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BEHAVIOUR OF THE FIRST EIGENVALUE OF THE p-LAPLACIAN IN A DOMAIN WITH A HOLE

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Abstract. We investigate the behaviour of a sequence λ_s, $s = 1, 2, \ldots$, of eigenvalues of the Dirichlet problem for the p-Laplacian in the domains Ω_s, $s = 1, 2, \ldots$, obtained by removing from a given domain Ω a set E_s whose diameter vanishes when $s \to \infty$. We estimate the deviation of λ_s from the eigenvalue of the limit problem. For the derivation of our results we construct an appropriate asymptotic expansion for the sequence of solutions of the original eigenvalue problem.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth bounded domain with boundary Γ. We denote by $B(x_0, \varepsilon_s)$ the ball inside Ω, centered at the point x_0 with radius ε_s, where ε_s, $s = 1, 2, \ldots$, is a sequence of positive numbers which converges to zero as $s \to \infty$. Let E_s be a set inscribed in $B(x_0, \varepsilon_s)$ and let $\Omega_s = \Omega \setminus E_s$ be the domain obtained by removing E_s from the domain Ω.

For $m \in [2, n)$, we consider the eigenvalue problem for the p-Laplacian in Ω_s, $s = 1, 2, \ldots$:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{m-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{m-2} u \quad \text{in } \Omega_s, \tag{1}$$

$$u(x) = 0 \quad \text{on } \partial\Omega_s, \tag{2}$$

where $x = (x_1, \ldots, x_n)$, $\partial\cdot$ denotes the boundary of a set $\cdot$, ∇u is the gradient of u. We shall denote by $W_m^1(\cdot)$ the standard Sobolev spaces in a domain $\cdot$, and by $\mathring{W}_m^1(\cdot)$ the set of functions in $W_m^1(\cdot)$ which vanish on $\partial\cdot$.

We shall call a number λ_s an *eigenvalue* of problem (1)–(2) if there exists a function $u_s \in \mathring{W}_m^1(\Omega_s)$, $u_s \not\equiv 0$, such that u_s is a weak solution of (1)–(2), i.e.,

$$\int_{\Omega_s} \sum_{i=1}^{n} |\nabla u_s|^{m-2} \frac{\partial u_s}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx = \lambda_s \int_{\Omega_s} |u_s|^{m-2} u_s \varphi\, dx \tag{3}$$

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whenever $\varphi \in \mathring{W}_m^1(\Omega_s)$. The function u_s is then referred to as an *eigenfunction* of (1)–(2) corresponding to λ_s.

The aim of this paper is to investigate the asymptotic behaviour of the sequence of first eigenvalues λ_s of (1)–(2) and the sequence of their corresponding eigenfunctions u_s. In particular, we give an estimate of the deviation $\lambda_s - \lambda$ of λ_s from the first eigenvalue λ of the limit problem in terms of the variational capacity of the ball E_s. The problem under investigation goes back to Samarskiĭ [4], who considered the Dirichlet problem for the Laplace operator, i.e., when $m = 2$. He gave an optimal asymptotic estimate of $\lambda_s - \lambda$ in terms of the harmonic capacity of E_s; Maz'ya *et al.* constructed in [3] a complete asymptotic expansion for the first eigenvalue of the classical boundary value problems for the Laplace operator in Ω_s. We refer to the bibliography in [3] for further references on the subject.

A nonlinear version of the theory elaborated in the above-mentioned papers for the p-Laplacian in a domain with a hole is not known to us. We note that when m is a natural number, the theory that we propose here leads to a sharper error estimate which, when $m = 2$, coincides modulo a multiplicative constant with the main term in the corresponding asymptotic estimate obtained in [4].

The existence of the first eigenvalue and its corresponding eigenfunction is well known. In this connection, we refer to P. Linqvist's paper [2] and the references therein. In particular, the first eigenvalue λ_s of (1)–(2) is determined by the formula

$$(4) \qquad \lambda_s = \inf_{v \in X_s} \int_{\Omega_s} |\nabla v|^m \, dx, \qquad X_s = \{v \in \mathring{W}_m^1(\Omega_s) : \|v\|_{L_m(\Omega_s)} = 1\}.$$

The infimum is attained if v is furthermore an eigenfunction of (1)–(2). By its definition, it is clear that the sequence $\{\lambda_s\}_{s>0}$ is bounded and positive. Indeed, a simple verification shows that if $\lambda_s = 0$, then the corresponding solution u_s of (1)–(2) is identically zero. Further, substituting $\varphi(x) = u_s(x)$ (an eigenfunction corresponding to λ_s) in (3), we readily see that there exists a constant K independent of s such that

$$(5) \qquad \|u_s\|_{\mathring{W}_m^1(\Omega_s)} \le K.$$

Extending the function u_s to Ω by setting $u_s(x) = 0$ in $\Omega \setminus \Omega_s$ we obtain a new function, again denoted by u_s, which belongs to $\mathring{W}_m^1(\Omega)$ and satisfies the inequality (5). This implies that there exists a subsequence of u_s (denoted by the same symbol) which converges weakly to a function u in $\mathring{W}_m^1(\Omega)$. Since λ_s is bounded, there exists a subsequence (again denoted by the same symbol) which converges to a number λ^*.

We introduce the concept of m-capacity (variational capacity) of a set (see e.g. Evans–Gariepy [1, Sect. 4.7]). We define the *m-capacity* of a set

$E \subset B(x_0, 1/2)$ to be the number

$$C_m[E] = \inf\Big\{ \int\limits_{B(x_0,1)} |\nabla\varphi|^m\, dx : \varphi \in \mathbb{C}_0^\infty(B(x_0,1)),\ \varphi(x) = 1 \text{ in } E\Big\}.$$

From the same reference, we have

$$\operatorname{meas} E \le C C_m[E]^{n/(n-m)}, \tag{6}$$

where C is a constant depending only on m and n, meas stands for the Lebesgue measure in $\mathbb{R}^n$. Now we are in a position to formulate our main result.

THEOREM 1. *Let λ_s, $s = 1, 2, \ldots$, be the sequence of first eigenvalues of problem* (1)–(2) *and u_s be eigenfunctions corresponding to λ_s. Assume that u_s converges weakly to u in $\overset{\circ}{W}{}^1_m(\Omega)$ and let $\lambda_s \to \lambda^*$ as $s \to \infty$. Then u_s strongly converges to u in $\overset{\circ}{W}{}^1_m(\Omega)$, λ^* is the first eigenvalue of the problem*

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|\nabla u|^{m-2}\frac{\partial u}{\partial x_i}\right) = \lambda^*|u|^{m-2}u \quad \text{in } \Omega, \tag{7}$$

$$u(x) = 0 \quad \text{on } \Gamma, \tag{8}$$

u is an eigenfunction corresponding to λ^*, and for s sufficiently large, the following error estimate holds:

$$\lambda_s - \lambda^* \le C\Big[C_m[B(x_0,\varepsilon_s)] \fint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\, dx\Big]^{1/m} \quad \text{for all } m \ge 2; \tag{9}$$

if furthermore m is a natural number, *then*

$$\lambda_s - \lambda^* \le C C_m[B(x_0,\varepsilon_s)] \fint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\, dx, \tag{10}$$

with the constant C depending only on the data. Here $\fint_B \star\, dx$ denotes the mean value of a function $\star$ over the set B.

For the proof of the theorem we shall introduce an auxiliary model problem following Skrypnik [5, Chap. 9]. Let $\psi \in C_0^\infty(B(x_0,1))$ be equal to 1 in $B(x_0, 1/2)$. For $\varepsilon_s < 1/2$, let $v_s \in W^1_m(B(x_0,1) \setminus E_s)$ be a solution of the problem

$$\begin{gathered} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|\nabla v|^{m-2}\frac{\partial v}{\partial x_i}\right) = 0 \quad \text{in } D_s = B(x_0,1) \setminus E_s, \\ v(x) = \psi(x - x_0) \quad \text{on } \partial D_s. \end{gathered} \tag{11}$$

Furthermore, we extend v_s to Ω by setting $v_s(x) = \psi(x - x_0)$ in $\Omega \setminus D_s$.

Arguing as in Skrypnik [5, Chap. 9, Theorem 2.2], we readily show that any solution v_s of problem (11) satisfies the following inequalities:

$$0 \le v_s(x) \le 1, \tag{12}$$

$$\int\limits_{B(x_0,1)} \left|\frac{\partial v_s}{\partial x}\right|^m dx \le \gamma C_m[E_s], \tag{13}$$

where γ is a constant independent of s.

2. Proof of Theorem 1. Let u_s be a sequence of eigenfunctions of (1)–(2) corresponding to the eigenvalues λ_s, such that $u_s \to u$ weakly in $\mathring{W}^1_m(\Omega)$ and $\lambda_s \to \lambda^*$. We look for a solution of (1)–(2) in the form

$$u_s(x) = u(x) + I_1^{(s)}(x) + R_s(x) \tag{14}$$

where $I_1^{(s)} = -v_s(x)u(x)$ and $R_s(x)$ is the remainder term.

In what follows we denote by C inessential constants depending only on the data and independent of s. We divide the proof of Theorem 1 in several steps.

Step 1. We start by showing that the function u_s as defined by (14) strongly converges to u in $\mathring{W}^1_m(\Omega)$. For that, we show that $I_1^{(s)}$ and R_s strongly converge to zero in $\mathring{W}^1_m(\Omega)$. We write

$$I_1^{(s)}(x) = (u^{(s)} - u(x))v_s(x) - u^{(s)}v_s(x),$$

where $u^{(s)}$ stands for the mean value of the function u over the ball $B(x_0, 2\varepsilon_s)$, i.e.,

$$u^{(s)} = \frac{1}{\operatorname{meas} B(x_0, 2\varepsilon_s)} \int\limits_{B(x_0,2\varepsilon_s)} u(x)\,dx.$$

We state the following inequality: for $x \in B(x_0, 2\varepsilon_s)$,

$$|u^{(s)} - u|^m \le C \fint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx. \tag{15}$$

Indeed, by Hölder's inequality and Poincaré's inequality,

$$\begin{aligned} |u^{(s)} - u(x)| &\le \fint\limits_{B(x_0,2\varepsilon_s)} |u(x) - u(y)|\,dy \\ &\le C\Big(\fint\limits_{B(x_0,2\varepsilon_s)} |u(x) - u(y)|^m\,dy\Big)^{1/m} \\ &\le C\Big(\fint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big)^{1/m}. \end{aligned}$$

This implies (15).

We have

$$\int_{\Omega} |\nabla I_1^{(s)}|^m \, dx \le \sup_{x \in B(x_0, 2\varepsilon_s)} |u^{(s)} - u(x)|^m \int_{B(x_0,2\varepsilon_s)} |\nabla v_s|^m \, dx + \int_{B(x_0,2\varepsilon_s)} v_s^m |\nabla u|^m \, dx + \int_{B(x_0,2\varepsilon_s)} |u^{(s)}|^m |\nabla v_s|^m \, dx \le C(\varepsilon_s^n + C_m[E_s]) \oint_{B(x_0,2\varepsilon_s)} |\nabla u|^m \, dx, \tag{16}$$

where we have used the inequalities (12), (13), (15) and Hölder's inequality. It follows from (16) that

$$\lim_{s\to\infty} \int_{\Omega} |\nabla I_1^{(s)}|^m \, dx = 0, \tag{16}$$

i.e., $I_1^{(s)}$ converges strongly to zero in $\overset{\circ}{W}{}_m^1(\Omega)$.

For the investigation of the behaviour of R_s, we note that by the expansion (14) and the properties of the functions v_s, we have $R_s \in \overset{\circ}{W}{}_m^1(\Omega_s)$ and $R_s = 0$ in E_s. Furthermore, in view of the weak convergence of u_s to u in $\overset{\circ}{W}{}_m^1(\Omega)$ and the strong convergence of $I_1^{(s)}$ to zero in $\overset{\circ}{W}{}_m^1(\Omega)$ it follows that R_s weakly converges to zero in $\overset{\circ}{W}{}_m^1(\Omega)$, hence R_s strongly converges to zero in $L_m(\Omega)$ by Sobolev's embedding theorem. We substitute $\varphi(x) = R_s(x)$ in the integral identity (3) and get

$$\int_{\Omega_s} \sum_{i=1}^n |\nabla u_s|^{m-2} \frac{\partial u_s}{\partial x_i} \frac{\partial R_s}{\partial x_i} \, dx = \lambda_s \int_{\Omega_s} |u_s|^{m-2} u_s R_s \, dx. \tag{18}$$

By the strong convergence of R_s to zero in $L_m(\Omega)$ it readily follows that

$$\lim_{s\to\infty} \lambda_s \int_{\Omega_s} |u_s|^{m-2} u_s R_s \, dx = 0. \tag{19}$$

Let us write the left-hand side of this equation as

$$\int_{\Omega_s} \sum_{i=1}^n |\nabla u_s|^{m-2} \frac{\partial u_s}{\partial x_i} \frac{\partial R_s}{\partial x_i} \, dx = I_{1s} + I_{2s}, \tag{20}$$

where

$$I_{1s} = \int_{\Omega} \sum_{i=1}^n \left[|\nabla u_s|^{m-2} \frac{\partial u_s}{\partial x_i} - |\nabla (u_s - R_s)|^{m-2} \frac{\partial (u_s - R_s)}{\partial x_i} \right] \frac{\partial R_s}{\partial x_i} \, dx,$$

$$I_{2s} = \int_{\Omega} \sum_{i=1}^n |\nabla I_1^{(s)}|^{m-2} \frac{\partial I_1^{(s)}}{\partial x_i} \frac{\partial R_s}{\partial x_i} \, dx.$$

We recall the following well-known inequality: For all $p, q \in \mathbb{R}^n$ with components p_i, q_i $(i = 1, \dots, n)$ respectively and $m \geq 2$,

$$\sum_{i=1}^{n} [|p|^{m-2} p_i - |q|^{m-2} q_i](p_i - q_i) \geq C|p - q|^m, \tag{21}$$

where C is a positive constant.

By (21), we have

$$C \int_{\Omega} |\nabla R_s|^m \, dx \leq I_{1s}. \tag{22}$$

By Hölder's inequality and (16) it readily follows that $\lim_{s\to\infty} I_{2s} = 0$. Hence from (18), (19), (20) and (22) we conclude that R_s strongly converges to zero in $\mathring{W}^1_m(\Omega)$. Thus we have shown the first assertion of the theorem.

STEP 2. Now we show that u and λ^* satisfy (7)–(8). Let $g \in C_0^\infty(\Omega)$. We consider the sequence of functions

$$g_s(x) = g(x) + L_{1s}(x), \qquad L_{1s}(x) = -g(x) v_s(x). \tag{23}$$

It is clear that $g_s \in \mathring{W}^1_m(\Omega_s)$. Furthermore, analogous arguments to those used in Step 1 show that L_{1s} strongly converges to zero in $\mathring{W}^1_m(\Omega)$. Hence g is the strong limit of g_s in $\mathring{W}^1_m(\Omega)$. Substituting $\varphi(x) = g_s(x)$ in the integral identity (3), and using the fact that u_s strongly converges to u in $\mathring{W}^1_m(\Omega)$, we readily show that, as $s \to \infty$, for all $g \in \mathring{W}^1_m(\Omega)$,

$$\int_{\Omega} \sum_{i=1}^{n} |\nabla u|^{m-2} \frac{\partial u}{\partial x_i} \frac{\partial g}{\partial x_i} \, dx = \lambda^* \int_{\Omega} |u|^{m-2} u g \, dx.$$

It is clear that if u_s is not identically zero then u fails to vanish identically. This means that λ^* is an eigenvalue of problem (7)–(8) and u the corresponding eigenfunction.

STEP 3. Further, we need to show that λ^* is indeed the first eigenvalue of (7)–(8), i.e., λ^* coincides with the number

$$\lambda = \inf_X \int_{\Omega} |\nabla v|^m \, dx, \qquad X = \{v \in \mathring{W}^1_m(\Omega) : \|v\|_{L_m(\Omega)} = 1\}. \tag{24}$$

The infimum is attained if v is furthermore an eigenfunction of (7)–(8). By the homogeneity of the equation (1), we can assume that $\|u_s\|_{L_m(\Omega_s)} = 1$, and subsequently that $\|u\|_{L_m(\Omega)} = 1$. From the definition of λ it is clear that $\lambda \leq \lambda^*$. We now prove the reverse inequality, that is,

$$\lambda^* \leq \lambda. \tag{25}$$

Hence the needed claim will be established. We consider the sequence U_s, $s = 1, 2, \dots$, of functions

$$U_s(x) = u(x) + I_1^{(s)}(x), \tag{26}$$

obtained from (14) by dropping the remainder term $R_s(x)$. Then u is a solution of (7)–(8) normalized as above. We introduce the sequence of functions

$$\varphi_s(x) = \frac{U_s(x)}{\|U_s\|_{L_m(\Omega_s)}}.$$

It is clear that $\varphi_s \in \mathring{W}_m^1(\Omega_s)$ and $\|\varphi_s\|_{L_m(\Omega_s)} = 1$, i.e., $\varphi_s \in X_s$. Thus from the definition of λ_s, we have

$$\lambda_s \le \int_{\Omega_s} |\nabla\varphi_s|^m\,dx = \frac{1}{\|U_s\|_{L_m(\Omega_s)}} \int_{\Omega_s} |\nabla U_s|^m\,dx. \tag{27}$$

Now we estimate the integral on the right-hand side of (27). We have

$$\int_{\Omega_s} |\nabla U_s|^m\,dx \le \int_{\Omega} |\nabla u|^m\,dx + H_{1s}, \tag{28}$$

where

$$H_{1s} = \int_{\Omega} [|\nabla U_s|^m - |\nabla u|^m]\,dx.$$

Next by Hölder's inequality we get

$$\begin{aligned} H_{1s} &\le C \int_{\Omega} [|\nabla U_s|^{m-1} + |\nabla u|^{m-1}]|\nabla(U_s - u)|\,dx \\ &\le C\Big\{ \int_{\Omega} [|\nabla U_s|^m + |\nabla u|^m]\,dx\Big\}^{(m-1)/m} \Big\{ \int_{\Omega} |\nabla(U_s - u)|^m\,dx\Big\}^{1/m}. \end{aligned} \tag{29}$$

The second factor in the last inequality approaches zero, since $U_s \to u$ strongly in $\mathring{W}_m^1(\Omega)$. Hence it follows that $H_{1s} \to 0$ as $s \to \infty$. Thus passing to the limit in (27) we obtain

$$\lambda^* \le \int_{\Omega} |\nabla u|^m\,dx = \lambda.$$

Hence the claim that λ^* is the first eigenvalue of (7)–(8) is proved.

STEP 4. Now we establish the error estimate (9). For that we continue the estimation of H_{1s} that was started in (29). In view of (16), for s sufficiently large we have

$$\begin{aligned} \int_{\Omega} |\nabla(U_s - u)|^m\,dx &\le \int_{\Omega} |\nabla I_{1s}|^m\,dx \\ &\le C\Big\{(\varepsilon_s^n + C_m[E_s]) \oint_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big\} \\ &\le CC_m[E_s] \oint_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx. \end{aligned} \tag{30}$$

This inequality and (29) imply that

$$H_{1s} \le C\Big[\Big[C_m[E_s] \oint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big]^{1/m} + C_m[E_s] \oint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big]. \tag{31}$$

Here we have used the fact that $\operatorname{meas} E_s \sim \varepsilon_s^n$ (since $\operatorname{meas} E_s \ne 0$) and relation (6). For s large enough,

$$\|U_s\|_{L_m(\Omega_s)} \ge 1 - \|U_s - u\|_{L_m(\Omega)} \ge 1 - K\|\nabla(U_s - u)\|_{\mathring{W}^1_m(\Omega)}, \tag{32}$$

where we have used Poincaré's inequality. Now by (27), (31) and (32), we see that for s sufficiently large

$$\lambda_s \le \lambda + C\Big\{C_m[E_s] \oint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big\}^{1/m}.$$

This proves (9).

Next let us assume that m is a natural number ≥ 2. In this case we can estimate the norm $\|U_s\|_{\mathring{W}^1_m(\Omega_s)}$ as

$$\int\limits_{\Omega_s} |\nabla U_s|^m\,dx \le \int\limits_{\Omega} |\nabla u|^m\,dx + H_{2s}, \tag{33}$$

with

$$H_{2s} = \int\limits_{\Omega} |\nabla I_{1s}|^m\,dx + \sum_{k=1}^{m-1} \binom{m}{k} \int\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^k |\nabla I_{1s}|^{m-k}\,dx,$$

where $\binom{m}{k} = m!/(k!(m-k)!)$. Applying Young's inequality to the terms in the sum, we get

$$\begin{aligned} H_{2s} &\le C\Big\{\int\limits_{B(x_0,2\varepsilon_s)} |\nabla I_{1s}|^m\,dx + \int\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx\Big\} \\ &\le CC_m[E_s] \oint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx. \end{aligned}$$

Thus in view of (33) and (32) and (27), we deduce that for large s,

$$\lambda_s - \lambda \le CC_m[E_s] \oint\limits_{B(x_0,2\varepsilon_s)} |\nabla u|^m\,dx.$$

This is the error estimate (10). The proof of Theorem 1 is complete.

Remark. The estimate (10) is sharper than (9) and when $m = 2$ it coincides modulo a multiplicative constant with the main term in the asymptotic estimate obtained by Samarskiĭ [4] who showed that when a set of small capacity E_s is removed from a domain $\Omega \subset \mathbb{R}^3$, the first eigenvalue λ_s of the

Laplace operator admits the asymptotically sharp estimate

$$\lambda_s - \lambda_0 \leq 4\pi\omega_s^2 c(E_s; \Omega) + O(c(E_s, \Omega)^2),$$

where ω_s is the maximal value of the first normalized eigenfunction of the Laplace operator in Ω over the set E_s, λ_0 is the first eigenvalue of the Laplace operator in Ω, and $c(E_s, \Omega)$ is the harmonic capacity of the set E_s relative to Ω.

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