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EXISTENCE AND CONSTRUCTION OF TWO-DIMENSIONAL INVARIANT SUBSPACES FOR PAIRS OF ROTATIONS

BY

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Abstract. By a rotation in a Euclidean space V of even dimension we mean an orthogonal linear operator on V which is an orthogonal direct sum of rotations in 2-dimensional linear subspaces of V by a common angle $\alpha \in [0, \pi]$. We present a criterion for the existence of a 2-dimensional subspace of V which is invariant under a given pair of rotations, in terms of the vanishing of a determinant associated with that pair. This criterion is constructive, whenever it is satisfied. It is also used to prove that every pair of rotations in V has a 2-dimensional invariant subspace if and only if the dimension of V is congruent to 2 modulo 4.

1. Statement of the main result. We agree that $0 \in \mathbb{N}$. For all $m \in \mathbb{N}$ we denote by $\mathbb{R}^{m\times m}$ the set of all real $m \times m$-matrices, and by $\mathbb{I}_m \in \mathbb{R}^{m\times m}$ the identity matrix. For all $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$ we use moreover the matrix notation

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad \mathbb{I}_m \otimes R_\alpha = \begin{pmatrix} R_\alpha & & \\ & \ddots & \\ & & R_\alpha \end{pmatrix} \in \mathbb{R}^{2m\times 2m}.$$

Let V be a Euclidean space of even dimension. By a *rotation* in V we mean an orthogonal linear operator $\sigma \in \mathrm{O}(V)$ whose matrix in some orthonormal basis of V is $\mathbb{I}_m \otimes R_\alpha$ for some $\alpha \in [0, \pi]$. Given any pair (σ, τ) of rotations in V, we call a linear subspace $W \subset V$ *invariant under* (σ, τ), or briefly (σ, τ)-*invariant*, if both inclusions $\sigma(W) \subset W$ and $\tau(W) \subset W$ hold. In that case, $\sigma(W) = W$ and $\tau(W) = W$. Moreover, (σ, τ) induces a pair (σ_W, τ_W) of linear operators on W. If $\{\sigma, \tau\} \not\subset \{\mathbb{I}_V, -\mathbb{I}_V\}$, where $\mathbb{I}_V$ denotes the identity operator on V, then $\dim W$ is even and (σ_W, τ_W) is a pair of rotations in W.

In [2], it is asked whether every pair of rotations in a 6-dimensional Euclidean space has a 2-dimensional invariant subspace. An affirmative answer plays in fact a crucial role in the classification of the 8-dimensional absolute-

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valued algebras with a non-zero central idempotent or a one-sided unity. Since these are (non-associative) real division algebras, their classification contributes substantially to the problem of classifying all finite-dimensional real division algebras, an old and hard problem which, originating from the work of Hamilton, Graves and Cayley, to date is still only partially solved and recently has attracted renewed interest (e.g. [3]–[9], [11]–[14]).

More than just establishing the desired affirmative answer to the above-mentioned question about 6-dimensional Euclidean spaces, the main result of the present article asserts the following.

THEOREM 1.1. *Let V be a Euclidean space of even dimension. Then every pair of rotations in V has a 2-dimensional invariant subspace if and only if the dimension of V is congruent to 2 modulo 4.*

As an immediate consequence we obtain the following decomposability criterion for pairs of orthogonal matrices.

COROLLARY 1.2. *Let $A, B \in \mathrm{O}(n)$ be real orthogonal $n \times n$-matrices, satisfying*

$$n \equiv 2 \pmod 4, \quad S^t A S = \mathbb{I}_m \otimes R_\alpha \quad \text{and} \quad T^t B T = \mathbb{I}_m \otimes R_\beta$$

for certain $S, T \in \mathrm{O}(n)$ and $\alpha, \beta \in [0, \pi]$. Then there exists a real orthogonal matrix $U \in \mathrm{O}(n)$ such that

$$U^t A U = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}, \quad U^t B U = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$$

for certain $A_1, B_1 \in \mathrm{O}(2)$ and $A_2, B_2 \in \mathrm{O}(n-2)$.

Proof. In terms of the notation introduced in Section 2, the hypothesis on A and B means that $\underline{A}$ and $\underline{B}$ are rotations in $\mathbb{E}^n$, where $\dim \mathbb{E}^n \equiv 2 \pmod 4$. By Theorem 1.1 there exists a 2-dimensional subspace $P \subset \mathbb{E}^n$ which is invariant under $(\underline{A}, \underline{B})$. It follows that $P^\perp$ also is invariant under $(\underline{A}, \underline{B})$. Hence for every choice of orthonormal bases (u_1, u_2) in P and $(u_3, \ldots, u_n)$ in $P^\perp$, the matrix $U \in \mathrm{O}(n)$ whose column series is $(u_1, \ldots, u_n)$ will do. ■

Corollary 1.2 can be reformulated in terms of modules over the free associative algebra $\Lambda = \mathbb{R}\langle X, Y\rangle$, as follows. Every pair (σ, τ) of linear operators on a real vector space V endows V with the structure of a left Λ-module $V = V(\sigma, \tau)$, given by $Xv = \sigma(v)$ and $Yv = \tau(v)$ for all $v \in V$.

COROLLARY 1.3. *For every pair (σ, τ) of rotations in a Euclidean space V of dimension $n \equiv 2 \pmod 4$, the left Λ-module $V = V(\sigma, \tau)$ decomposes orthogonally, $V = P \oplus P^\perp$, into a 2-dimensional Λ-submodule P and its associated $(n-2)$-dimensional Λ-submodule $P^\perp$. If in addition $n \geq 6$, then the Λ-module $V(\sigma, \tau)$ is not indecomposable.*

If one instead considers pairs (σ, τ) of rotations in a Euclidean space V of dimension $n \equiv 0 \pmod 4$, then the situation is more delicate. The Λ-module $V = V(\sigma, \tau)$ may or may not decompose as in Corollary 1.3. Precise information about what happens is given in Proposition 2.3.

In representation theory, the free associative algebra $\Lambda = \mathbb{R}\langle X, Y\rangle$ is known as the most prominent example of a wild algebra. Here the term "wild" signifies, among other phenomena, that for every finitely generated real associative algebra Γ, the category $\mathrm{mod}_{\mathrm{f}}\,\Gamma$ of finite-dimensional left Γ-modules admits a full and faithful embedding $\mathrm{mod}_{\mathrm{f}}\,\Gamma \hookrightarrow \mathrm{mod}_{\mathrm{f}}\,\Lambda$ (see [1]). Thus $\mathrm{mod}_{\mathrm{f}}\,\Lambda$ abounds with indecomposable objects. These are however far from being classified, and constructive approaches to the decomposition of objects in $\mathrm{mod}_{\mathrm{f}}\,\Lambda$ are very rare. Seen against this background and taking into account that $V(\sigma, \tau)$ forms an $(n^2 - n + 2)$-parameter family of Λ-modules as (σ, τ) ranges through all pairs of rotations in V (the dimension of the real Lie group $\mathrm{O}(n)$ being $(n^2 - n)/2$), and that the decomposition statement of Corollary 1.3 is constructive (cf. Section 2), Corollary 1.3 appears less innocuous than it may seem.

Let us also mention that Corollary 1.3 reveals but a special instance of a more general, yet presently poorly known and only little understood interrelation between the classification theory of (non-associative) real division algebras and the module theory over certain real associative algebras. This topic will be treated in greater generality in the forthcoming article [10].

Finally, it should be pointed out that Proposition 3.1 below is a statement on the Lie algebra $\mathfrak{o}(n)$ of the real Lie group $\mathrm{O}(n)$ which, to our knowledge, was not previously known within that classical theory. The present note thus comprises aspects of such diverse algebraic theories as (non-associative) real division algebras, modules over wild real associative algebras, and real Lie algebras. All of these aspects emerge naturally from the originally posed problem on the 6-dimensional Euclidean space, and they merge naturally into its solution, the proof of Theorem 1.1.

2. A determinant criterion for the existence of two-dimensional invariant subspaces. For the remainder of this article, $n = 2m$ is an even natural number, V is any Euclidean space of dimension n, and $\mathbb{E}^n$ is the particular Euclidean space $\mathbb{R}^n$ equipped with the standard scalar product $\langle v, w\rangle = v^t w$. Every matrix $M \in \mathbb{R}^{n\times n}$ determines a linear operator $\underline{M} : \mathbb{E}^n \to \mathbb{E}^n$, $\underline{M}(v) = Mv$. In our context, the matrix

$$I = \mathbb{I}_m \otimes R_{\pi/2} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

is of special importance. For brevity we set $\varrho_\alpha = \underline{\mathbb{I}_m \otimes R_\alpha}$ for all $\alpha \in [0,\pi]$, and $\iota = \underline{I} = \varrho_{\pi/2}$. In this notation, the rotations in $\mathbb{E}^n$ are precisely the linear operators of the form $\underline{S}\varrho_\alpha\underline{S}^{-1}$, where $S \in \mathrm{O}(n)$ and $\alpha \in [0,\pi]$. The notation $[a,b] = ab - ba$ is used for elements a, b in $\mathrm{Hom}_{\mathbb{R}}(V,V)$ or in $\mathbb{R}^{n\times n}$. By $\mathrm{sp}(v,w)$ we mean the $\mathbb{R}$-linear span of elements $v, w \in \mathbb{E}^n$.

If $\alpha \in \{0,\pi\}$ then $\varrho_\alpha = \pm\mathbb{I}_{\mathbb{E}^n}$, and hence every linear subspace of $\mathbb{E}^n$ is ϱ_α-invariant. In the generic situation where $0 < \alpha < \pi$, the 2-dimensional invariant subspaces for single rotations in $\mathbb{E}^n$ are described in the following lemma.

LEMMA 2.1. *Let $P \subset \mathbb{E}^n$ be a 2-dimensional linear subspace.*

(i) *For each $v \in \mathbb{E}^n \setminus \{0\}$, $\mathrm{sp}(v, Iv)$ is 2-dimensional and ι-invariant. Conversely, if P is ι-invariant, then $P = \mathrm{sp}(v, Iv)$ for each $v \in P \setminus \{0\}$.*
(ii) *For each $\alpha \in \,]0,\pi[$, P is ι-invariant if and only if P is ϱ_α-invariant.*
(iii) *For each $\alpha \in \,]0,\pi[$ and $S \in \mathrm{O}(n)$, P is ϱ_α-invariant if and only if $\underline{S}(P)$ is $\underline{S}\varrho_\alpha\underline{S}^{-1}$-invariant.*

Proof. (i) For each $v \in \mathbb{E}^n \setminus \{0\}$, the vectors v and Iv are non-proportional since I has no real eigenvalue. Thus $\dim \mathrm{sp}(v, Iv) = 2$. Moreover, we note that $\iota(\mathrm{sp}(v, Iv)) = \mathrm{sp}(Iv, -v) = \mathrm{sp}(v, Iv)$. Conversely, if P is ι-invariant and $v \in P \setminus \{0\}$, then $\mathrm{sp}(v, Iv) \subset P$ and $\dim \mathrm{sp}(v, Iv) = 2$, so $\mathrm{sp}(v, Iv) = P$.

(ii) follows immediately from the identities

$$\varrho_\alpha = (\cos\alpha)\mathbb{I}_{\mathbb{E}^n} + (\sin\alpha)\iota \quad \text{and} \quad \iota = -\frac{\cos\alpha}{\sin\alpha}\,\mathbb{I}_{\mathbb{E}^n} + \frac{1}{\sin\alpha}\,\varrho_\alpha.$$

(iii) Clearly, $\varrho_\alpha(P) = P$ if and only if $\underline{S}\varrho_\alpha\underline{S}^{-1}\underline{S}(P) = \underline{S}(P)$. ∎

Passing now to pairs of rotations, we begin with an easy necessary criterion for the existence of a 2-dimensional invariant subspace.

LEMMA 2.2. *If a pair (σ, τ) of rotations in V has a 2-dimensional invariant subspace, then the linear operator $[\sigma, \tau]$ is not invertible.*

Proof. Let $P \subset V$ be a 2-dimensional (σ,τ)-invariant subspace. Then (σ,τ) induces a pair (σ_P, τ_P) of rotations in P. Moreover, P being 2-dimensional, σ_P and τ_P commute. Hence $P \subset \ker[\sigma,\tau]$, and so $[\sigma,\tau]$ is not invertible. ∎

From the previous two lemmas we now derive a necessary and sufficient criterion for the existence of a 2-dimensional invariant subspace, in terms of the vanishing of a determinant associated with the pair of rotations in question.

PROPOSITION 2.3. *For every pair $(\sigma,\tau) = (\underline{S}\varrho_\alpha\underline{S}^{-1}, \underline{T}\varrho_\beta\underline{T}^{-1})$ of rotations in $\mathbb{E}^n$, with $S, T \in \mathrm{O}(n)$ and $0 < \alpha, \beta < \pi$, the following statements* *are equivalent*:

(i) *The pair* (σ, τ) *has a* 2*-dimensional invariant subspace.*
(ii) *The identity* $\det[I, S^tTIT^tS] = 0$ *holds true.*

Proof. (i)⇒(ii). Let $P \subset \mathbb{E}^n$ be a 2-dimensional (σ, τ)-invariant subspace. Set $U = S^tT$. Repeated application of Lemma 2.1 shows that $\underline{S}^{-1}(P)$ is $(\varrho_\alpha, \underline{U}\,\varrho_\beta \underline{U}^{-1})$-invariant, and thus even $(\iota, \underline{U}\iota\underline{U}^{-1})$-invariant. By Lemma 2.2, the linear operator $[\iota, \underline{U}\iota\underline{U}^{-1}] = [\underline{I}, \underline{UIU^t}]$ is not invertible. Equivalently, the matrix $[I, S^tTIT^tS]$ is not invertible.

(ii)⇒(i). If we set $U = S^tT$, hypothesis (ii) states that $\det[I, UIU^t] = 0$. On the other hand, the matrix identities

$$\begin{aligned}[I, UIU^t] &= IUIU^t - UIU^tI + (UIU^t)^2 - I^2 = (UIU^t + I)(UIU^t - I)\\ &= (UI + IU)U^t(UI - IU)U^t\end{aligned}$$

hold true. Accordingly, $\det(UI + \varepsilon IU) = 0$ for some $\varepsilon \in \{1, -1\}$. Hence there exists a $v \in \mathbb{E}^n \setminus \{0\}$ such that $UIv + \varepsilon IUv = 0$. Setting $w = Uv$, we obtain $\underline{U}(\mathrm{sp}(v, Iv)) = \mathrm{sp}(Uv, UIv) = \mathrm{sp}(Uv, IUv) = \mathrm{sp}(w, Iw)$. Application of Lemma 2.1 to the end terms of the latter chain of identities shows that $\mathrm{sp}(w, Iw)$ is 2-dimensional and $(\varrho_\alpha, \underline{U}\,\varrho_\beta\underline{U}^{-1})$-invariant. Hence

$$\underline{S}(\mathrm{sp}(w, Iw)) = \mathrm{sp}(Tv, SIS^tTv)$$

is 2-dimensional and $(\underline{S}\varrho_\alpha\underline{S}^{-1}, \underline{T}\varrho_\beta\underline{T}^{-1})$-invariant, i.e. (σ, τ)-invariant. ∎

Note that the proof of Proposition 2.3 actually contains a method of constructing 2-dimensional invariant subspaces for pairs of rotations, provided they exist. Indeed, if (σ, τ) has a 2-dimensional invariant subspace, then the matrix $S^tTI + \varepsilon IS^tT$ is not invertible for some $\varepsilon \in \{1, -1\}$, and for every $v \in \ker(\underline{S^tTI + \varepsilon IS^tT}) \setminus \{0\}$ the 2-dimensional subspace $\mathrm{sp}(Tv, SIS^tTv)$ is invariant under (σ, τ).

Corollary 2.4. *For any Euclidean space* V *of even dimension* n, *the following statements are equivalent*:

(i) *Every pair of rotations in* V *has a* 2*-dimensional invariant subspace.*
(ii) *Every pair of rotations in* $\mathbb{E}^n$ *has a* 2*-dimensional invariant subspace.*
(iii) *The identity* $\det[I, UIU^t] = 0$ *holds for all* $U \in \mathrm{O}(n)$.

Proof. (i)⇔(ii) holds because V and $\mathbb{E}^n$ are isomorphic Euclidean spaces.

(ii)⇒(iii). For every $U \in \mathrm{O}(n)$, the linear operator $\underline{UIU^t}$ is a rotation in $\mathbb{E}^n$. By hypothesis (ii) and Lemma 2.2, the linear operator $[\underline{I}, \underline{UIU^t}]$ is not invertible. Equivalently, the matrix $[I, UIU^t]$ is not invertible.

(iii)⇒(ii). Hypothesis (iii) ensures that $n \geq 2$. (Indeed, if $n = 0$, then $I = U = [I, UIU^t] = \mathbb{I}_0$, the unique matrix in $\mathbb{R}^{0\times 0}$. Since $\mathbb{I}_0$ corresponds to the identity operator on $\mathbb{R}^0 = \{0\}$, one defines $\det \mathbb{I}_0 = 1$.) Let $(\sigma, \tau) =$

$(\underline{S}\varrho_\alpha\underline{S}^{-1}, \underline{T}\varrho_\beta\underline{T}^{-1})$ be any pair of rotations in $\mathbb{E}^n$. If $\alpha \in \{0,\pi\}$ or $\beta \in \{0,\pi\}$, then (σ,τ) clearly has a 2-dimensional invariant subspace. If $0<\alpha,\beta<\pi$, then Proposition 2.3 guarantees the existence of a 2-dimensional (σ,τ)-invariant subspace, because $\det[I, S^tTIT^tS] = 0$ holds by hypothesis (iii). ∎

3. Proof of the main result. The following proposition tells us for which even natural number n the identity $\det[I,A] = 0$ holds for all skew-symmetric matrices $A \in \mathbb{R}^{n\times n}$. Thus it is a statement on the Lie algebra $\mathfrak{o}(n)$ of the real Lie group $\mathrm{O}(n)$, which seems to be of independent interest.

PROPOSITION 3.1. *Let n be an even natural number.*

(i) *If* $n \equiv 0 \pmod 4$, *then there exists a permutation matrix* $P \in \mathrm{O}(n)$ *such that* $\det[I, PIP^t] = 2^n$.
(ii) *If* $n \equiv 2 \pmod 4$, *then* $\det[I,A] = 0$ *for all skew-symmetric matrices* $A \in \mathbb{R}^{n\times n}$.

Proof. (i) Let $n = 2m = 4\ell$, where $\ell \in \mathbb{N}$. Set

$$D = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathbb{R}^{m\times m}, \qquad J = J_n = \left(\begin{array}{c|c} & -D \\ \hline D & \end{array}\right) \in \mathbb{R}^{n\times n}.$$

Calculation of IJ and JI respectively shows that

$$IJ = J_m \otimes E = -JI, \quad \text{where} \quad E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\det[I,J] = \det(IJ - JI) = \det(2IJ) = 2^n(\det I)(\det J) = 2^n.$$

To accomplish the proof of (i), it suffices to exhibit for each $\ell \in \mathbb{N}$ a permutation matrix $P_\ell \in \mathrm{O}(n)$ with $P_\ell I P_\ell^t = J$. We do this by induction on $\ell \in \mathbb{N}$.

If $\ell = 0$, then $I = J = \mathbb{I}_0$, the unique matrix in $\mathbb{R}^{0\times 0}$. Hence $P_0 = \mathbb{I}_0$ satisfies $P_0 I P_0^t = J \in \mathbb{R}^{0\times 0}$. If $\ell = 1$, then

$$P_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

is a permutation matrix in $\mathrm{O}(4)$ such that $P_1 I P_1^t = J \in \mathbb{R}^{4\times 4}$. If $\ell \geq 1$ and

$P_\ell \in \mathrm{O}(n)$ is a permutation matrix with $P_\ell I P_\ell^t = J \in \mathbb{R}^{n\times n}$, then

$$P_{\ell+1} = \left(\begin{array}{cc|c|cc} 1 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ \hline & & P_\ell & & \\ \hline 0 & 0 & & 0 & 1 \\ 0 & 1 & & 0 & 0 \end{array}\right)$$

is a permutation matrix in $\mathrm{O}(n+4)$ such that $P_{\ell+1} I P_{\ell+1}^t = J \in \mathbb{R}^{(n+4)\times(n+4)}$.

(ii) Let $n = 2m = 4\ell + 2$, where $\ell \in \mathbb{N}$, set $\underline{m} = \{1, \dots, m\}$, and let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric. Since I is skew-symmetric, so is $B = [I, A]$. We view B as an $m \times m$-matrix of 2×2-blocks B_{rs}. Denoting the entries of A by a_{ij}, we obtain for all $(r, s) \in \underline{m}^2$ the identities

$$\begin{aligned} B_{rs} = &\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{2r-1,2s-1} & a_{2r-1,2s} \\ a_{2r,2s-1} & a_{2r,2s} \end{pmatrix} \\ &- \begin{pmatrix} a_{2r-1,2s-1} & a_{2r-1,2s} \\ a_{2r,2s-1} & a_{2r,2s} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -c_{rs} & b_{rs} \\ b_{rs} & c_{rs} \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} b_{rs} \\ c_{rs} \end{pmatrix} = \begin{pmatrix} a_{2r-1,2s-1} - a_{2r,2s} \\ a_{2r-1,2s} + a_{2r,2s-1} \end{pmatrix}.$$

Since B is skew-symmetric and all B_{rs} are symmetric, we conclude that $B_{sr} = -B_{rs}$ for all $(r, s) \in \underline{m}^2$. Now we permute the columns of B by transposing each subsequent pair of columns that forms a block-column. Thus we arrive at

$$C = BP, \qquad \text{where } P = \mathbb{I}_m \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By construction, the 2×2-blocks of C are

$$C_{rs} = \begin{pmatrix} b_{rs} & -c_{rs} \\ c_{rs} & b_{rs} \end{pmatrix},$$

and they satisfy $C_{sr} = -C_{rs}$ for all $(r, s) \in \underline{m}^2$. Therefore, the matrix C admits the following complex interpretation.

Let $Z \in \mathbb{C}^{m\times m}$ be the complex matrix with entries $z_{rs} = b_{rs} + c_{rs}i$. Then Z is skew-symmetric and $m = 2\ell + 1$ is odd, so $\det Z = 0$. Equivalently, the $\mathbb{C}$-linear map $\underline{Z} : \mathbb{C}^m \to \mathbb{C}^m$, $\underline{Z}(z) = Zz$, is not invertible. Viewed as an $\mathbb{R}$-linear map, $\underline{Z}$ is all the same non-invertible. Equivalently, the matrix M of the $\mathbb{R}$-linear map $\underline{Z} : \mathbb{C}^m \to \mathbb{C}^m$, $\underline{Z}(z) = Zz$, in the standard basis of the real vector space $\mathbb{C}^m$ is not invertible. Note, however, that $M = C$.

Summarizing, we have found that $[I, A] = B = BP^2 = CP = MP$, where M is not invertible. So $\det[I, A] = (\det M)(\det P) = 0$. ∎

Finally, our main result, Theorem 1.1, falls out as a trivial consequence of Corollary 2.4 and Proposition 3.1.

Proof of Theorem 1.1. Let V be a Euclidean space of even dimension n.

If every pair of rotations in V has a 2-dimensional invariant subspace, then Corollary 2.4 implies that $\det[I, UIU^t] = 0$ for all $U \in \mathrm{O}(n)$. Proposition 3.1(i) implies further that $n \not\equiv 0 \pmod 4$. Hence $n \equiv 2 \pmod 4$.

Conversely, if $n \equiv 2 \pmod 4$, then $\det[I, A] = 0$ holds for all skew-symmetric matrices $A \in \mathbb{R}^{n\times n}$, by Proposition 3.1(ii). In particular, the identity $\det[I, UIU^t] = 0$ holds for all $U \in \mathrm{O}(n)$. Hence every pair of rotations in V has a 2-dimensional invariant subspace, by Corollary 2.4. ∎

Remark. An alternative approach to Theorem 1.1 can be found in the recent article [6].

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