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HARMONIC MAPS FROM COMPACT KÄHLER MANIFOLDS WITH POSITIVE SCALAR CURVATURE TO KÄHLER MANIFOLDS OF STRONGLY SEMINEGATIVE CURVATURE

BY

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Dedicated to Prof. Qian Min on the occasion of his eightieth birthday

Abstract. It is well known there is no non-constant harmonic map from a closed Riemannian manifold of positive Ricci curvature to a complete Riemannian manifold with non-positive sectional curvature. If one reduces the assumption on the Ricci curvature to one on the scalar curvature, such a vanishing theorem does not hold in general. This raises the question: What information can we obtain from the existence of a non-constant harmonic map? This paper gives an answer to this problem when both manifolds are Kähler; the results obtained are optimal.

1. Introduction. Using the technique that now bears his name, Bochner showed that if M is a closed Riemannian manifold with positive Ricci curvature, then every harmonic 1-form on M must vanish [B]. If the 1-form is the differential of a harmonic map ϕ from M to the unit circle S^1, then this is equivalent to saying that ϕ is a constant map. Using the same technique, Eells and Sampson generalized this kind of vanishing result to any harmonic map $\phi : M \to N$, where the target N is a complete Riemannian manifold whose sectional curvature is non-positive: under the same assumption on M as before, ϕ must be a constant map, or equivalently, $d\phi$ must be a vanishing ϕ^*TN-valued harmonic 1-form [ES, p. 124, Corollary].

The concepts of Kähler manifolds with strongly negative and strongly seminegative curvature were introduced by Siu [Siu1]. Using what he called the $\partial\overline{\partial}$-Bochner–Kodaira technique instead of the Laplacian of squared norm, Siu overcame the difficulty of the traditional Bochner technique involving the curvature terms of opposite signs of both manifolds, and proved that any harmonic map between compact Kähler manifolds must be holomorphic or anti-holomorphic if the target has strongly negative curvature. However, the $\partial\overline{\partial}$-Bochner–Kodaira technique fails if we weaken his assumption to the target N having strongly seminegative curvature.

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In this paper we use the traditional Bochner technique, but we only assume that the scalar curvature R of M is positive and the curvature tensor of N is strongly seminegative in the sense of Siu. We prove the following optimal result:

THEOREM. *Let M be a compact Kähler manifold of complex dimension m with scalar curvature $R > 0$, and N a complete Kähler manifold of strongly seminegative curvature. If there is a non-constant harmonic map $\phi : M \to N$, then*

$$\int_M \frac{1}{R}\left(\frac{R^2}{4m(m-1)} - |E|^2\right) e^{1/2}(\phi)\, d\mathrm{vol}_M \le 0, \tag{1.1}$$

where $e(\phi) = |\partial\phi|^2 + |\partial\overline{\phi}|^2$ denotes the energy density of ϕ and E denotes the trace-free Ricci tensor of M. Moreover, *equality is attained in* (1.1) *if and only if $\phi(M)$ is a geodesic Riemannian surface in N*, *the scalar curvature R of M is constant and the universal covering of M is holomorphically isometric to a direct product of $\mathbb{C}$ and a Kähler–Einstein manifold M' of constant scalar curvature R.*

The proof of this theorem is actually a refinement of the Bochner method, using some ideas of Gursky [Gur] who considered similar problems for conformal vector fields on four-dimensional Riemannian manifolds.

The paper is organized as follows. In Sections 2 and 3, we review the relevant facts regarding harmonic maps, derive the Bochner–Weitzenböck formulas by different methods (cf. [Lic] and [ES]) using a Ricci identity for any smooth map between Kähler manifolds, and then we present some useful lemmas. We finally give the proof of the Theorem in Section 4.

This is a continuation of my previous work [Yang] where I considered a similar problem in the Riemannian case.

2. Preliminaries. Let M and N be Kähler manifolds of complex dimensions m and n, respectively, and suppose further that M is compact. Let $\{z^i\}$ and $\{w^\alpha\}$ be local holomorphic coordinates on M and N respectively. Write the Kähler metrics in these local coordinates as

$$ds_M^2 = 2\,\mathrm{Re}(g_{i\overline{j}} dz^i d\overline{z}^j), \qquad ds_N^2 = 2\,\mathrm{Re}(h_{\alpha\overline{\beta}} dw^\alpha d\overline{w}^\beta),$$

where we adopt the Einstein summation convention.

Let $\phi : M \to N$ be a smooth map. The complexified differential

$$d^{\mathbb{C}}\phi : T^{\mathbb{C}}M \to \phi^* T^{\mathbb{C}}N$$

has a partial splitting in terms of complex types $(1,0)$ and $(0,1)$:

$$\partial\phi : T^{1,0}M \to \phi^*T^{1,0}N, \qquad \overline{\partial}\phi : T^{0,1}M \to \phi^*T^{1,0}N,$$
$$\partial\overline{\phi} : T^{1,0}M \to \phi^*T^{0,1}N, \qquad \overline{\partial\phi} : T^{0,1}M \to \phi^*T^{0,1}N.$$

The bundles $T^{1,0}M$ and $T^{1,0}N$ are Hermitian holomorphic vector bundles; the conjugate of the bundle $T^{1,0}M$ and the pull-back bundles $\phi^*T^{1,0}N$ and $\phi^*T^{0,1}N$ are Hermitian vector bundles as well [K]. Let $\nabla^{T^{\mathbb{C}}M}$ be the Hermitian connection on M and $\nabla^{\phi^{-1}T^{\mathbb{C}}N}$ the pull-back of the Hermitian connection on N. Denote the induced Hermitian connection on $\mathrm{Hom}(T^{\mathbb{C}}M, \phi^*T^{\mathbb{C}}N)$ by ∇. We have the decomposition $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla^{1,0} : \Gamma(T^{1,0}M) \otimes \mathrm{Hom}(T^{\mathbb{C}}M, \phi^*T^{\mathbb{C}}N) \to \mathrm{Hom}(T^{\mathbb{C}}M, \phi^*T^{\mathbb{C}}N),$$

and similarly for $\nabla^{0,1}$. For brevity in the following we denote $\nabla^{1,0}_{\partial/\partial z^i}$ by ∇_i and $\nabla^{1,0}_{\partial/\partial \bar{z}^i}$ by $\nabla_{\bar{i}}$. We have similar definitions for $\nabla^{T^{\mathbb{C}}M}_i, \nabla^{T^{\mathbb{C}}M}_{\bar{i}}, \nabla^{\phi^*T^{\mathbb{C}}N}_{\alpha}$, $\nabla^{\phi^*T^{\mathbb{C}}N}_{\overline{\alpha}}$. Denote the Christoffel symbols and the curvature tensors of M and N respectively by $\Gamma^i_{jk}, \Gamma^{\alpha}_{\beta\gamma}$ and $R_{i\bar{j}k\bar{l}}, R_{\alpha\bar{\beta}\gamma\bar{\delta}}$. Throughout, Γ and R are used for both manifolds; confusion is avoided by using Latin letters for coordinate indices of M and Greek letters for coordinate indices of N. We denote the Ricci tensor of M by $R_{i\bar{j}}$. In terms of the local holomorphic coordinates, the curvature tensor of N is given by

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \partial_\gamma \partial_{\overline{\delta}} g_{\alpha\overline{\beta}} - g^{\mu\overline{\nu}} \partial_\alpha g_{\gamma\overline{\nu}} \partial_{\overline{\beta}} g_{\mu\overline{\delta}}.$$

The sectional curvature of the 2-plane spanned by two tangent vectors

$$X = 2\,\mathrm{Re}(\xi^\alpha \partial_\alpha), \quad Y = 2\,\mathrm{Re}(\eta^\alpha \partial_\alpha)$$

is given by

$$-\frac{R_{\alpha\overline{\beta}\gamma\overline{\delta}}(\xi^\alpha\overline{\eta^\beta} - \eta^\alpha\overline{\xi^\beta})\overline{(\xi^\delta\overline{\eta^\gamma} - \eta^\delta\overline{\xi^\gamma})}}{|X \wedge Y|^2}. \tag{2.1}$$

The curvature $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is said to be *strongly negative* (resp. *strongly seminegative*) if

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}}(A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta})\overline{(A^\delta\overline{B^\gamma} - C^\delta\overline{D^\gamma})}$$

is positive (resp. non-negative) for arbitrary complex numbers A^α, B^α, C^α, D^α such that $A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta} \neq 0$ for at least one pair of indices (α, β). Comparing this with the definition of sectional curvature, by (2.1) we see that the strong negativity of the curvature tensor implies the negativity of the sectional curvature, and the strong seminegativity of the curvature tensor implies that the sectional curvature is non-positive.

View $\partial\phi$ as a section of the Hermitian bundle $\mathrm{Hom}(T^{1,0}M, \phi^*T^{1,0}N)$ and $\partial\overline{\phi}$ as a section of the Hermitian bundle $\mathrm{Hom}(T^{1,0}M, \phi^*T^{0,1}N)$. In local holomorphic coordinates we can write

$$\partial\phi = \phi^\alpha_i dz^i \otimes \partial_\alpha, \quad \partial\overline{\phi} = \phi^{\overline{\alpha}}_i dz^i \otimes \partial_{\overline{\alpha}},$$

where $\phi^\alpha_i = \partial_i \phi^\alpha$ and $\phi^{\overline{\alpha}}_i = \partial_i \overline{\phi^\alpha}$. The *partial energy densities* of ϕ are defined as the following squares of complex norms:

$$e'(\phi) = |\partial\phi|^2 = g^{i\bar{j}}\phi_i^\alpha\phi_{\bar{j}}^{\bar{\beta}}h_{\alpha\bar{\beta}}, \qquad e''(\phi) = |\partial\overline{\phi}|^2 = g^{i\bar{j}}\phi_i^{\overline{\alpha}}\phi_{\bar{j}}^{\beta}h_{\overline{\alpha}\beta}. \tag{2.2}$$

Write the covariant derivatives as $\nabla_{\bar{j}}(\partial\phi) = \phi^\alpha_{i,\bar{j}}dz^i \otimes \partial_\alpha$ and $\nabla_{\bar{j}}(\partial\overline{\phi}) = \phi^{\overline{\alpha}}_{i,\bar{j}}dz^i \otimes \partial_{\overline{\alpha}}$, where

$$\phi^\alpha_{i,\bar{j}} = \partial_i\partial_{\bar{j}}\phi^\alpha + \Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_{\bar{j}}^\gamma, \qquad \phi^{\overline{\alpha}}_{i,\bar{j}} = \partial_i\partial_{\bar{j}}\phi^{\overline{\alpha}} + \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\phi_i^{\overline{\beta}}\phi_{\bar{j}}^{\overline{\gamma}}.$$

Let $\tau'(\phi)$ and $\tau''(\phi)$ denote the traces of $\nabla^{0,1}\partial\phi$ and $\nabla^{0,1}\partial\overline{\phi}$ respectively. In local holomorphic coordinates we have

$$\tau'(\phi) = g^{i\bar{j}}(\partial_i\partial_{\bar{j}}\phi^\alpha + \Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_{\bar{j}}^\gamma), \qquad \tau''(\phi) = g^{i\bar{j}}(\partial_i\partial_{\bar{j}}\phi^{\overline{\alpha}} + \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\phi_i^{\overline{\beta}}\phi_{\bar{j}}^{\overline{\gamma}}).$$

The map ϕ is called *harmonic* if both $\tau'(\phi)$ and $\tau''(\phi)$ vanish. Let $\Delta_M = 2(\partial^*\partial + \partial\partial^*) = 2(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)$ be the complex Laplacian operator of the Kähler manifold M. Then $\tau'(\phi) = 0$ iff $\Delta_M(\partial\phi) = 0$, and $\tau''(\phi) = 0$ iff $\Delta_M(\partial\overline{\phi}) = 0$. Thus ϕ is harmonic iff $\partial\phi$ is a $\phi^*T^{1,0}N$-valued harmonic $(1,0)$-form and $\overline{\partial}\phi$ is a $\phi^*T^{0,1}N$-valued harmonic $(1,0)$-form.

By the definition of the covariant derivative, we have

$$(\phi^\alpha_{k,j\bar{i}} - \phi^\alpha_{k,\bar{i}j})dz^k \otimes \partial_\alpha = (\nabla_j\nabla_{\bar{i}} - \nabla_{\bar{i}}\nabla_j)(\phi^\alpha_k dz^k \otimes \partial_\alpha), \tag{2.3}$$

and since ∇ is the tensor product connection of $\nabla^{T^{\mathbb{C}}M}$ and $\nabla^{\phi^*T^{\mathbb{C}}N}$, we have

$$\begin{aligned}(\nabla_j\nabla_{\bar{i}} - \nabla_{\bar{i}}\nabla_j)(dz^k \otimes \partial_\alpha) = {}& ((\nabla_j^{T^{\mathbb{C}}M}\nabla_{\bar{i}}^{T^{\mathbb{C}}M} - \nabla_{\bar{i}}^{T^{\mathbb{C}}M}\nabla_j^{T^{\mathbb{C}}M})dz^k) \otimes \partial_\alpha \\ & + dz^k \otimes (\nabla_j^{\phi^*T^{\mathbb{C}}N}\nabla_{\bar{i}}^{\phi^*T^{\mathbb{C}}N} - \nabla_{\bar{i}}^{\phi^*T^{\mathbb{C}}N}\nabla_j^{\phi^*T^{\mathbb{C}}N})\partial_\alpha.\end{aligned} \tag{2.4}$$

Note that

$$(\nabla_j^{T^{\mathbb{C}}M}\nabla_{\bar{i}}^{T^{\mathbb{C}}M} - \nabla_{\bar{i}}^{T^{\mathbb{C}}M}\nabla_j^{T^{\mathbb{C}}M})dz^k = R^k_{lj\bar{i}}dz^l \tag{2.5}$$

and by the definition of pull-back connection, we have

$$\begin{aligned}&(\nabla_j^{\phi^*T^{\mathbb{C}}N}\nabla_{\bar{i}}^{\phi^*T^{\mathbb{C}}N} - \nabla_{\bar{i}}^{\phi^*T^{\mathbb{C}}N}\nabla_j^{\phi^*T^{\mathbb{C}}N})\partial_\alpha \\ &\quad = (\nabla^{T^{\mathbb{C}}N}_{\phi_*(\partial_j)}\nabla^{T^{\mathbb{C}}N}_{\phi_*(\partial_{\bar{i}})} - \nabla^{T^{\mathbb{C}}N}_{\phi_*(\partial_{\bar{i}})}\nabla^{T^{\mathbb{C}}N}_{\phi_*(\partial_j)})\partial_\alpha \\ &\quad = \phi_j^\beta\phi_{\bar{i}}^{\overline{\gamma}}(\nabla_\beta^{T^{\mathbb{C}}N}\nabla_{\overline{\gamma}}^{T^{\mathbb{C}}N} - \nabla_{\overline{\gamma}}^{T^{\mathbb{C}}N}\nabla_\beta^{T^{\mathbb{C}}N})\partial_\alpha \\ &\qquad + \phi_j^{\overline{\beta}}\phi_{\bar{i}}^{\gamma}(\nabla_{\overline{\beta}}^{T^{\mathbb{C}}N}\nabla_{\gamma}^{T^{\mathbb{C}}N} - \nabla_{\gamma}^{T^{\mathbb{C}}N}\nabla_{\overline{\beta}}^{T^{\mathbb{C}}N})\partial_\alpha \\ &\quad = -R^\delta_{\alpha\beta\overline{\gamma}}\phi_j^\beta\phi_{\bar{i}}^{\overline{\gamma}}\partial_\delta + R^\delta_{\alpha\gamma\overline{\beta}}\phi_j^{\overline{\beta}}\phi_{\bar{i}}^\gamma\partial_\delta = R^\delta_{\alpha\beta\overline{\gamma}}(\phi_{\bar{i}}^\beta\phi_j^{\overline{\gamma}} - \phi_j^\beta\phi_{\bar{i}}^{\overline{\gamma}})\partial_\delta.\end{aligned} \tag{2.6}$$

By (2.3)–(2.6) we get the following Ricci identity:

$$\phi^\alpha_{k,j\bar{i}} - \phi^\alpha_{k,\bar{i}j} = R^l_{kj\bar{i}}\phi^\alpha_l + R^\alpha_{\delta\beta\overline{\gamma}}(\phi_{\bar{i}}^\beta\phi_j^{\overline{\gamma}} - \phi_j^\beta\phi_{\bar{i}}^{\overline{\gamma}})\phi^\delta_k. \tag{2.7}$$

Similarly, noting that

$$
\begin{aligned}
(2.8)\qquad & (\nabla_j^{\phi^*T^{\mathbb{C}}N}\nabla_{\bar i}^{\phi^*T^{\mathbb{C}}N}-\nabla_{\bar i}^{\phi^*T^{\mathbb{C}}N}\nabla_j^{\phi^*T^{\mathbb{C}}N})\partial_{\overline{\alpha}}\\
&=-\overline{(\nabla_i^{\phi^*T^{\mathbb{C}}N}\nabla_{\bar j}^{\phi^*T^{\mathbb{C}}N}-\nabla_{\bar j}^{\phi^*T^{\mathbb{C}}N}\nabla_i^{\phi^*T^{\mathbb{C}}N})\partial_\alpha}\\
&=\overline{R^{\delta}_{\alpha\beta\overline{\gamma}}(\phi_i^\beta\phi_{\bar j}^{\overline{\gamma}}-\phi_{\bar j}^\beta\phi_i^{\overline{\gamma}})\partial_\delta}=R^{\overline{\delta}}_{\overline{\alpha}\gamma\overline{\beta}}(\phi_{\bar i}^{\overline{\beta}}\phi_j^\gamma-\phi_j^{\overline{\beta}}\phi_{\bar i}^\gamma)\partial_{\overline{\delta}},
\end{aligned}
$$

we get

$$
(2.9)\qquad \phi^{\overline{\alpha}}_{k,j\bar i}-\phi^{\overline{\alpha}}_{k,\bar i j}=R^l_{kj\bar i}\phi_l^{\overline{\alpha}}+R^{\overline{\alpha}}_{\overline{\delta}\gamma\overline{\beta}}(\phi_{\bar i}^{\overline{\beta}}\phi_j^\gamma-\phi_j^{\overline{\beta}}\phi_{\bar i}^\gamma)\phi_k^{\overline{\delta}}.
$$

We remark here that the Ricci identities (2.8) and (2.9) hold for any smooth map ϕ which is not necessarily harmonic.

To prove the theorem we need the following two algebraic lemmas. We refer the readers to [SW, p. 234] for the proof of Lemma 2.1.

Lemma 2.1. *Let $A=(a_{ij})_{m\times m}$ be a Hermitian symmetric matrix with zero trace and $z_1,\dots,z_m\in\mathbb{C}$ be complex numbers. Then*

$$
(2.10)\qquad \Big|\sum_{i,j}a_{ij}z_i\bar z_j\Big|\le\sqrt{\frac{m-1}{m}\sum_{i,j}|a_{ij}|^2}\sum_i|z_i|^2.
$$

Moreover, when $\sum_i|z_i|^2\neq 0$, equality in (2.10) is attained if and only if there exists a unitary $m\times m$ matrix U such that

$$
(2.11)\qquad UAU^{-1}=\begin{pmatrix}(m-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda\end{pmatrix}
$$

and the U-image of $(z_1,\dots,z_m)$ is $((m-1)\lambda,0,\dots,0)$ up to a factor, correspondingly $\sum_{i,j}a_{ij}z_i\bar z_j=(\operatorname{sgn}\lambda)\sqrt{\frac{m-1}{m}\sum_{i,j}|a_{ij}|^2}\sum_i|z_i|^2$.

Lemma 2.2. *Let $A=(a^{ij})_{m\times m}$ be a Hermitian symmetric matrix with zero trace and $H=(h_{\alpha\beta})_{n\times n}$ be a Hermitian positive definite matrix and $Z=(z_i^\alpha)_{m\times n}$ a complex matrix. Then*

$$
(2.12)\qquad \Big|\sum_{i,j,\alpha,\beta}a^{ij}z_i^\alpha\bar z_j^\beta h_{\alpha\beta}\Big|\le\sqrt{\frac{m-1}{m}}\,|A|\sum_{i,\alpha,\beta}z_i^\alpha\bar z_i^\beta h_{\alpha\beta},
$$

where $|A|=\sqrt{\sum_{i,j}|a^{ij}|^2}$. Moreover, when $\sum_i|z_i|^2\neq 0$, equality in (2.12) is attained iff there exists a unitary $m\times m$ matrix U such that UAU^{-1} is diagonalized as (2.11), and there is a non-singular $n\times n$ matrix $(p_{\alpha\delta})$ such *that the* U*-image of* $(z_1^\alpha p_{\alpha\delta}, \dots, z_m^\alpha p_{\alpha\delta})$ *is* $((m-1)\lambda, 0, \dots, 0)$ *up to a factor, for any* $1 \le \delta \le n$. *In particular, the rank of* Z *is* 1.

Proof. Since H is positive definite we have a decomposition $H = P\overline{P}^T$, where P is an $n \times n$ non-singular matrix and $\overline{P}^T$ is the conjugate transpose of P. Thus we have

$$\begin{aligned}
\sum_{i,j,\alpha,\beta} a^{ij} z_i^\alpha \overline{z}_j^\beta h_{\alpha\beta} &= \sum_{i,j,\alpha,\beta,\delta} a^{ij} z_i^\alpha \overline{z}_j^\beta p_{\alpha\delta}\overline{p}_{\beta\delta} \\
&= \sum_{i,j,\delta} a^{ij} \Big(\sum_\alpha z_i^\alpha p_{\alpha\delta}\Big)\Big(\sum_\beta \overline{z}_j^\beta \overline{p}_{\beta\delta}\Big) \\
&\le \sum_\delta \sqrt{\frac{m-1}{m}\sum_{ij} |a^{ij}|^2} \sum_{i,\alpha} |z_i^\alpha p_{\alpha\delta}|^2 \qquad (2.13) \\
&= \sqrt{\frac{m-1}{m}\sum_{ij} |a^{ij}|^2} \sum_{i,\alpha,\beta,\delta} z_i^\alpha \overline{z}_i^\beta p_{\alpha\delta}\overline{p}_{\beta\delta} \\
&= \sqrt{\frac{m-1}{m}}\, |A| \sum_{i,\alpha,\beta} z_i^\alpha \overline{z}_i^\beta h_{\alpha\beta}.
\end{aligned}$$

If $\sum_i |z_i|^2 \ne 0$, equality in (2.12) is attained iff it is in (2.13). By Lemma 2.1, equality in (2.13) is attained iff the U-image of $(z_1^\alpha p_{\alpha\delta}, \dots, z_m^\alpha p_{\alpha\delta})$ is $((m-1)\lambda, 0, \dots, 0)$ up to a factor for any $1 \le \delta \le n$. Since P is non-singular, the rank of $Z = (z_i^\alpha)$ must be 1 ∎

3. The Weitzenböck formula and some lemmas. Now let $\phi : M \to N$ be a harmonic map. Let

$$\begin{aligned}
2b'(\phi) &= |\nabla^{1,0}\partial\phi|^2 + |\nabla^{0,1}\partial\phi|^2 \qquad (3.1) \\
&= g^{i\overline{j}} g^{k\overline{l}} \phi^\alpha_{i,k} \phi^{\overline{\beta}}_{\overline{j},\overline{l}} h_{\alpha\overline{\beta}} + g^{i\overline{j}} g^{\overline{k}l} \phi^\alpha_{i,\overline{k}} \phi^{\overline{\beta}}_{\overline{j},l} h_{\alpha\overline{\beta}},
\end{aligned}$$

$$\begin{aligned}
2b''(\phi) &= |\nabla^{1,0}\partial\overline{\phi}|^2 + |\nabla^{0,1}\partial\overline{\phi}|^2 \qquad (3.2) \\
&= g^{i\overline{j}} g^{k\overline{l}} \phi^{\overline{\alpha}}_{i,k} \phi^{\beta}_{\overline{j},\overline{l}} h_{\overline{\alpha}\beta} + g^{i\overline{j}} g^{\overline{k}l} \phi^{\overline{\alpha}}_{i,\overline{k}} \phi^{\beta}_{\overline{j},l} h_{\overline{\alpha}\beta}.
\end{aligned}$$

By the compatibility conditions of Hermitian connections and Hermitian metrics of the bundles $\mathrm{Hom}(T^{1,0}M, \phi^*T^{1,0}N)$ and $\mathrm{Hom}(T^{1,0}M, \phi^*T^{0,1}N)$ and by (2.2), we have

$$\begin{aligned}
\tfrac{1}{2}\Delta_M e'(\phi) &= b'(\phi) + g^{k\overline{i}} g^{j\overline{l}} \phi^\alpha_{j,k\overline{i}} \phi^{\overline{\beta}}_{\overline{l}} h_{\alpha\overline{\beta}}, \\
\tfrac{1}{2}\Delta_M e''(\phi) &= b''(\phi) + g^{k\overline{i}} g^{j\overline{l}} \phi^{\overline{\alpha}}_{j,k\overline{i}} \phi^{\beta}_{\overline{l}} h_{\overline{\alpha}\beta}.
\end{aligned}$$

Noting that $\phi^\alpha_{j,k}$ is symmetric in j and k, by (2.7) we have

$$(3.3)\qquad \begin{aligned} g^{k\bar i}\phi^\alpha_{j,k\bar i} &= g^{k\bar i}\phi^\alpha_{k,j\bar i} \\ &= g^{k\bar i}(\phi^\alpha_{k,\bar i j} + R^l_{kj\bar i}\phi^\alpha_l + R^\alpha_{\delta\beta\overline{\gamma}}(\phi^\beta_{\bar i}\phi^{\overline{\gamma}}_j - \phi^\beta_j\phi^{\overline{\gamma}}_{\bar i})\phi^\delta_k) \\ &= (g^{k\bar i}\phi^\alpha_{k,\bar i})_{,j} + g^{k\bar i}(R^l_{kj\bar i}\phi^\alpha_l + R^\alpha_{\delta\beta\overline{\gamma}}(\phi^\beta_{\bar i}\phi^{\overline{\gamma}}_j - \phi^\beta_j\phi^{\overline{\gamma}}_{\bar i})\phi^\delta_k) \\ &= R^l_j\phi^\alpha_l + g^{k\bar i}R^\alpha_{\delta\beta\overline{\gamma}}(\phi^\beta_{\bar i}\phi^{\overline{\gamma}}_j - \phi^\beta_j\phi^{\overline{\gamma}}_{\bar i})\phi^\delta_k. \end{aligned}$$

Combining (3.1) with (3.3) we get the following Weitzenböck formula for harmonic maps:

$$\tfrac12\Delta_M e'(\phi) = b'(\phi) + R^{i\bar j}\phi^\alpha_i\phi^{\overline{\beta}}_{\bar j}h_{\alpha\overline{\beta}} + R^\alpha_{\delta\eta\overline{\gamma}}(\phi^\eta_{\bar i}\phi^{\overline{\gamma}}_j - \phi^\eta_j\phi^{\overline{\gamma}}_{\bar i})\phi^\delta_k\phi^{\overline{\beta}}_{\bar l}g^{k\bar i}g^{j\bar l}h_{\alpha\overline{\beta}},$$

or by lifting and lowering the indices,

$$(3.4)\quad \tfrac12\Delta_M e'(\phi) = b'(\phi) + R^{i\bar j}\phi^\alpha_i\phi^{\overline{\beta}}_{\bar j}h_{\alpha\overline{\beta}} + R_{\delta\overline{\beta}\eta\overline{\gamma}}(\phi^\eta_{\bar i}\phi^{\overline{\gamma}}_j - \phi^\eta_j\phi^{\overline{\gamma}}_{\bar i})\phi^\delta_k\phi^{\overline{\beta}}_{\bar l}g^{k\bar i}g^{j\bar l}.$$

Likewise by (2.9) we have

$$(3.5)\qquad g^{k\bar i}\phi^{\overline{\alpha}}_{j,k\bar i} = R^l_j\phi^{\overline{\alpha}}_l + g^{k\bar i}R^{\overline{\alpha}}_{\overline{\delta}\eta\overline{\gamma}}(\phi^{\overline{\gamma}}_{\bar i}\phi^\eta_j - \phi^{\overline{\gamma}}_j\phi^\eta_{\bar i})\phi^{\overline{\delta}}_k.$$

Thus we get

$$(3.6)\quad \tfrac12\Delta_M e''(\phi) = b''(\phi) + R^{i\bar j}\phi^{\overline{\alpha}}_i\phi^\beta_{\bar j}h_{\overline{\alpha}\beta} + R_{\delta\overline{\beta}\eta\overline{\gamma}}(\phi^\eta_j\phi^{\overline{\gamma}}_{\bar i} - \phi^\eta_{\bar i}\phi^{\overline{\gamma}}_j)\phi^\delta_{\bar l}\phi^{\overline{\beta}}_k g^{k\bar i}g^{j\bar l}.$$

Let

$$(3.7)\qquad e(\phi) = e'(\phi) + e''(\phi), \qquad b(\phi) = b'(\phi) + b''(\phi).$$

Then by (3.4) and (3.6) we get

$$(3.8)\quad \begin{aligned} \tfrac12\Delta_M e(\phi) &= b(\phi) + R^{i\bar j}(\phi^\alpha_i\phi^{\overline{\beta}}_{\bar j} + \phi^{\overline{\beta}}_i\phi^\alpha_{\bar j})h_{\overline{\alpha}\beta} \\ &\quad + R_{\delta\overline{\beta}\eta\overline{\gamma}}(\phi^\eta_j\phi^{\overline{\gamma}}_{\bar i} - \phi^\eta_{\bar i}\phi^{\overline{\gamma}}_j)\overline{(\phi^\beta_{\bar k}\phi^{\overline{\delta}}_l - \phi^\beta_l\phi^{\overline{\delta}}_{\bar k})}g^{k\bar i}g^{j\bar l}. \end{aligned}$$

Now suppose that the curvature tensor of N is strongly seminegative. Then the last curvature term on the right hand side of (3.8) is non-negative. On the other hand, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\mathrm{grad}(|\partial\phi|^2)| &\le |\partial\phi|(|\nabla^{1,0}\partial\phi| + |\nabla^{0,1}\partial\phi|), \\ |\mathrm{grad}(|\partial\overline{\phi}|^2)| &\le |\partial\overline{\phi}|(|\nabla^{1,0}\partial\overline{\phi}| + |\nabla^{0,1}\partial\overline{\phi}|), \end{aligned}$$

or equivalently

$$(3.9)\qquad b'(\phi) = \tfrac12(|\nabla^{1,0}\partial\phi|^2 + |\nabla^{0,1}\partial\phi|^2) \ge |\mathrm{grad}\,|\partial\phi|\,|^2,$$

$$(3.10)\qquad b''(\phi) = \tfrac12(|\nabla^{1,0}\partial\overline{\phi}|^2 + |\nabla^{0,1}\partial\overline{\phi}|^2) \ge |\mathrm{grad}\,|\partial\overline{\phi}|\,|^2.$$

Let $D = \{x \in M \mid \partial\phi(x) = \partial\overline{\phi}(x) = 0\}$. Applying the Aronszajn–Carleman unique continuation theorem for elliptic PDE to vector bundle valued harmonic forms [EL, p. 10, (1.26)], we find that the set D is of measure zero. Combining this fact with (3.8), (3.9) and (3.10) yields the following proposition.

PROPOSITION 3.1. *Let ϕ be a non-constant harmonic map between Kähler manifolds M and N. Assume that the curvature of N is strongly seminegative. Then*

$$\tag{3.11} \begin{aligned} &\tfrac{1}{2}\Delta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)\\ &\qquad \geq |\mathrm{grad}\,|\partial\phi|\,|^2+|\mathrm{grad}\,|\partial\overline{\phi}|\,|^2+R^{i\overline{j}}(\phi_i^\alpha\phi_{\overline{j}}^{\overline{\beta}}+\phi_i^{\overline{\beta}}\phi_{\overline{j}}^{\alpha})h_{\overline{\alpha}\beta}\end{aligned}$$

on M in the sense of distributions.

PROPOSITION 3.2. *Let ϕ be a non-constant harmonic map between Kähler manifolds M and N, and suppose that the curvature of N is strongly seminegative. For small $\varepsilon > 0$, define $D_\varepsilon = \{x\in M \mid \sqrt{|\partial\phi|^2+|\partial\overline{\phi}|^2}\geq\varepsilon\}$ and*

$$\varrho_\varepsilon(x)=\begin{cases}\sqrt{|\partial\phi|^2+|\partial\overline{\phi}|^2} & \text{if } x\in D_\varepsilon,\\ \varepsilon & \text{if } x\notin D_\varepsilon.\end{cases}$$

Then

$$\tag{3.12} \int_M R^{i\overline{j}}(\phi_i^\alpha\phi_{\overline{j}}^{\overline{\beta}}+\phi_i^{\overline{\beta}}\phi_{\overline{j}}^{\alpha})h_{\overline{\alpha}\beta}\,\frac{1}{\varrho_\varepsilon}\,d\mathrm{vol}_M\leq 0.$$

Proof. For brevity we omit $d\mathrm{vol}_M$ from the integrals that follow. Multiplying both sides of (3.11) by $1/\varrho_\varepsilon$ and then integrating over M, we get

$$\tag{3.13} \begin{aligned} &\int_M R^{i\overline{j}}(\phi_i^\alpha\phi_{\overline{j}}^{\overline{\beta}}+\phi_i^{\overline{\beta}}\phi_{\overline{j}}^{\alpha})h_{\overline{\alpha}\beta}\,\frac{1}{\varrho_\varepsilon}\\ &\leq\frac{1}{2}\int_M\frac{1}{\varrho_\varepsilon}\,\Delta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)-\int_M\frac{1}{\varrho_\varepsilon}\,(|\mathrm{grad}\,|\partial\phi|\,|^2+|\mathrm{grad}\,|\partial\overline{\phi}|\,|^2)\\ &\leq\frac{1}{2}\int_M\frac{1}{\varrho_\varepsilon}\,\Delta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)-\int_{D_\varepsilon}\frac{1}{\varrho_\varepsilon}\,(|\mathrm{grad}\,|\partial\phi|\,|^2+|\mathrm{grad}\,|\partial\overline{\phi}|\,|^2).\end{aligned}$$

Denoting by D_ε^c the complement of D_ε in M, we have

$$\tag{3.14} \begin{aligned}\int_M\frac{1}{\varrho_\varepsilon}\,\Delta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)&=\int_{D_\varepsilon}\frac{1}{\varrho_\varepsilon}\,\Delta_M\,(|\partial\phi|^2+|\partial\overline{\phi}|^2)\\ &\quad+\varepsilon^{-1}\int_{D_\varepsilon^c}\Delta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2).\end{aligned}$$

If $D_\varepsilon\neq M$ then the boundary $\partial D_\varepsilon=\partial D_\varepsilon^c$ is a closed hypersurface which may not be connected, but at each point $p\in D_\varepsilon$, the outer unit normal vectors $\vec{n}_2$ of ∂D_ε^c and $\vec{n}_1$ of ∂D_ε differ only in sign. Therefore integrating by parts, we see that the second term on the right hand side of (3.14) is

$$\varepsilon^{-1}\int_{D_\varepsilon^c} \varDelta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2) = \varepsilon^{-1}\int_{D_\varepsilon^c} \nabla^i\nabla_i(|\partial\phi|^2+|\partial\overline{\phi}|^2) = -\varepsilon^{-1}\int_{\partial D_\varepsilon^c} \operatorname{grad}(|\partial\phi|^2+|\partial\overline{\phi}|^2)\cdot\vec{n}_2, \tag{3.15}$$

and the first term on the right hand side of (3.14) is

$$\begin{aligned}&\int_{D_\varepsilon} \varrho_\varepsilon^{-1}\varDelta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)\\ &= \int_{D_\varepsilon} \nabla^i(\varrho_\varepsilon^{-1}\nabla_i(|\partial\phi|^2+|\partial\overline{\phi}|^2) - \int_{D_\varepsilon} \nabla^i\varrho_\varepsilon^{-1}\nabla_i(|\partial\phi|^2+|\partial\overline{\phi}|^2)\\ &= -\int_{\partial D_\varepsilon} \varrho_\varepsilon^{-1}\operatorname{grad}(|\partial\phi|^2+|\partial\overline{\phi}|^2)\cdot\vec{n}_1 - \int_{D_\varepsilon} \operatorname{grad}\varrho_\varepsilon^{-1}\cdot\operatorname{grad}(|\partial\phi|^2+|\partial\overline{\phi}|^2).\end{aligned} \tag{3.16}$$

Note that $\varrho_\varepsilon|_{D_\varepsilon} = \sqrt{|\partial\phi|^2+|\partial\overline{\phi}|^2}$ and $\varrho_\varepsilon|_{\partial D_\varepsilon} = \varepsilon$. The last equality of (3.14) is further simplified as

$$\begin{aligned}\int_{D_\varepsilon} \varrho_\varepsilon^{-1}\varDelta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2) &= 2\int_{D_\varepsilon} \frac{(|\partial\phi|(\operatorname{grad}|\partial\phi|)+|\partial\overline{\phi}|(\operatorname{grad}|\partial\overline{\phi}|))^2}{(\sqrt{|\partial\phi|^2+|\partial\overline{\phi}|^2})^3}\\ &\quad+\varepsilon^{-1}\int_{\partial D_\varepsilon} \operatorname{grad}(|\partial\phi|^2+|\partial\overline{\phi}|^2)\cdot\vec{n}_2.\end{aligned} \tag{3.17}$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned}&(|\partial\phi|(\operatorname{grad}|\partial\phi|)+|\partial\overline{\phi}|(\operatorname{grad}|\partial\overline{\phi}|))^2\\ &\qquad\le (|\partial\phi|^2+|\partial\overline{\phi}|^2)(|\operatorname{grad}|\partial\phi|\,|^2+|\operatorname{grad}|\partial\overline{\phi}|\,|^2).\end{aligned} \tag{3.18}$$

By (3.17), (3.18) we obtain

$$\begin{aligned}&\int_{D_\varepsilon} \varrho_\varepsilon^{-1}\varDelta_M(|\partial\phi|^2+|\partial\overline{\phi}|^2)\\ &\le 2\int_{D_\varepsilon} \frac{1}{\varrho_\varepsilon}(|\operatorname{grad}|\partial\phi|\,|^2+|\operatorname{grad}|\partial\overline{\phi}|\,|^2) + \varepsilon^{-1}\int_{\partial D_\varepsilon} \operatorname{grad}(|\partial\phi|^2+|\partial\overline{\phi}|^2)\cdot\vec{n}_2.\end{aligned} \tag{3.19}$$

Now (3.12) follows from (3.13), (3.14), (3.15) and (3.19). ■

4. Proof of the Theorem. Let M be a compact Kähler manifold of complex dimension m with scalar curvature $R > 0$, and N a complete Kähler manifold of strongly seminegative curvature. Let $\phi : M \to N$ be a non-constant harmonic map. At any given point x, we choose a local holomorphic coordinate system on M such that $g^{i\overline{j}} = \delta_{ij}$, so that the Ricci tensor at x satisfies $R_{i\bar{j}} = R^{i\bar{j}}$. Denote by E the trace-free part of the Ricci tensor $(R_{i\bar{j}})_{m\times m}$, that is, $E_{i\bar{j}} = R_{i\bar{j}} - (R/2m)\delta_{ij}$. Then by Lemma 2.1 we see that

$$
\begin{aligned}
&(4.1) \quad R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^{\alpha}) h_{\bar{\alpha}\beta} \\
&(4.2) \quad = E_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^{\alpha}) h_{\bar{\alpha}\beta} + \frac{R}{2m}\,\delta_{ij}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^{\alpha}) h_{\bar{\alpha}\beta} \\
&(4.3) \quad \geq -\sqrt{\frac{m-1}{m}}\,|E|(|\partial\phi|^2 + |\partial\overline{\phi}|^2) + \frac{R}{2m}\,(|\partial\phi|^2 + |\partial\overline{\phi}|^2) \\
&(4.4) \quad = \frac{m-1}{R}\Bigg[\bigg(|E| - \frac{R}{2\sqrt{m(m-1)}}\bigg)^2 \\
&\qquad\qquad - \bigg(|E|^2 - \frac{R^2}{4m(m-1)}\bigg)\Bigg](|\partial\phi|^2 + |\partial\overline{\phi}|^2) \\
&(4.5) \quad \geq \frac{m-1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)(|\partial\phi|^2 + |\partial\overline{\phi}|^2)
\end{aligned}
$$

Substituting (4.5) into (3.12) gives

$$
(4.6) \qquad \int_M \frac{m-1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)(|\partial\phi|^2 + |\partial\overline{\phi}|^2)\varrho_\varepsilon^{-1} \leq 0
$$

for small $\varepsilon > 0$. Note that

$$
\begin{aligned}
\int_M & \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)(|\partial\phi|^2 + |\partial\overline{\phi}|^2)\varrho_\varepsilon^{-1} \\
&= \int_M \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)e^{1/2}(\phi) \\
&\quad + \int_M \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)e^{1/2}(\phi)(e^{1/2}(\phi)\varrho_\varepsilon^{-1} - 1) \\
&= \int_M \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)e^{1/2}(\phi) \\
&\quad + \int_{D_\varepsilon^c} \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)e^{1/2}(\phi)(e^{1/2}(\phi)\varrho_\varepsilon^{-1} - 1)
\end{aligned}
$$

and $0 \leq e^{1/2}(\phi)\varrho_\varepsilon^{-1} \leq 1$ on D_ε^c and $\lim_{\varepsilon\to 0} \operatorname{vol}(D_\varepsilon^c) = 0$. From the compactness of M, we can let $\varepsilon \to 0$ in (4.6) to conclude that

$$
(4.7) \qquad \int_M \frac{1}{R}\bigg(\frac{R^2}{4m(m-1)} - |E|^2\bigg)e^{1/2}(\phi) \leq 0.
$$

Now we claim that if equality holds in (4.7), then equalities in (4.3) and (4.5) both hold at any point of M.

In fact, if equality holds in (4.7), but does not hold in (4.3) or (4.5) at some points of M, then from (4.1) to (4.5) we have the following inequality:

$$R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta} \geq \frac{m-1}{R}\left(\frac{R^2}{4m(m-1)} - |E|^2\right)(|\partial\phi|^2 + |\partial\overline{\phi}|^2),$$

and it is strict at some points on M. Thus

$$\frac{R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta}}{(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2}} \geq \frac{m-1}{R}\left(\frac{R^2}{4m(m-1)} - |E|^2\right)(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2},$$

and it is a strict inequality at some points of M. Hence

$$\int_M \frac{R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta}}{(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2}} > \int_M \frac{m-1}{R}\left(\frac{R^2}{4m(m-1)} - |E|^2\right)(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2}.$$

Now note that the right hand side of the above strict inequality is zero by our assumption that (4.7) is an equality. Hence

$$\int_M R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta} \frac{1}{(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2}} > 0.$$

However, by (3.12) we have

$$\int_M R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta} \frac{1}{\varrho_\varepsilon} \leq 0.$$

Letting $\varepsilon \to 0$, we get

$$\int_M R_{i\bar{j}}(\phi_i^\alpha \phi_{\bar{j}}^{\bar{\beta}} + \phi_i^{\bar{\beta}} \phi_{\bar{j}}^\alpha) h_{\overline{\alpha}\beta} \frac{1}{(|\partial\phi|^2 + |\partial\overline{\phi}|^2)^{1/2}} \leq 0, \tag{4.8}$$

which is a contradiction. Thus if equality holds in (4.7), then equalities in (4.3) and (4.5) both hold at any point of M.

By Lemma 2.2, E can be diagonalized at each point of M as

$$(E_{i\bar{j}})_{m\times m} = \begin{pmatrix} (m-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix} \tag{4.9}$$

and

$$|E| = \frac{R}{2\sqrt{m(m-1)}}.$$

(4.7) and (4.9) give $\lambda = -R/2m(m-1)$ and we must have $\phi^\alpha_{\bar{j}} = \phi^{\overline{\alpha}}_{\bar{j}} = 0$ for any α and $j > 1$, and the Ricci tensor is diagonalized as

$$(R_{i\bar{j}})_{m\times m} = \begin{pmatrix} 0 & & & \\ & \frac{R}{2(m-1)} & & \\ & & \ddots & \\ & & & \frac{R}{2(m-1)} \end{pmatrix}.$$

This implies that $R_{i\bar{j}}(\phi^\alpha_i \phi^{\overline{\beta}}_{\bar{j}} + \phi^{\overline{\beta}}_i \phi^\alpha_{\bar{j}}) h_{\overline{\alpha}\beta} = 0$ and from (3.8) we know $b(\phi) = 0$ and thus

$$|\nabla^{1,0}\partial\phi|^2 = |\nabla^{0,1}\partial\phi|^2 = |\nabla^{1,0}\partial\overline{\phi}|^2 = |\nabla^{0,1}\partial\overline{\phi}|^2 = 0.$$

So ϕ is totally geodesic and $\phi(M)$ is a geodesic Riemannian surface in N. In particular, $\partial\phi$ and $\partial\overline{\phi}$ are in fact parallel 1-forms on M and the scalar curvature R is constant. By the de Rham decomposition theorem for Kähler manifolds [KN, p. 171], the universal covering of M is holomorphically isometric to a direct product of $\mathbb{C}$ and a Kähler–Einstein manifold M' of Ricci curvature $R/2(m-1) > 0$.

This completes the proof of the Theorem. ∎

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