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RINGS WHOSE MODULES ARE FINITELY GENERATED OVER THEIR ENDOMORPHISM RINGS

BY

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Abstract. A module M is called finendo (cofinendo) if M is finitely generated (respectively, finitely cogenerated) over its endomorphism ring. It is proved that if R is any hereditary ring, then the following conditions are equivalent: (a) Every right R-module is finendo; (b) Every left R-module is cofinendo; (c) R is left pure semisimple and every finitely generated indecomposable left R-module is cofinendo; (d) R is left pure semisimple and every finitely generated indecomposable left R-module is finendo; (e) R is of finite representation type. Moreover, if R is an arbitrary ring, then (a)⇒(b)⇔(c), and any ring R satisfying (c) has a right Morita duality.

1. Introduction. Modules which are finitely generated over their endomorphism rings, also called *finendo modules*, were introduced by Faith [16], who used them to characterize several classes of rings (e.g. right self-injective rings, right pseudo-Frobenius rings, prime right Goldie rings). The finendo condition occurs naturally in several contexts, in general module theory and representation theory. Special classes of finendo modules, studied recently in the literature, include endofinite modules [11], product-complete modules [27], endocoherent modules [2], and tilting modules [3, 4].

One of the main questions we will consider in this paper is to characterize rings R with the property that all right R-modules are finendo. More specifically, we will try to answer the question raised in [13] whether such a ring R has *finite representation type*, i.e. R is an artinian ring with finitely many isomorphism classes of finitely generated indecomposable left and right modules. Our study of this question is related to and motivated by works of several authors that investigate the relationship between representation-theoretic properties of rings and endoproperties of their modules.

A right R-module M is called *endofinite* if M has finite length as a left module over its endomorphism ring. An important starting point, obtained independently by Crawley-Boevey [10], Huisgen-Zimmermann and Zimmermann [26] and Prest [29], is that every right R-module is endofinite if and

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only if R is of finite representation type. In view of this result, it is natural to ask if finite representation type of a ring R can be characterized by more restricted endo-chain conditions on right (or left) R-modules. In this direction, it has been proved recently that a ring R has finite representation type if and only if every pure-projective right R-module is endoartinian [14, Proposition 4.10], if and only if every pure-injective right R-module is endonoetherian [15, Theorem 5.8]. On the other hand, the ascending (descending) chain condition on finite matrix subgroups of all right R-modules is equivalent to R having left (respectively, right) pure global dimension zero, i.e. R is a left (respectively, right) pure semisimple ring (see [26, 29]).

As a dual notion to finendo modules, a left (or right) R-module M is said to be *cofinendo* if M is finitely cogenerated as a module over its endomorphism ring. Thus the finendo and cofinendo conditions are generalizations of the endonoetherian and endoartinian conditions, respectively. It was shown in [13, Theorem 3.10] that a ring R has finite representation type if and only if every right R-module is both finendo and cofinendo. However, it was left open whether the finendo condition or cofinendo condition alone would be sufficient for the result to hold (see [13, Questions 1 and 2, pp. 122–123]).

We will see in this paper that, for any ring R, if all right R-modules are finendo, then all left R-modules are cofinendo, and the latter condition holds precisely when R is left pure semisimple and every finitely generated indecomposable left R-module is cofinendo (Theorem 2.11, Corollary 2.12).

Recall that a ring R is *left pure semisimple* if every left R-module is a direct sum of finitely generated modules (see, e.g., [18, 33, 34]). Left and right pure semisimple rings are precisely the rings of finite representation type (see [6, 19, 31]). However, it has been a long-standing open problem, known as the *pure semisimplicity conjecture*, whether left pure semisimple rings are also right pure semisimple (see [24, 38] for historical surveys on the conjecture). Therefore, the question of characterizing rings R with all right R-modules finendo can be regarded as a restricted form of the pure semisimplicity conjecture.

A central part of this paper is devoted to the study of left pure semisimple rings R such that every finitely generated indecomposable left R-module is cofinendo or finendo, respectively. We show that if R is a left pure semisimple hereditary ring, then either the cofinendo or finendo condition on all finitely generated indecomposable left R-modules implies finite representation type. As a consequence, we obtain positive answers to [13, Questions 1 and 2, pp. 122–123] in the hereditary case (Theorem 3.5). For general rings R, the situation seems more complicated, and we are able to show that if R is left pure semisimple with all finitely generated indecomposable left R-modules cofinendo, then R has a right Morita duality, and the quotient ring $R/(J(R))^2$ is of finite representation type (Corollaries 4.2 and 4.3).

Our method of dealing with hereditary rings will be based on ideas developed by Simson in [35, 37, 40] that allow one to reduce the question to certain triangular matrix rings of the form $\left(\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right)$ induced by division rings F and G and a bimodule ${}_G B_F$, and the use of reflection functors (see, e.g., [30, 35, 37]) will be essential. We will also apply Auslander's theory of Grassmannians [5] and Herzog's ideas [21], to reduce our study of an arbitrary ring to the case of a hereditary (triangular matrix) ring.

The paper is organized as follows. In Section 2, we discuss general properties of finendo and cofinendo modules, and characterize rings with all left modules cofinendo. In Section 3, we prove our main result on hereditary rings. Finally, Section 4 is devoted to the same questions for general rings.

2. Finendo and cofinendo modules. Throughout this paper, R is an associative ring with identity. We denote by R-mod the category of finitely presented left R-modules, and by R-Mod the category of all left R-modules. The corresponding categories of right R-modules are denoted by mod-R and Mod-R. Homomorphisms between R-modules are assumed to operate on the side opposite to the scalars. A right R-module M_R can be regarded in a natural way as a left $\mathrm{End}(M_R)$-module, and similarly a left R-module ${}_R N$ is a right $\mathrm{End}({}_R N)$-module. We refer the reader to [1, 8, 28, 36, 42, 43] for general properties of rings, modules, and categories, and for all undefined notions used in the text.

Recall that a right (or left) R-module M is *finendo* if M is finitely generated as a module over its endomorphism ring (see [16, 17]). Dually, M is said to be *cofinendo* if it is finitely cogenerated over $\mathrm{End}(M_R)$. (Note that the term "cofinendo module" was also used in [3], but with a meaning different from ours.) A module M is *endofinite* (*endoartinian*, *endonoetherian*) if M is of finite length (artinian, noetherian, respectively) as a module over its endomorphism ring. For a right R-module M, a subgroup L of the Abelian group M is called a *matrix subgroup* of M if it is of the form

$$L = \mathrm{Hom}_R(Y, M)(x) = \{f(x) \mid f \in \mathrm{Hom}_R(Y, M)\}$$

where Y is a right R-module and $x \in Y$. If the module Y is finitely presented, then L is called a *finite matrix subgroup* of M.

Following [27], a module M is *product-complete* provided every product of copies of M is a direct summand of a direct sum of copies of M. It is well-known that every product-complete module is finendo (see [27, Proposition 4.2]). The following characterization reflects a similar behavior of finendo modules with regard to direct products.

PROPOSITION 2.1. *Let R be any ring and M a right R-module. Then M is finendo if and only if M generates any product of copies of M.*

Proof. Suppose first that M is finendo with endomorphism ring S, and let I be any index set. There is a finite generating set $\{u_1, \dots, u_n\}$ of the left S-module M. To show that M generates M^I, it suffices to see that the trace of M^n on M^I is M^I. If we take any element $x = (x_i)_{i\in I}$ of the direct product M^I, then for each $i \in I$ we have $x_i = f_{1i}u_1 + \dots + f_{ni}u_n$ for some endomorphisms $f_{1i}, \dots, f_{ni}$ of M. Then we define the homomorphism $g : M^n \to M^I$ as follows: $g = (g_i)_{i\in I}$, where g_i is a homomorphism from M^n to the ith component M of M^I such that if $y = (y_1, \dots, y_n) \in M^n$, then $g_i(y) = f_{1i}y_1 + \dots + f_{ni}y_n$. It is clear that $g_i(u_1, \dots, u_n) = x_i$, and hence $g(u_1, \dots, u_n) = x$, proving our claim.

Conversely, suppose that M generates any product of copies of M. Consider the module M^M and the element of this product whose x-component is x. By hypothesis, there exists some homomorphism $g : M^n \to M^M$ and an element $(u_1, \dots, u_n)$ of M^n whose image under g is the given element $(x_x)_x \in M^M$. This clearly shows that elements $u_1, \dots, u_n$ generate M as a module over its endomorphism ring. ∎

We deduce the following immediate consequence.

COROLLARY 2.2. *Let R be any ring. The property of being finendo as a right R-module is preserved under taking finite direct sums, and arbitrary direct products of copies of a single module. Moreover, if M is a generator in* Mod-R, *then M is finendo.*

We now turn to the question of when a direct sum of cofinendo modules is again cofinendo. The next result shows that this holds for finite direct sums of indecomposable modules of finite length. For any R-module M with endomorphism ring S, the S-socle of M will be referred to as the *endosocle* of M, and denoted as $\mathrm{Esoc}(M)$.

LEMMA 2.3. *Let R be any ring and $M = \bigoplus_{k=1}^n M_k$ be a finite direct sum of left R-modules with local endomorphism rings. If each M_k has a finitely generated endosocle, then M has a finitely generated endosocle. In particular, if each M_k is cofinendo indecomposable of finite length, then $M = \bigoplus_{k=1}^n M_k$ is cofinendo.*

Proof. Let B be the endosocle of $M = \bigoplus_{k=1}^n M_k$. Because each M_k has a local endomorphism ring, it follows from [25, Lemma B] that there is a decomposition $B = \bigoplus_{k=1}^n B_k$ where $B_k = B \cap M_k$, and each B_k is the intersection of all kernels of non-isomorphisms from M_k to M_j with $j = 1, \dots, n$. In particular, B_k is an endosubmodule of M_k. Note that clearly each B_k is contained in the endosocle of M_k which is finitely generated by hypothesis. Thus, B_k is finitely generated over $\mathrm{End}(M_k)$. But each element of $\mathrm{End}(M_k)$ can also be viewed as an element of $S = \mathrm{End}(M)$. It follows that each B_k has a finite generating set as a left S-module. Therefore the endosocle B of M has a finite generating set as left S-module, as required.

For the second part of the lemma, assume that each M_k is cofinendo indecomposable of finite length. The left R-module $M = \bigoplus_{k=1}^{n} M_k$ is of finite length, hence it has a semiprimary endomorphism ring. Thus M has an essential endosocle. By the above, we see that M has a finitely generated endosocle. Therefore M is finitely cogenerated over its endomorphism ring. ■

Following Auslander [7], for a left R-module N with $S = \mathrm{End}({}_RN)$, the *local dual* of N is defined as the right R-module $D(N) = \mathrm{Hom}_S(N_S, C_S)$, where C_S is a minimal injective cogenerator of Mod-S. The following results give useful connections between the finendo and cofinendo conditions for certain left and right R-modules through the local duality.

PROPOSITION 2.4. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules. Let $M_i = D(N_i)$ be the local dual of N_i and $M = \bigoplus_{i\in I} M_i$. Then M is cofinendo if and only if N is finendo and $N/\mathrm{Rad}(N_S)$ is a semisimple right S-module, where S is the endomorphism ring of N.*

Proof. See [15, Proposition 4.8]. ■

PROPOSITION 2.5. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules each with a local endomorphism ring. Let $M_i = D(N_i)$ be the local dual of N_i and $M = \bigoplus_{i\in I} M_i$. If M is finendo, then N is cofinendo.*

Proof. See [15, Proposition 4.9]. ■

The second part of the result below was proved in [15, Proposition 4.10] under the additional hypothesis that the left functor ring A of R is right semiartinian. We now give an alternative proof for arbitrary rings. Note that when ${}_RN = R$, the result says that if the ring R has a finitely generated essential right socle, then the minimal injective cogenerator E of Mod-R is finendo, a fact that also follows from Beachy [9].

PROPOSITION 2.6. *Let R be any ring, N a left R-module, M a right R-module and suppose that the lattice of endosubmodules of N is anti-isomorphic to the lattice of matrix subgroups of M. If N is cofinendo, then M is finendo. In particular, if $N = \bigoplus_{i\in I} N_i$ is a direct sum of finitely presented left R-modules, set $M_i = D(N_i)$, the local dual of N_i, and $M = \bigoplus_{i\in I} M_i$. If N is cofinendo, then M is finendo.*

Proof. Since N is cofinendo, any family of endosubmodules of N which has zero intersection contains a finite subfamily having zero intersection (see, e.g., [43, 14.7]). It follows that the lattice of matrix subgroups of M is compact, i.e. whenever M is the join of a family of matrix subgroups, it is the join of a finite subfamily.

Let $M = \sum_{j \in J} X_j$ be a sum of finitely generated endosubmodules of M. Since finitely generated endosubmodules are matrix subgroups, the join in the lattice of matrix subgroups of the X_j is again M, and thus M is the join of a finite subfamily $\{X_j \mid j \in F\}$. But the usual sum of finitely many finitely generated endosubmodules of M is a finitely generated endosubmodule, hence a matrix subgroup of M, so the join of the finite subfamily $\{X_j \mid j \in F\}$ is $\sum_{j \in F} X_j$. Therefore $M = \sum_{j \in F} X_j$ is finitely generated as a module over its endomorphism ring.

Now, if $N = \bigoplus_{i \in I} N_i$ is a direct sum of finitely presented left R-modules, and $M = \bigoplus_{i \in I} D(N_i)$, then [15, Theorem 4.1] implies that the lattice of endosubmodules of N is anti-isomorphic to the lattice of matrix subgroups of M, and thus the second assertion of the proposition follows. ∎

Remark 2.7. We note that the finendo and cofinendo properties are not preserved under direct summands, in general. Indeed, if M is any left module over an arbitrary ring R and M is not finendo, then the left R-module $R \oplus M$ is a generator in R-Mod, hence it is finendo by Corollary 2.2. This shows that direct summands of finendo modules need not be finendo, in general. As for the cofinendo property, let R be a left artinian hereditary ring that is not right artinian (for example, $R = \begin{pmatrix} F & F \\ 0 & K \end{pmatrix}$, where K is a subfield of a field F, and F is infinite-dimensional over K). Then there is a finitely generated projective left R-module M that is not finendo (see Lemma 3.3), and the left R-module $L = R \oplus M$ is finendo, as above. By Proposition 2.4, the local dual $D(M)$ of M is not cofinendo. On the other hand, it follows from [32, Theorem 1.6] that $D(L) = D(R \oplus M) \cong D(M) \oplus D(R)$, because the endomorphism ring of $R \oplus M$ is semiprimary. Note that, since L is finitely generated over its semiprimary endomorphism ring, L is semisimple modulo its endoradical (see, e.g., [43, 42.3]). Applying again Proposition 2.4, we find that $D(R \oplus M)$ is cofinendo. Hence the cofinendo right R-module $D(M) \oplus D(R)$ contains a direct summand $D(M)$ that is not cofinendo.

In this paper, we are interested in characterizing rings over which all or certain classes of right (or left) modules are finendo or cofinendo. The following results will be used.

Lemma 2.8. *Let R be a ring. Then R is right artinian if and only if every quotient ring R/I for a two-sided ideal I of R has a finitely generated essential right socle.*

Proof. See Beachy [9]. ∎

Corollary 2.9. *Let R be a left artinian ring such that every finitely generated indecomposable left R-module is cofinendo. Then R is right artinian.*

Proof. By Lemma 2.3, every finitely generated left R-module is cofinendo. In particular, for every two-sided ideal I of R, the ring R/I is finitely cogenerated as a right module over itself. Hence Lemma 2.8 shows that R is right artinian. ■

LEMMA 2.10. *Let R be a right artinian ring. Then R is left pure semisimple if and only if every pure-projective right R-module is finendo.*

Proof. See [13, Lemma 3.8]. ■

We now give a characterization of rings over which every left module is cofinendo.

THEOREM 2.11. *The following conditions are equivalent for a ring R:*

(a) *Every left R-module is cofinendo.*
(b) *R is left pure semisimple, and every finitely generated indecomposable left R-module is cofinendo.*
(c) *R is left pure semisimple, and every indecomposable pure-injective right R-module is finendo.*
(d) *R is left pure semisimple, and every pure-injective right R-module is finendo.*
(e) *R is left artinian, and every direct sum of a family of indecomposable pure-injective right R-modules is finendo.*

Proof. (a)⇒(b). Suppose that (a) holds. Then, in particular, the quotient ring R/I has a finitely generated essential right socle for every two-sided ideal I of R. Hence R is right artinian by Lemma 2.8. Let $N = \bigoplus_{i\in I} N_i$ be any direct sum of finitely presented right R-modules N_i. Consider the direct sum $M = \bigoplus_{i\in I} M_i$, where $M_i = D(N_i)$ is the local dual of N_i. Then by Proposition 2.4, the fact that the left R-module M is cofinendo implies that N is finendo. Therefore, R is right artinian and every pure-projective right R-module is finendo (keep in mind that over the right artinian ring R, pure-projective right R-modules are precisely the direct sums of finitely presented right R-modules). Applying Lemma 2.10, we conclude that R is left pure semisimple.

(b)⇒(a). Assume that R is left pure semisimple and every finitely generated indecomposable left R-module is cofinendo. Let M be any left R-module; we need to show that M is finitely cogenerated over its endomorphism ring. By [25, Lemma A], because M is Σ-pure-injective, M is semiartinian as a module over its endomorphism ring. In particular, M has an essential endosocle. Therefore it is sufficient to show that the endosocle B of M is finitely generated. Let $M = \bigoplus_{i\in I} M_i$ be an indecomposable decomposition of M, each M_i having a local endomorphism ring. By [25, Lemma B(2)], there is a direct sum decomposition $B = \bigoplus_{i\in I} B_i$, where $B_i = B\cap M_i$, and B_i is the $\mathrm{End}(M_i)$-submodule of M_i consisting of all elements of M_i annihilated by all non-isomorphisms in $\bigcup_{j\in I}\mathrm{Hom}_R(M_i, M_j)$. We will first show that there are only finitely many non-isomorphic modules M_i such that B_i is non-zero.

Since R is left pure semisimple, by [23, Theorem A] (cf. [12, Corollary 3.7]), the family $\{M_i \mid i \in I\}$ has a finite cogenerating set $\mathcal{A} = \{M_{i_1}, \ldots, M_{i_n}\}$, i.e. each M_i can be embedded into a finite direct sum of modules in $\mathcal{A}$. Suppose that some M_i is not isomorphic to any of the modules in $\mathcal{A}$. Then the reject of the family $\mathcal{A}$ in M_i is zero, i.e. the intersection of all kernels of homomorphisms from M_i to the modules in $\mathcal{A}$ is zero. Note that homomorphisms from M_i to the modules in $\mathcal{A}$ are non-isomorphisms. It follows from the description of B_i above that B_i is zero, proving the claim.

By grouping the isomorphic indecomposable modules M_i together, we obtain a disjoint partition $I = \bigcup_{\alpha\in\Omega} I_\alpha$ such that if $i, j \in I_\alpha$ then $M_i \cong M_j$, and if $i \in I_\alpha$, $j \in I_\beta$ with $\alpha \neq \beta$, then $M_i \not\cong M_j$. Set $N_\alpha = \bigoplus_{i\in I_\alpha} M_i$. Then $M = \bigoplus_{\alpha\in\Omega} N_\alpha$. Moreover, as observed above, $B \cap N_\alpha \neq 0$ for only finitely many distinct $\alpha \in \Omega$. Note that, by hypothesis, each M_i has a finitely generated endosocle, and [25, Lemma B(1)] implies that for any index set K, $\mathrm{Esoc}(M_i^{(K)}) = (\mathrm{Esoc}(M_i))^{(K)}$ is also finitely generated. Thus each N_α has a finitely generated endosocle. For each $i \in I$, because the endosocle of M_i is the intersection of all kernels of non-isomorphisms from M_i to M_i, it is clear that B_i is contained in the endosocle of M_i. Hence

$$B \cap N_\alpha = \bigoplus_{i\in I_\alpha}(B \cap M_i) \subseteq \bigoplus_{i\in I_\alpha}\mathrm{Esoc}(M_i) = \mathrm{Esoc}\Big(\bigoplus_{i\in I_\alpha} M_i\Big) = \mathrm{Esoc}(N_\alpha)$$

and so $B \cap N_\alpha$ is a finitely generated endosubmodule of the endosocle of N_α. Keeping in mind that each endomorphism of N_α can also be identified in a natural way with an endomorphism of M, it follows easily that $B = \bigoplus_{\alpha\in\Omega}(B \cap N_\alpha)$, with only finitely many non-zero terms, is indeed finitely generated over the endomorphism ring S of M.

(b)⇒(c). Suppose that (b) holds. Since R is left pure semisimple, we know that every indecomposable pure-injective right R-module is the local dual of a finitely presented indecomposable left R-module (see [15, Proposition 5.6]). Thus it follows from Proposition 2.6 that every indecomposable pure-injective right R-module is finendo.

(c)⇒(d). Assume that (c) holds. If N is any finitely generated indecomposable left R-module, then its local dual $D(N)$ is an indecomposable pure-injective right R-module. Since $D(N)$ is finendo by hypothesis, Proposition 2.5 implies that N is cofinendo. This shows that (b) holds.

Now let M be any pure-injective right R-module. By [15, Proposition 5.6] M is the pure-injective envelope of a direct sum $\bigoplus_{i\in I} D(N_i)$, each N_i being a finitely presented indecomposable left R-module. By the implication (b)⇒(a) proved above, the left R-module $\bigoplus_{i\in I} N_i$ is cofinendo. Let A be the left functor ring of R, and $T : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ be the canonical embedding functor (see, e.g., [15, p. 375] for definitions and basic properties). Note that $T(M)$ is the injective hull of $T(\bigoplus_{i\in I} D(N_i)) \cong \bigoplus_{i\in I} T(D(N_i))$. By [15, Lemma 2.1], the torsion theory of $\mathrm{Mod}(A)$ cogenerated by $T(M)$ is the same as the torsion theory cogenerated by $\bigoplus_{i\in I} T(D(N_i))$. By applying [15, Proposition 3.2] to M and $\bigoplus_{i\in I} D(N_i)$, we see that the lattice of matrix subgroups of M is isomorphic to the lattice of matrix subgroups of $\bigoplus_{i\in I} D(N_i)$, which in turn is anti-isomorphic to the lattice of endosubmodules of $N = \bigoplus_{i\in I} N_i$ (see [15, Theorem 4.1]). Since N is cofinendo, we conclude that M is finitely generated as a module over its endomorphism ring, by Proposition 2.6.

(d)⇒(e). Suppose that (d) holds. Clearly (d)⇒(c), and we have already established (c)⇒(b) in the proof of (c)⇒(d) above. Hence (b) holds. Then we know by the implication (b)⇒(a) that every left R-module is cofinendo. On the other hand, because R is left pure semisimple, we know by [15, Proposition 5.6] that every direct sum of indecomposable pure-injective right R-modules is a direct sum of local duals of finitely presented indecomposable left R-modules. Therefore it follows from Proposition 2.6 that every direct sum of indecomposable pure-injective right R-modules is finendo, proving (e).

(e)⇒(a). Suppose that (e) holds. Let N be any pure-projective left R-module. Because R is left artinian, there is a direct sum decomposition $N = \bigoplus_{i\in I} N_i$ with each N_i finitely presented with local endomorphism ring. Take the local dual $M_i = D(N_i)$ of each N_i, and set $M = \bigoplus_{i\in I} M_i$. It is well-known that each M_i is an indecomposable pure-injective right R-module. Hence M is finendo by (e), and Proposition 2.5 implies that N is cofinendo. Thus R is a left artinian ring with the property that each pure-projective left R-module is cofinendo. It follows from [13, Lemma 3.9] that R is left pure semisimple. Thus every left R-module is pure-projective and (a) follows. ∎

We deduce the following consequence that gives a relationship between the two open questions mentioned in the introduction.

COROLLARY 2.12. *Let R be a ring, and consider the following two conditions.*

(a) *Every right R-module is finendo.*
(b) *Every left R-module is cofinendo.*

Then we have the implication (a)⇒(b).

Proof. Suppose that (a) holds. In particular, every quasi-injective right R-module is finendo, hence we know by Faith [16, Theorem 17A] that R is right artinian. Because every pure-projective right R-module is finendo, by Lemma 2.10 it follows that R is left pure semisimple. Now Theorem 2.11 above shows that every left R-module is cofinendo. ∎

3. The case of hereditary rings. In this section, we study left pure semisimple hereditary rings R with the property that every finitely generated indecomposable left R-module is cofinendo, or the property that every finitely generated indecomposable left R-module is finendo. A related version of these properties was also studied by Simson [35, Corollary 3.2]. We show that any left pure semisimple hereditary ring satisfying either of the above properties is of finite representation type, and we give positive answers to [13, Questions 1 and 2, pp. 122–123] in the hereditary case.

The proof of our result is based on some lemmas. The first lemma below is valid without the hereditary hypothesis.

LEMMA 3.1. *Let A and B be Morita equivalent rings. Then every finitely generated left A-module is finendo* (*cofinendo*) *if and only if every finitely generated left B-module is finendo* (*respectively*, *cofinendo*).

Proof. Let $G : B\text{-Mod} \to A\text{-Mod}$ be a category equivalence, let M be any finitely generated left B-module, and set $N = G(M)$. If ${}_AP$ is any finitely generated projective left A-module, then $\mathrm{Hom}_A(P, N)$ is a direct summand of a finite direct sum of copies of $\mathrm{Hom}_A(A, N) \cong N$, as a right $\mathrm{End}(N)$-module. Since $G(B)$ is finitely generated projective, it follows that if N is finendo (respectively, cofinendo), then $\mathrm{Hom}_A(G(B), N)$ is finitely generated (respectively, finitely cogenerated) as a right $\mathrm{End}(N)$-module.

Since G is full and faithful, $\mathrm{Hom}_A(G(B), N) \cong \mathrm{Hom}_B(B, M)$, and this is a semilinear isomorphism relative to the ring isomorphism $\mathrm{End}(M) \cong \mathrm{End}(N)$. It is clear that a semilinear isomorphism preserves the lattices of submodules, and thus if N is finendo (respectively, cofinendo), then $\mathrm{Hom}_B(B, M) \cong M$ is finitely generated (respectively, finitely cogenerated) as a right $\mathrm{End}(M)$-module. ∎

The following characterization of hereditary rings of finite representation type, due to Simson [35, 40], will give an important induction step in our study of left pure semisimple hereditary rings.

LEMMA 3.2. *Let R be a basic indecomposable left pure semisimple hereditary ring. Then the following conditions are equivalent*:

(a) *R is of finite representation type.*
(b) *For any pair of indecomposable projective direct summands $P_i \ncong P_j$ of ${}_RR$, there is a ring isomorphism*

$$R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix} \cong \mathrm{End}(P_i \oplus P_j)$$

where $F = \mathrm{End}_R P_i$ and $G = \mathrm{End}_R P_j$ are division rings, $B = \mathrm{Hom}_R(P_i, P_j)$, and R_B is a ring of finite representation type.

Proof. See [40, Theorem 3.4]. ∎

We also need the following simple but useful fact.

LEMMA 3.3. *Let R be a left artinian hereditary ring. Suppose that every finitely generated indecomposable projective left R-module is finendo. Then R is right artinian.*

Proof. Let P be any finitely generated indecomposable projective left R-module, and let $f : P \to P$ be any non-zero homomorphism. Since R is left hereditary, $\mathrm{Im}(f)$ is a projective left R-module, hence f splits, implying that $\mathrm{Ker}(f)$ is a direct summand of P. Since P is indecomposable and f is non-zero, we see that $\mathrm{Ker}(f) = 0$, so f is a monomorphism. As P is of finite length, f must be an isomorphism. This shows that the endomorphism ring S of P is a division ring, and because P is finendo by hypothesis, it follows that P is endofinite. Since endofinite modules are preserved under taking finite direct sums [11, Proposition 4.3], the ring R is endofinite as a left module over itself, i.e. R is right artinian. ∎

The next proposition, which might be of independent interest, is a key step in the proof of our main result.

PROPOSITION 3.4. *Let F, G be division rings, and let ${}_G B_F$ be a non-zero bimodule. Suppose that the ring $R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix}$ is left and right artinian. Set ${}_F M_G = \mathrm{Hom}_F(B, F)$ and consider the triangular matrix ring $R_M = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix}$.*

(a) *If every finitely generated indecomposable left R_B-module is cofinendo, then every finitely generated indecomposable left R_M-module is cofinendo.*
(b) *If every finitely generated indecomposable left R_B-module is finendo, then every finitely generated indecomposable left R_M-module is finendo.*

Proof. (a) We assume that R_B is left and right artinian, and every finitely generated indecomposable left R_B-module is cofinendo.

First note that if A is any ring and L is any left A-module, then L is cofinendo if and only if for any finitely generated left A-module X, $\mathrm{Hom}_A(X, L)$ is finitely cogenerated over $\mathrm{End}(L)$. Indeed, L being cofinendo means that $\mathrm{Hom}_A(A, L)$ is finitely cogenerated over $\mathrm{End}(L)$, hence $\mathrm{Hom}_A(A^m, L)$ is finitely cogenerated over $\mathrm{End}(L)$ for any positive integer m. There is an exact sequence $A^n \to X \to 0$ in A-Mod which induces the exact sequence $0 \to \mathrm{Hom}_A(X, L) \to \mathrm{Hom}_A(A^n, L)$ in Mod-$\mathrm{End}(L)$, implying that $\mathrm{Hom}_A(X, L)$ is finitely cogenerated over $\mathrm{End}(L)$. The "if" part of the claim is trivial.

We recall (see, e.g., [21, p. 174]) that there is an equivalence between the category R_B-Mod of left R_B-modules and the category whose objects are the triples (X, Y, λ), where X, Y are left F- or G-modules, respectively, and $\lambda : B \otimes_F X \to Y$ is a G-homomorphism (equivalently, we could take instead the F-homomorphism $\overline{\lambda} : X \to \mathrm{Hom}_G(B, Y)$). The equivalence associates to such an object (X, Y, λ) the left R_B-module whose elements are column vectors $\binom{x}{y}$ with $x \in X$, $y \in Y$ with the usual matrix operations and using λ to define the product $B \times X \to Y$. Moreover, the morphisms in that category between two objects (X, Y, λ) and (X', Y', μ) are given by the pairs (f, g) of linear maps $f : {}_FX \to {}_FX'$ and $g : {}_GY \to {}_GY'$ such that $g \circ \lambda = \mu \circ (1 \otimes f)$. From now on, we identify each left R_B-module with the corresponding triple.

The construction of the Bernstein–Gelfand–Ponomarev *reflection functors* $S^+ : R_B\text{-mod} \to R_M\text{-mod}$ and $S^- : R_M\text{-mod} \to R_B\text{-mod}$ is given, for example, in Ringel [30], Simson [35, 37], or Herzog [21]. These functors can be defined as follows. Given a finitely generated left R_B-module (X, Y, λ), we use the canonical isomorphism $B \otimes_F X \cong \mathrm{Hom}_F(M, X)$ to write $\widetilde{\lambda} : \mathrm{Hom}_F(M, X) \to Y$, and take its kernel $K = \mathrm{Ker}(\widetilde{\lambda})$, giving the left R_M-module $S^+(X, Y, \lambda) = (K, X, u)$, with u corresponding to the inclusion $\overline{\mu} : K \to \mathrm{Hom}_F(M, X)$. Similarly, given a finitely generated left R_M-module (U, V, μ) with $U \xrightarrow{\overline{\mu}} \mathrm{Hom}_F(M, V)$, where U and V are finite-dimensional G- (respectively, F-) vector spaces, we use the same isomorphism above to obtain $\widetilde{\mu} : U \to B \otimes_F V$. Then we take its cokernel C, giving $S^-(U, V, \mu) = (V, C, p)$, where $p : B \otimes_F V \to C$ is the projection. In view of the equivalence of categories shown in [21, Proposition 6.8] (see also [37, Lemma 3.1]), we have the following properties:

(i) If X is an indecomposable module in R_M-mod, then $S^-(X) = 0$ if and only if X is isomorphic to $(G, 0, 0)$, a simple injective left R_M-module which we shall denote as Q. Moreover, if $S^-(X)$ is non-zero, then $S^+S^-(X)$ is isomorphic to X.

(ii) If X and Y are indecomposable modules in R_M-mod such that $S^-(X)$ and $S^-(Y)$ are non-zero, then there is an isomorphism of abelian groups
$$\mathrm{Hom}_{R_M}(X, Y) \cong \mathrm{Hom}_{R_B}(S^-(X), S^-(Y)).$$

In particular, if $X \cong Y \ncong Q$, then (ii) implies that X and $S^-(X)$ have isomorphic endomorphism rings. Moreover, $\mathrm{Hom}_{R_M}(X, Y)$ is a right $\mathrm{End}(Y)$-module, $\mathrm{Hom}_{R_B}(S^-(X), S^-(Y))$ is a right $\mathrm{End}(S^-(Y))$-module, and the above abelian group isomorphism is also a semilinear isomorphism relative to this ring isomorphism.

Now, let C be any finitely generated indecomposable left R_M-module, and we want to show that C is cofinendo. If C is isomorphic to the simple left R_M-module Q above, then C is the module of column vectors $\binom{x}{0}$ with

$x \in G$, with the usual matrix multiplication; and $\mathrm{End}(C)$ is canonically isomorphic to $\mathrm{End}({}_G G) \cong G$, so that C is isomorphic to G_G as a module over its endomorphism ring. It follows that C is endofinite.

Thus, we will assume below that C is not isomorphic to Q. It is enough to show that for any finitely generated indecomposable left R_M-module X, $\mathrm{Hom}_{R_M}(X, C)$ is finitely cogenerated over $\mathrm{End}(C)$ (because if L is any finitely generated left R_M-module, then $\mathrm{Hom}_{R_M}(L, C)$ is a finite direct sum of modules of the form $\mathrm{Hom}_{R_M}(X, C)$ with X indecomposable, and the property of being finitely cogenerated as $\mathrm{End}(C)$-modules is preserved under taking finite direct sums). If X is not isomorphic to Q, then in view of (ii) above, $\mathrm{Hom}_{R_M}(X, C)$ is isomorphic to $\mathrm{Hom}_{R_B}(S^-(X), S^-(C))$, and because $\mathrm{Hom}_{R_B}(S^-(X), S^-(C))$ is finitely cogenerated over $\mathrm{End}(S^-(C))$ by hypothesis, we conclude that $\mathrm{Hom}_{R_M}(X, C)$ is finitely cogenerated over $\mathrm{End}(C)$. Finally, if X is isomorphic to Q, then because Q is simple injective and C is indecomposable non-injective, it is clear that $\mathrm{Hom}_{R_M}(Q, C) = 0$ in this case, completing our proof.

(b) We now assume that R_B is left and right artinian and that all finitely generated indecomposable left R_B-modules are finendo. We will show that every finitely generated indecomposable left R_M-module is finendo.

Thus let ${}_G U$ and ${}_F V$ be finite-dimensional left vector spaces such that

$$U \xrightarrow{\overline{\mu}} \mathrm{Hom}_F(M, V)$$

corresponds to a finitely generated indecomposable left R_M-module $X = (U, V, \mu)$, and assume that the homomorphism $\overline{\mu}$ is injective. If we view X as a left R_M-module, the elements of X are column vectors $\binom{u}{v}$ with $u \in U$, $v \in V$, and each endomorphism of X takes that element to $\binom{u'}{v'}$ for some $u' \in U$ and $v' \in V$. Therefore X is, as a right module over its endomorphism ring, isomorphic to a direct sum $U \oplus V$. To show that X is finendo, it is enough to prove that each of the endosubmodules U, V is finitely generated over $\mathrm{End}(X)$.

By the construction of the functor S^-, we have the exact sequence

$$U \xrightarrow{\overline{\mu}} \mathrm{Hom}_F(M, V) \cong B \otimes_F V \xrightarrow{p} C \to 0$$

and $S^-(X) = (V, C, p)$, being a finitely generated indecomposable left R_B-module, is finendo. By the same argument used in the preceding paragraph for the endosubmodules U, V of X, both V and C are finitely generated over the endomorphism ring of $S^-(X)$, which is canonically isomorphic to $\mathrm{End}(X)$. It is not hard to see that this entails that V is finitely generated over the endomorphism ring of X, and that also $B \otimes_F V$ is finitely generated over $\mathrm{End}(S^-(X)) \cong \mathrm{End}(X)$. But U, being a direct summand of $B \otimes_F V$, is then finitely generated over the endomorphism ring $\mathrm{End}(X)$, and we are done.

We now turn to the case when the homomorphism $\overline{\mu}: U \to \mathrm{Hom}(M, V)$ is not injective. But, since the module X is indecomposable, we have $V = 0$ and $U \cong G$, and the module is then isomorphic to $(G, 0, 0)$. And we have already seen in part (a) that this is a finendo module. Hence we have shown that every finitely generated indecomposable left R_M-module is finendo. ∎

We are now in a position to prove the main result of this section, giving positive answers to Questions 1 and 2 in [13, pp. 122–123] for hereditary rings. The implications (c)⇒(e) and (d)⇒(e) also sharpen [14, Theorem 4.1] in the hereditary case, where finitely generated indecomposable left R-modules were assumed to be endofinite (see also [35, Corollary 3.2]).

Theorem 3.5. *Let R be any hereditary ring. Then the following conditions are equivalent.*

(a) *Every right R-module is finendo.*
(b) *Every left R-module is cofinendo.*
(c) *R is left pure semisimple, and every finitely generated indecomposable left R-module is cofinendo.*
(d) *R is left pure semisimple, and every finitely generated indecomposable left R-module is finendo.*
(e) *R is of finite representation type.*

Proof. (a)⇒(b) and (b)⇔(c) follow from Corollary 2.12 and Theorem 2.11, respectively (even without the hereditary hypothesis on the ring R).

(c)⇒(e). Suppose that (c) holds, i.e. R is left pure semisimple and every finitely generated indecomposable left R-module is cofinendo. Then, by Lemma 2.3, every finitely generated left R-module is cofinendo.

Now assume on the contrary that R is not of finite representation type. In view of Lemma 3.1, we can take R to be a basic and indecomposable ring. Moreover, we know by Lemma 3.2 that there exist a pair of indecomposable projective direct summands $P_i \not\cong P_j$ of ${}_RR$ and a ring isomorphism

$$R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix} \cong \mathrm{End}(P_i \oplus P_j)$$

where $B = \mathrm{Hom}_R(P_i, P_j)$, $F = \mathrm{End}({}_RP_i)$ and $G = \mathrm{End}({}_RP_j)$ are division rings, such that the ring R_B is not of finite representation type. Note that, by Simson [36, Theorem 17.46], there is a fully faithful embedding functor T: R_B-mod $\to R$-mod. It follows easily that R_B is a left pure semisimple ring. Note also that every finitely generated indecomposable left R_B-module can be seen as a finitely generated left R-module having the same endomorphism ring. Consequently, every finitely generated indecomposable left R_B-module is cofinendo. By Corollary 2.9 we see that R_B is right artinian.

Let $M = \mathrm{Hom}_F(B, F)$. Then M is a F-G-bimodule, and we can consider the triangular matrix ring

$$R_M = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix}.$$

Then R_M is a left pure semisimple ring of infinite representation type (see [21, 37]), and by Proposition 3.4(a) every finitely generated indecomposable left R_M-module is cofinendo. Applying Corollary 2.9, we find that R_M is right artinian, and hence the vector space M_G is finite-dimensional. We can take $M_1 = \mathrm{Hom}_G(M, G)$ which is a G-F-bimodule, and consider the triangular matrix ring

$$R_{M_1} = \begin{pmatrix} F & 0 \\ M_1 & G \end{pmatrix}.$$

Again, using arguments similar to the above, we deduce that R_{M_1} is left pure semisimple representation-infinite, and every finitely generated indecomposable left R_{M_1}-module is cofinendo, so R_{M_1} is right artinian, again by Corollary 2.9.

We may extend this process so that the iterated dual bimodules are always finite-dimensional on both sides. Then, by applying [35, Theorem 3.1] or [37, Theorem 3.4], we conclude that the matrix ring R_B must be of finite representation type, which is a contradiction.

(d)⇒(e). Suppose that (d) holds. Then Corollary 2.2 implies that every finitely generated left R-module is finendo. As in the proof of (c)⇒(e), we can use Lemma 3.1 to assume that R is basic and indecomposable, and then Lemma 3.2 allows us to start with a left pure semisimple representation-infinite ring of the form $R_B = \left(\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right)$ that satisfies (d). Note that, by Lemma 3.3, it follows that any hereditary ring satisfying (d) must be right artinian. Now using Proposition 3.4(b) repeatedly, as in the proof of (c)⇒(e), we will get a contradiction by applying again [35, Theorem 3.1] or [37, Theorem 3.4].

(e)⇒(a). If R is of finite representation type, then every right R-module is endofinite (see [10, 26, 29]), hence (a) follows. ■

Remark 3.6. As pointed out to us by Professor Daniel Simson, the implications (c)⇒(e) and (d)⇒(e) of Theorem 3.5 can also be proved using [40, Corollary 2.11] (cf. [35, 37]), from which it follows that a left pure semisimple ring of the form $R_B = \left(\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right)$ is of finite representation type if every indecomposable preprojective left R-module is endofinite.

4. The case of non-hereditary rings. Given an arbitrary ring R, we have seen in Section 2 that if all right R-modules are finendo, then R is left pure semisimple and all finitely generated indecomposable left R-modules are cofinendo (Theorem 2.11 and Corollary 2.12). So, the questions mentioned in the introduction can be reduced to the problem whether a left pure semisimple ring R with all finitely generated indecomposable left R-modules cofinendo has to be of finite representation type. We know that the answer is positive when R is hereditary (Theorem 3.5). In the general case, we can prove that R is *right Morita*, i.e. R is right artinian and every indecomposable injective right R-module is finitely generated. This is the goal of this section, where we follow the steps in [21, Section 6], and, in order to do this, we first state and prove some results about triangular matrix rings.

Let R be a semiprimary ring with Jacobson radical $J = J(R)$. Suppose that I is a two-sided ideal of R such that $JI = IJ = 0$. Then we may construct the triangular matrix ring

$$T_I(R) = \begin{pmatrix} R/J & 0 \\ I & R/J \end{pmatrix}.$$

As in the preceding section, we shall use the identification of the left $T_I(R)$-modules with the triples (X, Y, λ), where X, Y are left R/J-modules and $\lambda : I \otimes_{R/J} X \to Y$ is an R/J-homomorphism. Following [5, Section 2] or [21, p. 175], we shall say that a left $T_I(R)$-module (X, Y, λ) is *Grassmannian* when there are no non-zero elements $x \in X$ with $\lambda(a \otimes x) = 0$ for all $a \in I$. Note that, as observed in [21, p. 175], any left $T_I(R)$-module is a direct sum of a Grassmannian module and a module of the form $(X, 0, 0)$. For any left R-module X, let $\mathrm{soc}(X)$ denote the socle of X, and $\mathrm{ann}_X(I)$ the annihilator $\{x \in X \mid Ix = 0\}$.

We now define the following functor, which is closely related to the functors F or J appearing in [5, Section 3] (see also [20, p. 99] and [41]); in the form below, the definition was given in [21]):

$$\mathrm{Gr} : R\text{-Mod} \to T_I(R)\text{-Mod}, \qquad \mathrm{Gr}(X) = (X/\mathrm{ann}_X(I), \mathrm{soc}(X), f),$$

where the mapping $f : I \otimes_{R/J} (X/\mathrm{ann}_X(I)) \to \mathrm{soc}(X)$ is canonical.

LEMMA 4.1. *Let R be a left artinian ring with Jacobson radical $J = J(R)$, and let I be a two-sided ideal of R such that $I \subseteq J$ and $JI = IJ = 0$. Let $T_I(R)$ be the triangular matrix ring constructed above.*

(a) *Each finitely generated indecomposable left $T_I(R)$-module is isomorphic either to $(X, 0, 0)$ for some simple left R/J-module X or to $\mathrm{Gr}(X)$ for some finitely generated indecomposable left R-module X.*
(b) *If every finitely generated indecomposable left R-module is cofinendo, then every finitely generated indecomposable left $T_I(R)$-module is cofinendo.*

Proof. (a) It follows from the observation above on Grassmannian $T_I(R)$-modules that a finitely generated indecomposable left $T_I(R)$-module is either

Grassmannian or isomorphic to $(X, 0, 0)$. In this second case, it is obvious that X has to be a simple left R/J-module.

Suppose next that the finitely generated indecomposable left $T_I(R)$-module M is Grassmannian. By the density of the functor J in [5, Theorem 3.1, b)], there exists a left R-module X with injective hull Q so that M is isomorphic to $(X/\mathrm{ann}_X(I), \mathrm{soc}(Q), f)$, where the mapping $f : I \otimes_{R/J} (X/\mathrm{ann}_X(I)) \to \mathrm{soc}(Q)$ is canonical. Since $\mathrm{soc}(Q) = \mathrm{soc}(X)$, we infer that M is isomorphic to $\mathrm{Gr}(X)$.

Now, suppose that X is not indecomposable. Then $X = X_1 \oplus X_2$ implies that $\mathrm{soc}(X) = \mathrm{soc}(X_1) \oplus \mathrm{soc}(X_2)$, hence the given $T_I(R)$-module would be a non-trivial direct sum, since R is left artinian. This proves (a).

(b) We start by noting that every indecomposable left $T_I(R)$-module of the form $(X, 0, 0)$ is isomorphic to X as a module over its endomorphism ring. Since X is an R/J-module and R/J is semisimple, we deduce that it is endofinite, and, in particular, cofinendo.

By (a), it remains to see that any indecomposable Grassmannian left $T_I(R)$-module M is cofinendo. In view of [5, Theorem 3.1], the restriction of the functor Gr to those left R-modules X such that $\mathrm{ann}_X(I)$ is injective as a left R/I-module is a full functor. Since it is also dense, we can choose an indecomposable finitely generated left R-module X such that $\mathrm{Gr}(X) \cong M$ and there is a surjective ring homomorphism

$$\phi : E = \mathrm{End}({}_R X) \to H = \mathrm{End}({}_{T_I(R)}\mathrm{Gr}(X)).$$

Let $K = \mathrm{Ker}(\phi)$. Both E and H are local rings. If U_E is the injective hull of the unique simple right E-module, then it is not hard to see that $\mathrm{ann}_U(K)_H$ is the injective hull of the unique simple right H-module. Therefore $D(X) = \mathrm{Hom}_E(X, U)$ and $D(\mathrm{Gr}(X)) = \mathrm{Hom}_H(\mathrm{Gr}(X), \mathrm{ann}_U(K))$.

Our plan for the proof is the following. Since ${}_R X$ is cofinendo, it follows from Proposition 2.6 that $D(X) = \mathrm{Hom}_E(X, U)$ is finendo. We will use this to show that $D(\mathrm{Gr}(X))$ is a finendo right $T_I(R)$-module, and by Proposition 2.5 we will conclude that $\mathrm{Gr}(X)$ is cofinendo.

We know from [23, Remark 2, p. 312] that the endomorphism ring of $D(X)$ is isomorphic to $\mathrm{End}(U_E)$ in the natural way, and, similarly, the endomorphism ring of $D(\mathrm{Gr}(X))$ is isomorphic to $\mathrm{End}(\mathrm{ann}_U(K)_H)$. Now, $\mathrm{Gr}(X)$ is, as a right H-module, isomorphic to the direct sum of the modules $\overline{X} = X/\mathrm{ann}_X(I)$ and $\mathrm{soc}(X) = IX$. Specifically, the structure of $D(\mathrm{Gr}(X))$ as a right $T_I(R)$-module is easily seen to be given by the triple $(\mathrm{Hom}_H(IX, \mathrm{ann}_U(K)), \mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K)), \mu)$, where μ is the surjective canonical map

$$\mu : \mathrm{Hom}_H(IX, \mathrm{ann}_U(K)) \otimes_{R/J} I \to \mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K)).$$

Thus, to see that $D(\mathrm{Gr}(X))$ is finendo, it will suffice to show that each of the modules $\mathrm{Hom}_H(IX, \mathrm{ann}_U(K))$ and $\mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K))$ is finitely generated over the endomorphism ring of $\mathrm{ann}_U(K)$.

By hypothesis, $\mathrm{Hom}_E(X, U)$ is finitely generated over $\mathrm{End}(U_E)$. Thus there exist elements $f_1, \ldots, f_s \in \mathrm{Hom}_E(X, U)$ so that each $f : X_E \to U_E$ has the form $f = \sum_{i=1}^{s} \alpha_i \circ f_i$ for certain $\alpha_i \in \mathrm{End}(U_E)$. By noting that $K = \mathrm{Hom}_R(X, \mathrm{ann}_X(I))$, it follows easily that any $f : X_E \to U_E$ restricts to a homomorphism $\widehat{f} : (IX)_E \to (\mathrm{ann}_U(K))_E$. On the other hand, every homomorphism $g : (IX)_E \to (\mathrm{ann}_U(K))_E$ can be extended to $X_E \to U_E$, because U_E is injective. It follows immediately that $\widehat{f_1}, \ldots, \widehat{f_s}$ is a system of generators of $\mathrm{Hom}_E(IX, \mathrm{ann}_U(K)) = \mathrm{Hom}_H(IX, \mathrm{ann}_U(K))$ as a module over $\mathrm{End}(\mathrm{ann}_U(K)_H)$.

Finally, we consider $\mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K))$. Take a set $a_1, \ldots, a_r$ of generators of I as a left R/J-module. Then each $\widehat{f_i} \otimes a_j$ gives $\mu(\widehat{f_i} \otimes a_j) = \phi_{ij} \in \mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K))$. Each element $g \in \mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K))$ can be written as $g = \sum_{j=1}^{r} \mu(h_j \otimes a_j)$. In turn, $h_j = \sum_{i=1}^{s} \alpha_{ij} \widehat{f_i}$, for certain endomorphisms α_{ij} of $\mathrm{ann}_U(K)$, in view of the preceding paragraph. Therefore

$$g = \sum_{i,j} \alpha_{ij} \mu(\widehat{f_i} \otimes a_j) = \sum_{i,j} \alpha_{ij} \phi_{ij},$$

which shows that the module $\mathrm{Hom}_H(\overline{X}, \mathrm{ann}_U(K))$ is finitely generated over $\mathrm{End}(\mathrm{ann}_U(K))$, as was to be shown. ∎

We are now ready to deduce the central result of this section.

THEOREM 4.2. *If R is a left pure semisimple ring such that every finitely generated indecomposable left R-module is cofinendo, then R is right Morita.*

Proof. Suppose, to the contrary, that there is some left pure semisimple ring A which is not right Morita and such that every finitely generated indecomposable left A-module is cofinendo.

Note that every quotient ring Q of A has the same property, i.e., every finitely generated indecomposable left Q-module is cofinendo. This is because the restriction of the scalars functor Q-Mod $\to$ A-Mod views each left Q-module as a left A-module, with the same endomorphisms. As in [21, Section 5], we may use the fact that A is left artinian to obtain a quotient R of A which is left pure semisimple and not right Morita, and such that all the proper quotients of R are right Morita. Then every finitely generated indecomposable left R-module is cofinendo.

We now follow the proof of [21, Proposition 5.8] to show that this ring R has a unique minimal two-sided ideal I. Indeed, if there are non-zero two-sided ideals I_1, I_2 of R with $I_1 \cap I_2 = 0$, and E is the injective hull of a simple right R-module, then the natural monomorphism of right R-modules

$R \to R/I_1 \oplus R/I_2$ induces an epimorphism

$$p : \mathrm{Hom}_R(R/I_1, E) \oplus \mathrm{Hom}_R(R/I_2, E) \to E.$$

But $E_j = \mathrm{Im}(p\,|_{R/I_j})$ is a submodule of E which is indecomposable injective as a right R/I_j-module. Since R/I_j is right Morita, E_j is finitely generated also as a right R-module. Hence E is finitely generated, and R is right Morita, which is a contradiction.

We next claim that if R any left pure semisimple ring such that every finitely generated indecomposable left R-module is cofinendo, and $I \subseteq J$ is a two-sided ideal of R such that $IJ = JI = 0$ and R/I is right Morita, then R is also right Morita. The application of this claim to the left pure semisimple and non-right Morita ring R above will yield the desired contradiction and complete our proof.

The proof of the claim could also be obtained by adapting the proofs of [21, Proposition 6.2, Theorem 6.3], but we give a direct proof for the convenience of the reader. First we note that, under the assumptions in the claim, $T_I(R)$ is left pure semisimple. By [22, Corollary 2], it is enough to show that each left $T_I(R)$-module is a direct sum of indecomposable modules. In turn, it will suffice to show the property for Grassmannian modules, according to Lemma 4.1. But this follows easily from the facts that the functor Gr preserves indecomposable modules [5] and direct sums, and every left R-module is a direct sum of indecomposable modules.

By Lemma 4.1, every finitely generated indecomposable left $T_I(R)$-module is cofinendo. As $T_I(R)$ is a hereditary ring, we infer by Theorem 3.5 that $T_I(R)$ is of finite representation type. Note that R is right artinian, by Corollary 2.9. Now let M be any indecomposable injective right R-module. Since $\mathrm{ann}_M(I)$ is an injective indecomposable right R/I-module [16, Proposition 8], it is finitely generated, as R/I is right Morita. Moreover, $\mathrm{Gr}(M)$ is indecomposable by [21, Theorem 6.1]. Therefore $\mathrm{Gr}(M)$ is finitely generated, because $T_I(R)$ is of finite representation type, hence $M/\mathrm{ann}_M(I)$ is finitely generated as a right R-module, and hence so is M. This shows that R is right Morita. ∎

Proposition 4.3. *Let R be a left pure semisimple ring such that every finitely generated indecomposable left R-module is cofinendo. Then the quotient ring $R/(J(R))^2$ is of finite representation type.*

Proof. Set $R' = R/(J(R))^2$. Thus R' is a left pure semisimple ring such that every finitely generated indecomposable left R'-module is cofinendo. If J is the Jacobson radical of R', then $J^2 = 0$, and we may consider the matrix ring $M_J(R') = \left(\begin{smallmatrix} R'/J & 0 \\ J & R'/J \end{smallmatrix}\right)$.

By Lemma 4.1, every finitely generated indecomposable left $M_J(R')$-module is cofinendo. Moreover, $M_J(R')$ is left pure semisimple, and it is of infinite representation type if R' is of infinite representation type, by [5, Corollary 4.3]. But it is hereditary, and this contradicts Theorem 3.5. Therefore R' is of finite representation type. ∎

We summarize our findings for the class of rings of our main interest, in the general case.

Corollary 4.4. *Let R be a ring such that every right R-module is finendo. Then R is a left pure semisimple ring with a right Morita duality, and the quotient ring $R/(J(R))^2$ is of finite representation type.*

Proof. The result follows by combining Corollary 2.12, Theorem 2.11, Theorem 4.2, and Proposition 4.3. ∎

We conclude the paper with the following remark (observed independently by Professor Daniel Simson, who suggested including it here).

Remark 4.5. Following Simson [39], a right artinian ring R is said to have an *infinite right Morita sequence* if there is an infinite sequence of right artinian rings $\{R_n\}_{n=0}^{\infty}$ such that $R_0 = R$, and there is a Morita duality $\mathcal{D}_n$: mod-$R_n \to R_{n+1}$-mod for each $n \geq 0$. Such a sequence, if it exists, is uniquely determined by the ring R. Examples of artinian rings having an infinite right Morita sequence include artinian rings with self-duality, artinian PI-rings, and rings of finite representation type. Simson [39, Theorem 2.5] has shown that left pure semisimple rings R having an infinite right Morita sequence are of finite representation type. Now, suppose that R is a ring with all right R-modules finendo. By Corollary 4.4, R is right Morita, hence there is a ring R_1 and a Morita duality $\mathcal{D}$: mod-$R \to R_1$-mod, and clearly R_1 is left pure semisimple. If the finendo property of all right R-modules is transferred through the Morita duality to the right R_1-modules, that would show the existence of an infinite right Morita sequence for such a ring R. This in turn would imply that rings R with all right R-modules finendo must have finite representation type, in view of Simson's result above. However, we do not know if the finendo property of all right R-modules is preserved through the Morita duality in this situation.

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