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LOCAL-GLOBAL PRINCIPLE FOR ANNIHILATION OF GENERAL LOCAL COHOMOLOGY

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Abstract. Let A be a Noetherian ring, let M be a finitely generated A-module and let Φ be a system of ideals of A. We prove that, for any ideal $\mathfrak{a}$ in Φ, if, for every prime ideal $\mathfrak{p}$ of A, there exists an integer $k(\mathfrak{p})$, depending on $\mathfrak{p}$, such that $\mathfrak{a}^{k(\mathfrak{p})}$ kills the general local cohomology module $H^j_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ for every integer j less than a fixed integer n, where $\Phi_\mathfrak{p} := \{\mathfrak{a}_\mathfrak{p} : \mathfrak{a} \in \Phi\}$, then there exists an integer k such that $\mathfrak{a}^k H^j_\Phi(M) = 0$ for every $j < n$.

Introduction. Throughout this paper A denotes a commutative Noetherian ring (with non-zero identity), M a finitely generated A-module and $\mathfrak{a}$, $\mathfrak{b}$ ideals of A. The ith local cohomology module of M with respect to $\mathfrak{a}$ is defined by

$$H^i_\mathfrak{a}(M) = \varinjlim_n \operatorname{Ext}^i_A(A/\mathfrak{a}^n, M).$$

One of the basic problems concerning local cohomology theory is the following question, which will be referred to as the local-global principle for annihilation of local cohomology [15]: if, for every prime ideal $\mathfrak{p}$ of A, there exists an integer $k(\mathfrak{p})$, depending on $\mathfrak{p}$, such that $\mathfrak{b}^{k(\mathfrak{p})}$ kills the local cohomology module $H^j_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})$ for every integer j less than a fixed integer n, then does there exist an integer k such that $\mathfrak{b}^k H^j_\mathfrak{a}(M) = 0$ for every $j < n$? This problem has been investigated by Faltings [6, 7], Brodmann [5] and Raghavan [14, 15].

One of the interesting results in this connection is Faltings' Lemma [6, Lemma 3] which reads as follows: For an ideal $\mathfrak{a}$ of a Noetherian ring A, a finitely generated A-module M, and a positive integer n, the following conditions are equivalent:

(1) $H^j_\mathfrak{a}(M)$ is finitely generated for all $j < n$;
(2) $H^j_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for $j < n$ and all prime ideals $\mathfrak{p}$ of A;
(3) there exists an integer k such that $\mathfrak{a}^k H^j_\mathfrak{a}(M) = 0$ for all $j < n$.

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There are some generalizations of local cohomology theory. The following one is given in [2]. *A system of ideals* of A is a non-empty set Φ of ideals (of A) such that, whenever $\mathfrak{x}, \mathfrak{y} \in \Phi$, there exists $\mathfrak{z} \in \Phi$ with $\mathfrak{z} \subseteq \mathfrak{x}\mathfrak{y}$. Such a system Φ determines the Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(A) \to \mathcal{C}(A)$ (where $\mathcal{C}(A)$ denotes the category of all A-modules and A-homomorphisms). This is the subfunctor of the identity functor on $\mathcal{C}(A)$ for which

$$\Gamma_\Phi(G) = \{g \in G : \mathfrak{a}g = 0 \text{ for some } \mathfrak{a} \in \Phi\}$$

for each A-module G. Note that in [2], Γ_Φ is denoted by L_Φ and called the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the ith right derived functor of Γ_Φ is denoted by H^i_Φ. It is shown in [2] that the study of torsion theories over A is equivalent to studying this general local cohomology theory.

Our main theme in this paper is to prove the following theorem, which gives a generalization of Faltings' Lemma in the context of general local cohomology modules.

THEOREM. *Let Φ be a system of ideals of A. For an ideal $\mathfrak{a} \in \Phi$ and a positive integer n, the following conditions are equivalent*:

(1) *there exists an integer k such that $\mathfrak{a}^k H^j_\Phi(M) = 0$ for all $j < n$*;

(2) *for every prime ideal $\mathfrak{p}$ of A, there exists an integer $k(\mathfrak{p})$, depending on $\mathfrak{p}$, such that $\mathfrak{a}^{k(\mathfrak{p})} H^j_{\Phi_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $j < n$, where $\Phi_\mathfrak{p} := \{\mathfrak{a}_\mathfrak{p} : \mathfrak{a} \in \Phi\}$.*

The proof begins with a description of the finiteness dimension $f_\Phi(M)$ of M relative to Φ, in terms of sequence conditions. Next, we will show that the set $\mathrm{Ass}_A(H^{f_\Phi(M)}_\Phi(M)) \cap V(\mathfrak{a})$ is finite, which provides a powerful tool for proving the main theorem. Finally, we will show that in many cases the set $\mathrm{Ass}_A(H^{f_\Phi(M)}_\Phi(M))$ is not finite (see Example 7).

The results. Throughout, Φ will denote a system of ideals of A. A sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is said to be an $\mathfrak{a}$-*filter regular M-sequence* if

$$\mathrm{Supp}\left(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of A containing $\mathfrak{a}$. In the particular case when A is local, this notion has been studied in [16], [19], [20] and has led to some interesting results. It is easy to see that the analogue of [19, Appendix, 2(ii)] holds true whenever A is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so, if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular M-sequence. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular M-sequence of length n.

Our first two results will provide motivation for the formal introduction of the concept of finiteness dimension in its general sense. In fact, they suggest a probably close connection between finiteness dimension of general local cohomology and filter regular sequences.

LEMMA 1. *Let n be a positive integer and let $\mathfrak{a}$ be an ideal in Φ such that $\mathfrak{a}H^j_\Phi(M) = 0$ for all $j < n$. Then, for each i with $0 \le i < n$, we have*

$$\mathfrak{a}^{\alpha_i} H^j_\Phi\left(\frac{M}{(x_1, \ldots, x_i)M}\right) = 0$$

for all $j = 0, 1, \ldots, n - i - 1$, all $\mathfrak{b} \in \Phi$ and all $\mathfrak{b}$-filter regular M-sequences $x_1, \ldots, x_n$ where $\alpha_i = 3^i$.

Proof. This can be obtained by a slight modification of the proof of Lemma 2.1 in [11].

DEFINITION 2. A sequence $x_1, \ldots, x_n$ of elements of A is said to be an $\mathfrak{a}$-*weak M-sequence* if $(x_1, \ldots, x_{i-1})M :_M x_i \subseteq (x_1, \ldots, x_{i-1})M :_M \mathfrak{a}$ for all $i = 1, \ldots, n$. Clearly, every $\mathfrak{a}$-weak M-sequence in $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular M-sequence.

THEOREM 3. *For a system of ideals Φ of a Noetherian ring A, a finitely generated A-module M, and a positive integer n, the following conditions are equivalent*:

(1) *$H^j_\Phi(M)$ is finitely generated for all $j < n$*;

(2) *there exists an ideal $\mathfrak{a}$ in Φ such that $\mathfrak{a}H^j_\Phi(M) = 0$ for all $j < n$*;

(3) *there exists an ideal $\mathfrak{c}$ in Φ such that, for all $\mathfrak{b} \in \Phi$, every $\mathfrak{b}$-filter regular M-sequence of length n is a $\mathfrak{c}$-weak M-sequence.*

Proof. The equivalence (1)⇔(2) can be easily obtained by extending the proof of Faltings' Lemma (see [6, Lemma 3]) *mutatis mutandis* to this general case.

(2)⇒(3). We use induction on n. When $n = 1$, let x be a $\mathfrak{b}$-filter regular M-sequence, for some $\mathfrak{b} \in \Phi$. Since, by [11, 1.2], $H^0_\mathfrak{b}(M) \cong H^0_{(x)}(M)$ and $H^0_\mathfrak{b}(M) \subseteq H^0_\Phi(M)$, it follows from the assumption that $\mathfrak{a}H^0_{(x)}(M) = 0$. Therefore x is an $\mathfrak{a}$-weak M-sequence. Hence, in this case, it suffices to put $\mathfrak{c} = \mathfrak{a}$. Now suppose, inductively, that $n > 1$ and the result has been proved for positive integers smaller than n. So there exists an ideal $\mathfrak{d}$ in Φ such that, for all $\mathfrak{b} \in \Phi$, every $\mathfrak{b}$-filter regular M-sequence of length $n - 1$ is a $\mathfrak{d}$-weak M-sequence. Since Φ is a system of ideals of A, there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}^{3^{n-1}}\mathfrak{d}$. Let $\mathfrak{b} \in \Phi$ and let $x_1, \ldots, x_n$ be a $\mathfrak{b}$-filter regular M-sequence. Our inductive hypothesis ensures that $x_1, \ldots, x_{n-1}$ is a $\mathfrak{c}$-weak M-sequence. Set

$$M_{n-1} := \frac{M}{(x_1, \ldots, x_{n-1})M}.$$

Since x_n is a $\mathfrak{c}$-filter regular M_{n-1}-sequence, it follows from [11, 1.2] that $H^0_{\mathfrak{c}}(M_{n-1}) \cong H^0_{(x_n)}(M_{n-1})$. Hence, in view of Lemma 1, x_n is an $\mathfrak{a}^{3^{n-1}}$-weak M_{n-1}-sequence and so a $\mathfrak{c}$-weak M_{n-1}-sequence. Therefore $x_1, \ldots, x_n$ is a $\mathfrak{c}$-weak M-sequence. This completes the inductive step.

In order to prove the implication (3)⇒(2), by [4, 2.1], it suffices to show that for all $\mathfrak{b} \in \Phi$, $\mathfrak{c}H^j_{\mathfrak{b}}(M) = 0$ for all $j < n$. This immediately follows from the following lemma.

Lemma 4. *Let* $\mathfrak{b}$, $\mathfrak{c}$ *be ideals of* A *such that every* $\mathfrak{b}$*-filter regular* M*-sequence of length* n *is a* $\mathfrak{c}$*-weak* M*-sequence. Then* $\mathfrak{c}H^j_{\mathfrak{b}}(M) = 0$ *for all* $j < n$.

Proof. We prove this by induction on n. To begin, note that the case $n = 1$ is clear. Now suppose, inductively, that $n > 1$ and the result has been proved for positive integers smaller than n. To complete the inductive step, it is enough to show that $\mathfrak{c}H^{n-1}_{\mathfrak{b}}(M) = 0$. Set $\overline{M} := M/H^0_{\mathfrak{b}}(M)$. Since $H^{n-1}_{\mathfrak{b}}(M) \cong H^{n-1}_{\mathfrak{b}}(\overline{M})$, and every $\mathfrak{b}$-filter regular $\overline{M}$-sequence of length n is a $\mathfrak{c}$-weak $\overline{M}$-sequence, we may assume that M is a $\mathfrak{b}$-torsion-free A-module. So we can deduce that $\mathfrak{b}$ contains an element r which is a non-zero divisor on M. Let $m \in H^{n-1}_{\mathfrak{b}}(M)$. Since $H^{n-1}_{\mathfrak{b}}(M)$ is a $\mathfrak{b}$-torsion module, there exists a positive integer t such that $r^t m = 0$. Now from the exact sequence

$$0 \to M \xrightarrow{r^t} M \to \frac{M}{r^t M} \to 0$$

we obtain the induced exact sequence

$$H^{n-2}_{\mathfrak{b}}\left(\frac{M}{r^t M}\right) \to H^{n-1}_{\mathfrak{b}}(M) \xrightarrow{r^t} H^{n-1}_{\mathfrak{b}}(M),$$

which in turn, by applying the inductive hypothesis to the module $M/r^t M$, yields $\mathfrak{c}m = 0$. Now, it follows that $\mathfrak{c}H^{n-1}_{\mathfrak{b}}(M) = 0$. The inductive step is therefore complete.

The previous theorem provides some motivation for the following definition. Here, we adopt the convention that the infimum of the empty set of integers is ∞.

Definition 5. Let M be a finitely generated A-module. In the light of Theorem 3, we define the *finiteness dimension* $f_\Phi(M)$ of M relative to Φ by

$$\begin{aligned} f_\Phi(M) &= \inf\{i \in \mathbb{N} : H^i_\Phi(M) \text{ is not finitely generated}\} \\ &= \inf\{i \in \mathbb{N} : \mathfrak{a}H^i_\Phi(M) \neq 0 \text{ for all } \mathfrak{a} \in \Phi\}. \end{aligned}$$

Proposition 6. *Let* n *be a positive integer such that* $H^j_\Phi(M)$ *is finitely generated for all* $j < n$. *Then, for all* $\mathfrak{a} \in \Phi$, *the set* $\operatorname{Ass}_A(H^n_\Phi(M)) \cap V(\mathfrak{a})$ *is finite.*

Proof. It follows from Theorem 3 that there exists an ideal $\mathfrak{c}$ in Φ such that every $\mathfrak{b}$-filter regular M-sequence of length n is a $\mathfrak{c}$-weak M-sequence for all $\mathfrak{b} \in \Phi$. Let $\mathfrak{a} \in \Phi$. Since Φ is a system of ideals of A, we obtain an ideal $\mathfrak{c}' \in \Phi$ such that $\mathfrak{c}' \subseteq \mathfrak{a}\mathfrak{c}$. Hence

$$\mathrm{Ass}_A(H^n_\Phi(M)) \cap V(\mathfrak{a}) \subseteq \mathrm{Ass}_A(H^n_\Phi(M)) \cap V(\mathfrak{c}')$$

and every $\mathfrak{b}$-filter regular M-sequence of length n is a $\mathfrak{c}'$-weak M-sequence for all $\mathfrak{b} \in \Phi$. So we may assume without loss of generality that, for all $\mathfrak{b} \in \Phi$, every $\mathfrak{b}$-filter regular M-sequence is an $\mathfrak{a}$-weak M-sequence. Now, let $\mathfrak{p} \in \mathrm{Ass}_A(H^n_\Phi(M)) \cap V(\mathfrak{a})$. Then, in view of [3, 2.1], there exists $\mathfrak{b}' \in \Phi$ such that $\mathfrak{p} \in \mathrm{Ass}_A(H^n_\mathfrak{d}(M))$ for all ideals $\mathfrak{d} \in \Phi$ with $\mathfrak{d} \subseteq \mathfrak{b}'$. Assume that $\mathfrak{d}'$ is an ideal in Φ such that $\mathfrak{d}' \subseteq \mathfrak{b}'\mathfrak{a}$. Hence $\mathfrak{p} \in \mathrm{Ass}_A(H^n_{\mathfrak{d}'}(M))$. Now, let $x_1, \ldots, x_n$ be a $\mathfrak{d}'$-filter regular M-sequence. (Note that the existence of such sequences is explained at the beginning of this section.) By [11, 1.2], $H^n_{\mathfrak{d}'}(M) \cong H^0_{\mathfrak{d}'}(H^n_{(x_1,\ldots,x_n)}(M))$ and so $\mathfrak{p} \in \mathrm{Ass}_A(H^n_{(x_1,\ldots,x_n)}(M))$. Since the sequence $x_1, \ldots, x_n$ is an $\mathfrak{a}$-weak M-sequence, it becomes an $\mathfrak{a}$-filter regular M-sequence and hence, by [11, 1.2] again, we obtain $H^n_\mathfrak{a}(M) \cong H^0_\mathfrak{a}(H^n_{(x_1,\ldots,x_n)}(M))$. On the other hand, $\mathfrak{p} \in V(\mathfrak{a})$. Hence, by [1, p. 138], $\mathfrak{p} \in \mathrm{Ass}_A(H^n_\mathfrak{a}(M))$. So the required assertion follows from [12, Theorem B(β)].

It is an open problem, in local cohomology theory, whether every local cohomology module has finitely many associated primes (see [9]). Huneke and Sharp [10] have shown that, for each $j \in \mathbb{N}$, the set $\mathrm{Ass}_A(H^j_\mathfrak{a}(A))$ is finite if A is regular local ring of positive characteristic. Also, in [12, Theorem B(β)], it is shown that the set $\mathrm{Ass}_A(H^n_\mathfrak{a}(M))$ is finite whenever $H^j_\mathfrak{a}(M)$ is finitely generated for all $j < n$. Now we propose an example to show that, for positive integers n, the set of associated primes of $H^n_\Phi(A)$ is not finite in many cases. This example is based on the theory of modules of generalized fractions which was introduced in [17]. The reader is referred to [17, 18] for details of the following brief résumé of this theory.

Let k be a positive integer. We denote by $D_k(A)$ the set of all $k \times k$ lower triangular matrices with entries in A; for $H \in D_k(A)$, the determinant of H is denoted by $|H|$; and we use $^{\mathrm{T}}$ to denote matrix transpose.

A *triangular subset* of A^k is a non-empty subset U of A^k such that

(i) whenever $(u_1, \ldots, u_k) \in U$, then $(u_1^{n_1}, \ldots, u_k^{n_k}) \in U$ for all choices of positive integers $n_1, \ldots, n_k$, and

(ii) whenever $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ are in U, then there exist $(w_1, \ldots, w_k) \in U$ and $H, K \in D_k(A)$ such that

$$H[u_1, \ldots, u_k]^{\mathrm{T}} = [w_1, \ldots, w_k]^{\mathrm{T}} = K[v_1, \ldots, v_k]^{\mathrm{T}}.$$

Given such a U and an A-module M, R. Y. Sharp and H. Zakeri have constructed the module of generalized fractions $U^{-k}M$ of M with respect to U as follows.

Let $\alpha = ((u_1, \dots, u_k), x)$, $\beta = ((v_1, \dots, v_k), y) \in U \times M$. Then we write $\alpha \sim \beta$ when there exist $(w_1, \dots, w_k) \in U$ and $H, K \in D_k(A)$ such that

$$H[u_1, \dots, u_k]^{\mathrm{T}} = [w_1, \dots, w_k]^{\mathrm{T}} = K[v_1, \dots, v_k]^{\mathrm{T}}$$

and

$$|H|x - |K|y \in \Big(\sum_{i=1}^{k-1} Aw_i\Big)M.$$

This is an equivalence relation on the set $U \times M$. For $x \in M$ and $(u_1, \dots, u_k) \in U$, we define the formal symbol $x/(u_1, \dots, u_k)$ to be the equivalence class of $((u_1, \dots, u_k), x)$ and let $U^{-k}M$ denote the set of all these equivalence classes. Then $U^{-k}M$ has an A-module structure described as follows. If $x, y \in M$ and $(u_1, \dots, u_k), (v_1, \dots, v_k) \in U$, then

$$x/(u_1, \dots, u_k) + y/(v_1, \dots v_k) = (|H| + |K|)/(w_1, \dots, w_k)$$

for any choice of $(w_1, \dots, w_k) \in U$ and $H, K \in D_k(A)$ such that

$$H[u_1, \dots, u_k]^{\mathrm{T}} = [w_1, \dots, w_k]^{\mathrm{T}} = K[v_1, \dots, v_k]^{\mathrm{T}}.$$

Also, with the above notation, and for $a \in A$,

$$a(x/(u_1, \dots, u_k)) = ax/(u_1, \dots, u_k).$$

Example 7. Let A be a Gorenstein ring of dimension at least 3. For $i = 1, 2$, set

$$U_i := \{(x_1, \dots, x_i) : x_1, \dots, x_i \text{ is an } A\text{-weak } A\text{-sequence}\}.$$

It is easy to see that U_i is a triangular subset of A; and so one can construct the module of generalized fractions $U_i^{-i}A$ of A with respect to U_i. By [18, 5.8] and [13, 18.8], $U_2^{-2}A \cong \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} E(A/\mathfrak{p})$. Therefore the set $\mathrm{Ass}_A(U_2^{-2}A)$ is not finite. Now, for $i = 1, 2$, set $\Phi_i := \{x_1A + \ldots + x_iA : (x_1, \dots, x_i) \in U_i\}$. It is routine to check that Φ_i is a system of ideals of A, that $H^0_{\Phi_1}(A) = 0$ and that $H^0_{\Phi_2}(A) = H^1_{\Phi_2}(A) = 0$. Moreover, by using the exactness theorem in [18, 3.3], the complex

$$0 \to A \to U_1^{-1}A \to U_2^{-2}A$$

is exact. Hence, by [8, 3.3], we have the exact sequence

$$0 \to U_1[1]^{-2}A \to U_2^{-2}A \to U_2[1]^{-3}A \to 0$$

where the triangular set $U_i[1]$ is as follows:

$$U_i[1] = \{(x_1, \dots, x_i, 1) \in A^{i+1} : (x_1, \dots, x_i) \in U_i\}$$

for $i = 1, 2$. Hence either the set $\mathrm{Ass}_A(U_1[1]^{-2}A)$ or the set $\mathrm{Ass}_A(U_2[1]^{-3}A)$ is not finite. Therefore, by [4, Theorem], either the set $\mathrm{Ass}_A(H^1_{\Phi_1}(A))$ or the set $\mathrm{Ass}_A(H^2_{\Phi_2}(A))$ is not finite, as claimed.

Definition 8. Let Φ be a system of ideals of A and let $\mathfrak{a}$ be an ideal of A. We define the $\mathfrak{a}$-*finiteness dimension* $f_{\Phi}^{\mathfrak{a}}(M)$ of M relative to Φ by

$$f_{\Phi}^{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N} : \mathfrak{a} \not\subseteq \sqrt{(0 : H_{\Phi}^{i}(M))}\}.$$

Note that $f_{\Phi}^{\mathfrak{a}}(M)$ is either a positive integer or ∞.

We are now in a position to prove the main theorem, mentioned in the Introduction.

Theorem 9. *Let $\mathfrak{a} \in \Phi$ and let $n \in \mathbb{N}$. Then $f_{\Phi}^{\mathfrak{a}}(M) \geq n$ if and only if $f_{\Phi_{\mathfrak{p}}}^{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq n$ for all prime ideals $\mathfrak{p}$ of A, where $\Phi_{\mathfrak{p}} := \{\mathfrak{a}_{\mathfrak{p}} : \mathfrak{a} \in \Phi\}$.*

Proof. Only the "if" part requires proof. We prove it by induction on n. To begin, let $n = 1$. Suppose that $\mathrm{Ass}_A(H_{\Phi}^{0}(M)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Since for all prime ideals $\mathfrak{p}$ of A, $f_{\Phi_{\mathfrak{p}}}^{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ for each $i = 1, \ldots, t$, there exists $k_i \in \mathbb{N}$ such that

$$\mathfrak{a}^{k_i}(H_{\Phi}^{0}(M))_{\mathfrak{p}_i} = 0.$$

Let $k = \max\{k_1, \ldots, k_t\}$. Then

$$(\mathfrak{a}^{k} H_{\Phi}^{0}(M))_{\mathfrak{p}_i} = 0 \quad \text{for all } i = 1, \ldots, t.$$

Therefore $\mathfrak{a}^k H_{\Phi}^{0}(M) = 0$ since $\mathrm{Ass}_A(\mathfrak{a}^k H_{\Phi}^{0}(M)) \subseteq \mathrm{Ass}_A(H_{\Phi}^{0}(M))$. Now suppose, inductively, that $n > 1$ and the result has been proved for smaller values of n. It follows from the assumption that for each prime ideal $\mathfrak{p}$ of A there exists an integer $k(\mathfrak{p})$ such that $\mathfrak{a}^{k(\mathfrak{p})} H_{\Phi_{\mathfrak{p}}}^{n-1}(M_{\mathfrak{p}}) = 0$. So $\mathrm{Supp}(H_{\Phi}^{n-1}(M)) \subseteq V(\mathfrak{a})$. Hence, by the inductive hypothesis in conjunction with Theorem 3 and Proposition 6, $\mathrm{Ass}_A(H_{\Phi}^{n-1}(M))$ is finite. We are therefore able to complete the inductive step by applying arguments similar to those used when $n = 1$. This completes the proof.

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