U_i.*

Proof. Let $q : U \to U/\!\!/G$ be the quotient morphism and assume that $U/\!\!/G$ is an A_2-variety. Choose a finite covering of the quotient space $U/\!\!/G$ by affine open subsets

$$U/\!\!/G = W_1 \cup \ldots \cup W_n$$

such that, for any two points $x, y \in U/\!\!/G$, there exists an index i such that $x, y \in W_i$. Such a finite covering may be chosen in the following way: since $U/\!\!/G$ is an A_2-variety, it follows that for any two points $x, y \in U/\!\!/G$ there exists an open affine neighborhood $W_{x,y}$ of x, y in $U/\!\!/G$. Then $(x, y) \in W_{x,y} \times W_{x,y} \subset U/\!\!/G \times U/\!\!/G$. Therefore $U/\!\!/G \times U/\!\!/G$ is covered by open sets $W \times W$ where W is an affine open subset of $U/\!\!/G$. We choose a finite covering $W_i \times W_i$, $i = 1, \ldots, n$. Then the collection W_i, $i = 1, \ldots, n$, is a covering of $U/\!\!/G$ with the desired property.

For any $x \in U/\!\!/G$, let $S(x) = \{i : x \in W_i\}$. Choose a finite set of points $x_1, \ldots, x_l \in U/\!\!/G$ such that the sets

$$V_i = \bigcap_{j \in S(x_i)} W_j$$

form a covering of $U/\!\!/G$. Let $\mathcal{J} = \{1, \ldots, n\}$. Then for any $j \in \mathcal{J}$ the set $U_j = q^{-1}(W_j)$ is an affine G-invariant subset of U. Let Π be a collection of cells $S(x_i), i = 1, \ldots, l$, with vertices in $\mathcal{J}$. Then

$$U(S(x_i)) = \bigcap_{j \in S(x_i)} U_j = q^{-1}\Big(\bigcap_{j \in S(x_i)} W_j \Big) \tag{3}$$

is saturated in U_j for any $j \in S(x_i)$. It is easy to see that Π is a good collection of cells. In fact (3) implies the condition (2) of 2.4 since the boundary of any cell in Π is empty, and condition (1) is satisfied by the choice of the covering $\{W_i : i = 1, \dots, n\}$. Obviously $U(\Pi) = U$. ■

PROPOSITION 4.6. *Let X be a G-variety and let Π be a good collection of cells with vertices in $\mathcal{J}$ and assume that, for every $k \in \mathcal{J}$, $U_k /\!\!/ G$ is an A_2-variety. Then $U(\Pi)/\!\!/G$ is an A_2-variety.*

Proof. According to Corollary 2.8 we can assume without loss of generality that all cells of Π are minimal in Π, hence, for any cell $I \in \Pi$, the set $U(I)$ is G-saturated in $U(\Pi)$. Notice that then, for any $I_1, I_2 \in \Pi$, there exists a vertex $k \in \mathcal{J}$ such that $U(I_1) \cup U(I_2)$ is contained as a saturated subset in U_k. Hence $(U(I_1) \cup U(I_2))/\!\!/G$ exists and is an A_2-variety. Let $x, y \in U(\Pi)$. There exist $I_1, I_2 \in \Pi$ such that $x \in U(I_1)$, $y \in U(I_2)$. Then $(U(I_1) \cup U(I_2))/\!\!/G$ is an open A_2-neighborhood of x and y. Thus the points x, y are contained in an open affine subset of $U(\Pi)/\!\!/G$. ■

DEFINITION 4.7. Let Π be a collection of cells with vertices in the set $\mathcal{J}$. The collection Π is *n-good* if it is good and, for any cells $I_1, \dots, I_n \in \Pi$, the intersection $\bigcap_{i=1}^{n} I_i$ is not empty.

PROPOSITION 4.8. *Let G act on X and let U be a subset of X such that there exists a good A_n-quotient. Then there exists an n-good collection Π of cells with vertices in the set of affine G-sets such that $U = U(\Pi)$.*

Proof. We proceed exactly as in the proof of Proposition 4.5.

5. The case of $\mathrm{Pic}(X) = \mathbb{Z}$**.** In this section we assume that X is projective, smooth and $\mathrm{Pic}(X) = \mathbb{Z}$ and we use the combinatorial construction described in Section 2 to build all $(G, 2)$-maximal subsets of X. We use sets with projective quotients as "building blocks".

First notice that there are finitely many open subsets with projective quotient:

PROPOSITION 5.1 (see Example 5.1 of [1]). *Let X be a normal projective variety and G be a reductive group acting on X. There exist only finitely many sets of semistable points corresponding to all G-linearized ample line bundles on X.*

This implies the following

COROLLARY 5.2. *Let X be a projective smooth variety with an action of a reductive group G and suppose that* $\mathrm{Pic}(X) = \mathbb{Z}$*. Then the number of open subsets with projective quotients is finite.*

Proof. In this case any open subset with a projective quotient space is the set of semistable points of some G-linearized ample line bundle. This is a consequence of the following

LEMMA 5.3 (Lemma 7.14 of [5]). *Let G be a reductive group acting on a projective smooth variety X with* $\mathrm{Pic}(X) = \mathbb{Z}$. *Let U be a G-invariant open subset of X such that there exists a good quotient $U \to U/\!\!/G$ and $U/\!\!/G$ is quasiprojective. Then there exists a G-linearized ample line bundle $\mathcal{L}$ on X such that U is contained in $X^{\mathrm{ss}}(\mathcal{L})$ as a saturated subset.*

Hence the number of open subsets of X with projective quotients is finite.

We shall need the following lemma:

LEMMA 5.4. *Let X be an algebraic variety with an action of a reductive group G and assume that $U_1, \ldots, U_k$ are G-invariant open subsets in X such that, for $j = 1, \ldots, k$, there exists a good quotient $U_j/\!\!/G$ and it is quasiprojective. Then there exists a good quotient of $U = \bigcap_{j=1}^k U_j$ and it is quasiprojective.*

Proof. We shall prove that there exists a G-linearized line bundle $\mathcal{L}$ on U such that $U = U^{\mathrm{ss}}(\mathcal{L})$. According to 1.13 of [8], for $i = 1, \ldots, k$, there exists a G-linearized line bundle $\mathcal{L}_i$ on U_i such that $U_i = (U_i)^{\mathrm{ss}}(\mathcal{L}_i)$. Let $x \in U$ and let $s_i \in \Gamma(U_i, \mathcal{L}_i)$ be a G-invariant section with affine support such that $s_i(x) \neq 0$. (Maybe we have to choose s_i as a section of some tensor power of $\mathcal{L}_i$.) Let $\mathcal{L}' = \mathcal{L}_1|U \otimes \ldots \otimes \mathcal{L}_k|U$. Then the support of $s_1|U \cdot \ldots \cdot s_k|U \in \Gamma(U, \mathcal{L}')$ is the intersection of the supports of s_i, hence is affine and contains x. It follows that $U = U^{\mathrm{ss}}(\mathcal{L}')$, hence there exists a good quotient $U \to U/\!\!/G$ and $U/\!\!/G$ is quasiprojective. ■

DEFINITION 5.5. Let $\mathcal{J}_{\mathrm{ss}}$ be the indexing set for the collection of all open G-invariant subsets of X with projective quotients. For $i \in \mathcal{J}_{\mathrm{ss}}$, let V_i be the corresponding open subset. For any $U \subset X$ let

$$I_{\mathrm{ss}}(U) = \{i \in \mathcal{J}_{\mathrm{ss}} : U \subset V_i\} \quad \text{and} \quad C_{\mathrm{ss}}(U) = \bigcap_{i \in I_{\mathrm{ss}}(U)} V_i.$$

Notice that, according to 5.2, $\mathcal{J}_{\mathrm{ss}}$ is finite.

LEMMA 5.6. *Assume X is a projective smooth G-variety with* $\mathrm{Pic}(X) = \mathbb{Z}$. *Let $U \subset X$ be an open G-invariant subvariety such that there exists a good quotient $U \to U/\!\!/G$ and the quotient space is quasiprojective. Then*

(i) $I_{\mathrm{ss}}(U) \neq \emptyset$,
(ii) *the set $I_{\mathrm{ss}}(U)$ is finite*,
(iii) *there exists a good quotient $C_{\mathrm{ss}}(U) \to C_{\mathrm{ss}}(U)/\!\!/G$*,
(iv) *the quotient space $C_{\mathrm{ss}}(U)/\!\!/G$ is quasiprojective*,
(v) *U is a saturated subset of $C_{\mathrm{ss}}(U)$.*

Proof. The first statement follows from Lemma 5.3. The second one is a consequence of Proposition 5.1. Statements (iii) and (iv) follow from 5.4. Finally notice that by (iv) and 5.3 there exists $i \in I_{\mathrm{ss}}(U)$ such that U is a saturated subset of V_i, hence U is saturated in $C_{\mathrm{ss}}(U) \subset V_i$. ∎

THEOREM 5.7. *Let X be a projective smooth algebraic variety with $\mathrm{Pic}(X) = \mathbb{Z}$. Assume that U is $(G,2)$-maximal. Then there exists a good collection $\widetilde{\Pi}$ of cells with vertices in $\mathcal{J}_{\mathrm{ss}}$ such that $U = U(\widetilde{\Pi})$.*

Proof. Assume that $U \subset X$ has a good quotient and the quotient space is an A_2-variety. By Proposition 4.5 we can find a finite collection $\mathcal{J}$ of G-invariant affine sets U_j, $j \in \mathcal{J}$, and a good collection of cells $\Pi = \{I_1, \ldots, I_m\}$ with vertices in $\mathcal{J}$ such that $U = U(\Pi)$. Therefore $U = \bigcup_{i=1}^m U(I_i)$. Moreover we can choose Π in such a way that for any $i = 1, \ldots, m$ the set $U(I_i)$ is saturated in U_j for every $j \in I_i$. For any cell I_i let

$$\widetilde{I}_i = I_{\mathrm{ss}}(U(I_i)).$$

We claim that $\widetilde{\Pi} = \{\widetilde{I}_i : i = 1, \ldots, m\}$ is a good collection of cells with vertices in $\mathcal{J}_{\mathrm{ss}}$.

We have to prove that, for any $i, j = 1, \ldots, m$,

$$\emptyset \neq \widetilde{I}_i \cap \widetilde{I}_j \not\subseteq \delta(I_i).$$

Consider cells $I_{i_1}, I_{i_2} \in \Pi$ and the corresponding cells $\widetilde{I}_{i_1}, \widetilde{I}_{i_2} \in \widetilde{\Pi}$. Let $W_1 = U(I_{i_1})$ and $W_2 = U(I_{i_2})$. By the assumptions on Π, the intersection $I_{i_1} \cap I_{i_2}$ is nonempty and W_1, W_2 are saturated in the affine set U_j for any $j \in I_{i_1} \cap I_{i_2}$. Let $V = W_1 \cup W_2$. Then V is a saturated subset of the affine variety U_j. According to [8], Thm. 1.1, there exists a good quotient $U_j \to U_j /\!\!/ T$ and it is an affine variety. Therefore there exists a good quotient $V /\!\!/ T$ and it is quasiaffine. By 5.6(v) we see that U_j is saturated in $C_{\mathrm{ss}}(U_j)$. According to 5.4, $C_{\mathrm{ss}}(U_j)$ has a quasiprojective good quotient, hence, by 5.3, there exists $k \in \mathcal{J}_{\mathrm{ss}}$ such that $C_{\mathrm{ss}}(U_j)$ is a saturated subset of V_k. It follows that V is a saturated subset of V_k, hence $k \in \widetilde{I}_{i_1} \cap \widetilde{I}_{i_2}$, but k is not contained in $\delta(\widetilde{I}_{i_j})$ for $j = 1, 2$. This shows that $\widetilde{\Pi}$ is a good collection of cells.

We now prove that U is a saturated subset of $U(\widetilde{\Pi})$. Let $x \in U$ and $y \in \overline{G \cdot x} \subset U(\widetilde{\Pi})$ and assume that $x \in W_1 = U(I_{i_1})$ and $y \in W_2 = U(\widetilde{I}_{i_2})$. As before we can find $l \in \mathcal{J}_{\mathrm{ss}}$ such that W_1, W_2 are saturated subsets of V_l, hence $y \in W_1 \subset U$. This ends the proof of Theorem 5.7. ∎

REMARK 5.8. Assume that, as before, X is a projective smooth G-variety with $\mathrm{Pic}(X) = \mathbb{Z}$ and U is (G,k)-maximal for $k > 1$. Then using the same method as above we can find a k-good collection Π of cells with vertices in $\mathcal{J}_{\mathrm{ss}}$ such that $U = U(\Pi)$.

6. Corollaries and examples. In this section (as before) we assume that X is projective smooth with $\operatorname{Pic}(X)=\mathbb{Z}$.

EXAMPLE 6.1. Let T be an algebraic torus acting on $\mathbb{P}^n$ and let U be a T-maximal set. It follows from 1.12 and 2.5 that there exists a good collection Π of cells with vertices in the set of elementary polytopes (see Example 2.3) such that $U=U(\Pi)$. This result follows immediately from Theorem 5.7. In fact, by Corollary 2.5 of [9] (see 1.13) the quotient space $U/\!\!/T$ is a toric variety, hence an A_2-variety.

PROPOSITION 6.2. *Assume that there is an action of an algebraic torus T on X where X is projective and smooth with* $\operatorname{Pic}(X)=\mathbb{Z}$. *Let U be a $(T,2)$-maximal subset of X. Then there exists an equivariant embedding $\iota: X\hookrightarrow\mathbb{P}^n$ and a T-maximal set $W\subset\mathbb{P}^n$ such that $\iota(U)=W\cap\iota(X)$.*

Proof. Let $\widetilde{\Pi}$ be a good collection of cells with vertices in $\mathcal{J}_{\rm ss}$ such that $U=U(\widetilde{\Pi})$. The existence of $\widetilde{\Pi}$ follows from Theorem 5.7. We embed X in $\mathbb{P}^n$ in such a way that, for any $j\in\mathcal{J}_{\rm ss}$, V_j is the intersection of X with a T-invariant set in $\mathbb{P}^n$ with a projective quotient. We shall identify X with $\iota(X)$. We define a collection Δ of distinguished polytopes in $X_{\mathbb{R}}(T)$ (see 1.6) in the following way: a distinguished polytope P belongs to Δ if there exists $x\in X$ such that $P(x)\subset P$.

We prove that Δ is a good collection of polytopes (see 1.11). For any $p\in X_{\mathbb{R}}(T)$, let $U(p)$ be as in 1.9. Then $U(p)/\!\!/T$ is projective. For any $x,y\in U$, there exists an open T-invariant neighborhood $U_{x,y}$ of x,y in U and a point $p_0\in X_{\mathbb{R}}(T)$ such that $U_{x,y}$ is a saturated subset in $U(p_0)\cap X$. Then $U(p_0)\cap X$ is a closed subset in $U(p_0)$, therefore $U_{x,y}$ is saturated in $U(p_0)$. Hence, by 1.9, $P(x_1)\cap P(x_2)\neq\emptyset$. Assume now that $F=P(x_1)\cap P(x_2)$ is a face of $P(x_1)$. Then $p_0\in F$. There exists $z\in\overline{T\cdot x}\subset X$ such that $P(z)=F$. Then $z\in U(p_0)\cap X$. It follows that $z\in U_{x,y}$. Hence $P(z)=F\in\Delta$. This proves that Δ is a good collection of distinguished polytopes. Then (by 1.12) there exists a good quotient $U(\Delta)\to U(\Delta)/\!\!/T$ and hence $U(\Delta)$ is contained in a T-maximal subset W of $\mathbb{P}^n$ as a saturated subset. As in the proof of Corollary 1.13, the quotient space $W/\!\!/T$ is a toric variety. Therefore, there exists a good quotient of $X\cap W$ by T and the quotient space is a closed subset of a toric variety, hence the quotient space is an A_2-variety. Now, $U(\Delta)$ is saturated in W and $U=U(\Delta)\cap X$, hence U is saturated in $W\cap X$. Since U is $(T,2)$-maximal, $U=W\cap X$. ∎

COROLLARY 6.3. *Let a reductive group G act on a smooth projective variety X with* $\operatorname{Pic}(X)=\mathbb{Z}$. *Let T be a fixed maximal torus of G.*

(I) *Let V be a $(T,2)$-maximal subset of X and $U=\bigcap_{g\in G}g\cdot V$. Then U is open, G-invariant and there exists a good quotient $U/\!\!/G$.*

(II) *For any* $(G,2)$*-maximal set* U *there exists a* $(T,2)$*-maximal set* V *containing* U *as a* T*-saturated subset and, for any such* V, *the quotient set* $U/\!\!/G$ *is an* A_2*-maximal subset of* $(\bigcap_{g\in G} g\cdot V)/\!\!/G$.

Proof. According to Corollary 6.2 we can embed X in $\mathbb{P}^n$ in such a way that $V = W\cap X$, where W is a T-maximal set in $\mathbb{P}^n$. By Theorem C of [5] the set

$$W_1 = \bigcap_{g\in G} g\cdot W$$

is open and there exists a good quotient $W_1/\!\!/G$. Then $U = W_1\cap X$ and therefore there exists a good quotient $U/\!\!/G$. This ends the proof of (I).

There is a good quotient $q_G : V \to V/\!\!/G$, hence by Proposition 2.1 of [3] there exists a good quotient $q_T : V \to V/\!\!/T$. For any two points $x, y \in V$ there exists an affine open subset W in $V/\!\!/G$ such that $q_G(x), q_G(y) \in W$. The quotient morphism q_G is affine, hence $q_G^{-1}(W)\subset V$ is affine and G-saturated (in V). It follows that $V_{x,y} = q_G^{-1}(W)$ is T-saturated in V. Hence the quotient space $V_{x,y}/\!\!/T$ is an affine neighborhood of $q_T(x)$, $q_T(y)$. It follows that $V/\!\!/T$ is an A_2-variety. Now (II) follows immediately from (I). ∎

Example 6.4. We give an example of an open subset U of a smooth projective T-variety X with $\mathrm{Pic}(X) = \mathbb{Z}$ such that $U \to U/\!\!/T$ exists and is complete but the quotient space is not an A_2-variety. It follows from 4.6 that U is not defined by a good collection of cells with vertices in $\mathcal{J}_{\mathrm{ss}}$. Let X be the Grassmann variety $X = G(2,4)$ of planes in A^4 with the action of a one-dimensional torus T induced by the action of T on A^4 given by the matrix

$$t \to \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & t^2 & 0\\ 0 & 0 & 0 & t^3 \end{pmatrix}.$$

It was noticed in Remark 1.6 of [2] that in this case there exist open subsets with good quotients by the torus T such that the quotient spaces are complete but not projective. In fact, we can embed X in the projective space $P(\bigwedge^2 A^4)$ with T acting by

$$t \mapsto \mathrm{diag}(t, t^2, t^3, t^3, t^4, t^5).$$

There are six fixed points of T on X: $p_1 = e_1\wedge e_2$, $p_2 = e_1\wedge e_3$, $p_3 = e_1\wedge e_4$, $p_4 = e_2\wedge e_3$, $p_5 = e_2\wedge e_4$, $p_6 = e_3\wedge e_4$ and the partial order on this set is given by the diagram

$$\begin{tikzcd} & p_1 \arrow[d] & \\ & p_2 \arrow[dl] \arrow[dr] & \\ p_3 \arrow[dr] & & p_4 \arrow[dl] \\ & p_5 \arrow[d] & \\ & p_6 & \end{tikzcd}$$

Let $A_- = \{p_1, p_2, p_3\}$, $A_+ = \{p_4, p_5, p_6\}$ and

$$U = \{x \in X : \lim_{t\to 0} t \cdot x \in A_-,\ \lim_{t\to\infty} t \cdot x \in A_+\}.$$

Then there exists a good quotient $q : U \to U/\!\!/T$ and it is complete but not projective. Moreover T acts on U with closed orbits. Consider points $x_1, x_2 \in P^4$ corresponding to the planes $\mathrm{lin}(e_1, e_2+e_4)$ and $\mathrm{lin}(e_2, e_1+e_3)$ respectively. Then $x_1, x_2 \in U$ but there is no affine T-invariant neighborhood of x_1 and x_2 in U. To see this, assume that there exists an affine T-invariant open $V \subset X$ such that $x_1, x_2 \in V \subset U$. Since $\mathrm{Pic}(X) = \mathbb{Z}$ and X is smooth, by Lemma 5.3, V would be a saturated subset of $X^{\mathrm{ss}}(L)$ for some linearized ample line bundle L. There is only one set of semistable points which contains x_1, x_2 and the orbits $T \cdot x_i$ are not closed in this set, contradicting the assumption that V is saturated in $X^{\mathrm{ss}}(L)$. It follows that there is no affine neighborhood of $q(x_1)$, $q(x_2)$ in $U/\!\!/T$, hence the quotient space $U/\!\!/T$ is not an A_2-variety and, therefore, cannot be embedded in a toric variety. ■

REFERENCES

[1] A. Białynicki-Birula, *Finiteness of the number of maximal open sets with a good quotient*, Transformation Groups 3 (1998), 301–319.

[2] A. Białynicki-Birula and J. Święcicka, *Complete quotients by algebraic torus actions*, in: Group Actions and Vector Fields, Lecture Notes in Math. 956, Springer, 1982, 10–20.

[3] —, —, *A reduction theorem for existence of good quotients*, Amer. J. Math. 113 (1990), 189–201.

[4] —, —, *A combinatorial approach to geometric invariant theory*, in: Proc. Sophus Lie Memorial Conf., Scand. Univ. Press, Oslo, 1992, 115–127.

[5] —, —, *Open subsets of projective spaces with a good quotient by an action of a reductive group*, Transformation Groups 1 (1996), 153–185.

[6] —, —, *Three theorems on existence of good quotients*, Math. Ann. 307 (1997), 143–149.

[7] S. L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. 84 (1966), 293–344.

[8] D. Mumford, *Geometric Invariant Theory*, Ergeb. Math. Grenzgeb. 34, Springer, 1982.

[9] J. Święcicka, *Quotients of toric varieties by actions of subtori*. Colloq. Math. 82 (1999), 105–116.

[10] J. Włodarczyk, *Embeddings in toric varieties and prevarieties*, J. Algebraic Geom. 2 (1993), 705–726.

[11] —, *Maximal quasiprojective subsets and the Kleiman–Chevalley quasiprojectivity criterion*, J. Math. Sci. Univ. Tokyo 6 (1999), 41–47.

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: jswiec@mimuw.edu.pl

Received 31 May 1999;
revised 5 February 2000 (3772)